SUBMERGED AND BURIED APERTURE ANTENNAS

by

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I. INTRODUCTION

The purpose of this report is to demonstrate the advantages of aperture-type antennas over the conventional linear antennas used in submerged and buried applications. In order to minimize losses in the lossy medium it is necessary to use frequencies lower than 10^6 cps. Thus, the antennas are necessarily small compared to a free-space wavelength. In free space, for antennas that are small compared to wavelength, aperture antennas have no particular advantage over linear antennas. However, in lossy media, as is shown in this report, the antenna efficiency is approximately proportional to its area.

Although, in some cases, it is possible to make the submerged antenna large compared to the wavelength as measured in the lossy medium and thereby enabling a certain degree of directivity within the lossy medium, this directivity is lost in the air above the lossy medium because of the spreading of the rays at the interface because of the large change in index of refraction at the interface. However, the aperture antenna can achieve a "free" improvement by a factor of 2 over the linear antenna because the aperture antenna, by means of its directivity in the lossy medium, eliminates that portion of power which is lost by radiation down deeper into the lossy medium.

Thus, no great improvements in directivity in the air are possible at the low frequencies used in submerged and buried antenna applications. Usually, the only practical route of communcation for submerged and buried antennas is through the air (such as the surface wave route or the ionospheric bounce route). Therefore, it is not practical to make significant improvements in submerged or buried antenna performance by means of directivity.

The only other way of improving the efficiency of submerged and buried antennas is to minimize the dissipation in the surrounding lossy medium. Since the power dissipation is proportional to the field intensity squared, it can be minimized by distributing the field intensity as uniformly as possible. This is shown to be the mechanism which accounts for the main advantage of the aperture-type submerged or buried antennas.

In the following analysis the formula which determines the efficiency of a general aperture antenna in a lossy medium is determined and is used to demonstrate that the efficiency is approximately proportional to the area of the antenna when comparing aperture distributions which are geometrically similar to each other. For illustration purposes, the efficiency is calculated for a specific example. It is shown how these results can be expected on the basis of physical intuition. In conclusion, the method of constructing a typical aperture submerged or buried antenna is described.

II. ANALYSIS OF SUBMERGED AND BURIED APERTURE ANTENNAS

The geometry of the aperture antenna in a lossy medium is shown in Fig. 1.

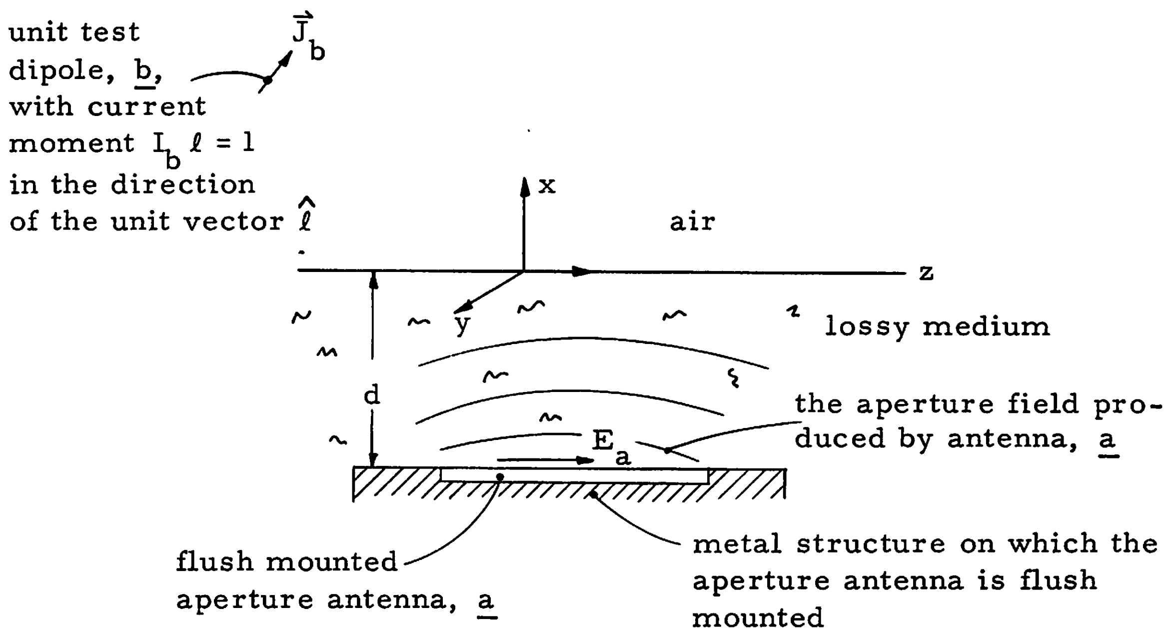

Fig. 1. Geometry of aperture antenna in a lossy medium.

Since the operating frequency of the submerged or buried antenna must be low (~ 10 kc to 10 Mc), the antenna radiates approximately isotropically in the air. Thus the communication effectiveness of the antenna a can be determined by measuring its field at any location in the air by the test dipole, b:

$$\hat{\ell} \cdot \vec{E}_a = \frac{\iiint \vec{J}_b \cdot \vec{E}_a \, dv}{I_b \ell} = \iiint \vec{J}_b \cdot \vec{E}_a \, dv \,. \tag{1}$$

The integral on the right side of Eq. (1) is called the reaction of source a on source b [1].

[1] V. H. Rumsey, "The reaction concept in electromagnetic theory," Phys. Rev., 94, pp. 1483-1491, (June 15, 1954).

By Huygens' principle [2], a source which will produce the same field as antenna, a, is a magnetic current $\vec{K}_a = \vec{E}_a \times \hat{n}$ placed over the aperture and backed by an electric conductor. ($\vec{E}_a$ is the aperture field and $\hat{n}$ is the unit vector normal to the aperture; in this case $\hat{n} = \hat{x}$.) The equivalent source is shown in Fig. 2.

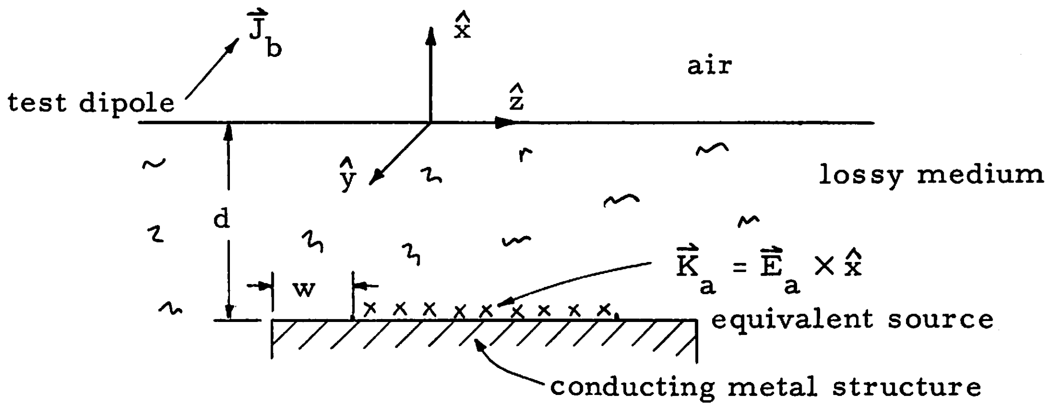

Fig. 2. Huygens' equivalent source for the aperture antenna.

If the distance from the edge of the aperture to the edge of the metal structure, w, is greater than the skin depth of the lossy medium, one can approximate the radiation of the magnetic current $\vec{K}_a$ as though it were on an infinite conducting plane as shown in Fig. 3.

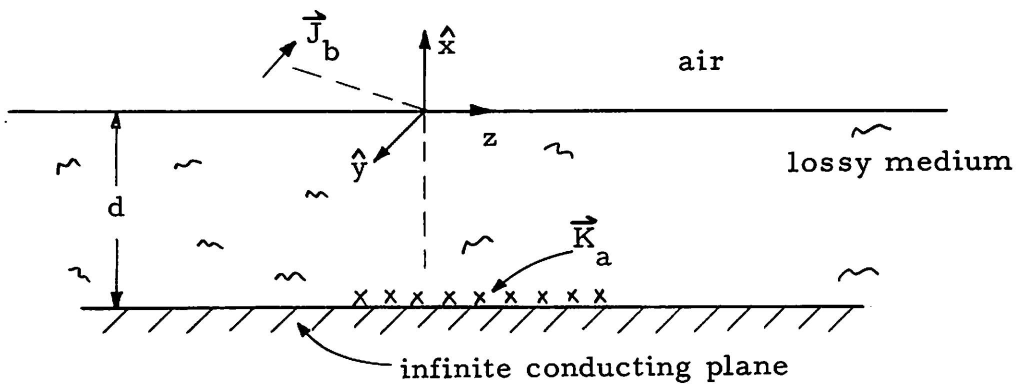

Fig. 3. Equivalent geometry.

[2] V. H. Rumsey, "Some new forms of Huygens' principle," IRE Trans. Antennas and Propagation, Vol. AP-7, pp. S103-S116, December 1959.

By means of the reciprocity theorem [1], we have from Eq. (1) that:

$$\hat{\ell} \cdot \vec{E}_a = -\iint_{\text{aperture}} \vec{K}_a \cdot \vec{H}_b^{*} \, dS \,. \tag{2}$$

Because of the large index of refraction of the lossy medium, the wave from the test dipole b enters the lossy medium approximately perpendicular to the interface. If $\vec{J}_b$ is aligned with $\vec{E}_a$ (assuming $\vec{K}_a$ to be linearly polarized) then the magnetic field $\vec{H}_b$ will be parallel to $\vec{K}_a$ and Eq. (2) gives

$$\left|\vec{E}_a\right| = \left|\vec{H}_b\right| \left|\iint_{ap} \vec{K}_a \, dS\right| \,. \tag{3}$$

We will impose the constraint that the transmitted field $\vec{E}_a$ remain unchanged for the various apertures to be considered, i.e.,

$$\left|\iint_{ap} \vec{K}_a \, dS\right| = \text{constant} \,. \tag{4}$$

Thus, if we scale down an aperture by a factor, s, then the new magnetic current distribution, $\vec{K}_a'(y, z)$, must be related to the old distribution, $\vec{K}_a(y, z)$, by the relation:

$$\vec{K}_a(y, z) = s^2 \vec{K}_a(sy, sz) \,. \tag{5}$$

The input power into antenna a is:

$$P_{in} = \frac{1}{2}\mathrm{Re}\left\{\iint_{ap} \vec{E}_a^* \times \vec{H}_a \cdot \hat{x}\, dy\, dz\right\} = \frac{1}{2}\mathrm{Re}\left\{\iint_{ap} \hat{x} \times \vec{E}_a^* \cdot \vec{H}_a\, dy\, dz\right\}$$

$$P_{in} = -\frac{1}{2}\mathrm{Re}\left\{\iint_{ap} \vec{K}_a^* \cdot \vec{H}_a\, dy\, dz\right\}. \tag{6}$$

We now express $\vec{K}_a$ and $\vec{H}_a$ in terms of their Fourier transforms:

$$\left.\begin{aligned}
\vec{K}_a(y, z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z\, \vec{\mathcal{K}}_a(k_y, k_z)\, e^{jk_y y + jk_z z} \\
\vec{\mathcal{K}}_a(k_y, k_z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, \vec{K}_a(y, z)\, e^{-jk_y y - jk_z z} \\
\vec{H}_a(y, z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_y' \int_{-\infty}^{\infty} dk_z'\, \vec{\mathcal{H}}_a(k_y', k_z')\, e^{jk_y' y + jk_z' z} \\
\vec{\mathcal{H}}_a(k_y', k_z') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, \vec{H}_a(y, z)\, e^{-jk_y' y - jk_z' z}
\end{aligned}\right\} \tag{7}$$

Then by a generalization of Parseval's formula as given in Morse and Feshbach [3], we have from (6)

[3] P. M. Morse and H. Feshbach, Methods of Theoretical Physics, Part I, McGraw-Hill Book Co., Inc., p. 458 (1953).

$$P_{in} = -\frac{1}{2}\,\mathrm{Re}\left\{\int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z\, \vec{\mathcal{K}}_a(k_y, k_z) \cdot \vec{\mathcal{H}}_a(k_y, k_z)\right\} . \tag{8}$$

Each Fourier component $\vec{\mathcal{K}}_a(k_y, k_z)$ is a magnetic current which produces a Fourier component $\vec{\mathcal{H}}_a(k_y, k_z)$ of the magnetic field which is linearly related to $\vec{\mathcal{K}}_a(k_y, k_z)$ so that

$$\vec{\mathcal{K}}_a^{*}(k_y, k_z) \cdot \vec{\mathcal{H}}_a(k_y, k_z) = |\vec{\mathcal{K}}_a(k_y, k_z)|^2\, h(k_y, k_z) , \tag{9}$$

where $h(k_y, k_z)$ is independent of all properties of the aperture antenna except its depth.

Thus, Eq. (8) becomes:

$$P_{in} = -\frac{1}{2}\,\mathrm{Re}\left\{\int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z\, |\vec{\mathcal{K}}_a(k_y, k_z)|^2\, h(k_y, k_z)\right\} . \tag{10}$$

If we now compare this aperture antenna to an antenna which is scaled down by a factor s, we have from Eqs. (5) and (7):

$$\begin{aligned}\vec{\mathcal{K}}_a(k_y, k_z) &= \frac{s^2}{2\pi}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, \vec{K}_a(sy, sz)\, e^{-jk_y y - jk_z z} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d(sy) \int_{-\infty}^{\infty} d(sz)\, \vec{K}_a(sy, sz)\, e^{-j\frac{k_y}{s}(sy) - j\frac{k_z}{s}(sz)} \\ &= \vec{\mathcal{K}}_a\left(\frac{k_y}{s}, \frac{k_z}{s}\right) ,\end{aligned} \tag{11}$$

and

$$P'_{in} = -\frac{1}{2}\,\mathrm{Re}\left\{\int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z \left|\vec{\mathcal{H}}_a \;\frac{k_y}{s}, \frac{k_z}{s}\;\right|^2 h(k_y, k_z)\right\} .$$

set $\quad k'_y = \frac{k_y}{s}$; $k'_z = \frac{k_z}{s}$; then

$$P'_{in} = -\frac{s^2}{2}\,\mathrm{Re}\left\{\int_{-\infty}^{\infty} dk'_y \int_{-\infty}^{\infty} dk'_z \,|\vec{\mathcal{H}}_a(k'_y, k'_z)|^2\, h(sk'_y, sk'_z)\right\} . \tag{12}$$

Thus, if one neglects the effect of $h(k_y, k_z)$, it is seen, by comparing Eqs. (10) and (12), that an antenna scaled down by an amount s will require s^2 times as much input power to transmit the same signal intensity as the original antenna.

On the basis of the above assumptions, the efficiency of the antenna, ξ, is seen to be proportional to the area of the aperture:

$$\xi = \frac{\text{transmitted power}}{\text{input power}} \sim \text{Area} . \tag{13}$$

Equation (13) is intuitively expected, since the aperture field intensity must be approximately inversely proportional to the aperture area in order to provide the same transmitted signal (see Eq. (3)). However, the dissipation in the lossy medium goes as the field intensity squared so that the dissipated power is proportional to:

$$P_{\text{dissipated}} \sim (\text{field intensity})^2 \times \text{Area} \sim \frac{1}{\text{Area}} . \tag{14}$$

The plausibility of (13) will now be demonstrated by a simple example. It will be evident from this example that in order to neglect the variations in $h(k_y, k_z)$, it is necessary that the medium be highly conductive (as in submerged or buried applications).

In this example we assume that the excitation of the aperture at $x = -d$, is uniform with y and

$$\left.\begin{aligned} \vec{E}_a &= \begin{cases} \hat{z}\,\frac{1}{L} & \text{if } -\frac{L}{2} < z < \frac{L}{2} \\ 0 & \text{if } |z| > \frac{L}{2}, \end{cases} \\ \text{thus} \qquad \vec{K}_a &= \begin{cases} \hat{y}\,\frac{1}{L} & \text{if } -\frac{L}{2} < z < \frac{L}{2} \\ 0 & \text{if } |z| > \frac{L}{2}. \end{cases} \end{aligned}\right\} \tag{15}$$

It is seen that this choice of aperture illumination satisfies the constraint (4) of constant transmitted signal for various L. From Eq. (7),

$$\vec{\mathcal{K}}_a(k_z) = \hat{y}\,\frac{1}{\sqrt{2\pi}} \int_{-L/2}^{L/2} \frac{e^{-jk_z z}}{L}\, dz = \hat{y}\sqrt{\frac{2}{\pi}}\, \frac{\sin k_z\, L/2}{k_z L}, \tag{16}$$

and from the appendix we have;

$$h(k_z) = \frac{\omega\,\epsilon_2}{k_{x_2}}\; \frac{q[\exp(-2jk_{x_2} d)] - 1}{q[\exp(-2jk_{x_2} d)] + 1}, \tag{17}$$

$$\text{where } q = \frac{k_{x_1} - k_{x_2}(\epsilon_1/\epsilon_2)}{k_{x_1} + k_{x_2}(\epsilon_1/\epsilon_2)}$$

$$k_{x_1} = \sqrt{\omega^2 \mu \epsilon_1 - k_z^2} \qquad \text{Re } k_{x_1} > 0,$$

$$k_{x_2} = \sqrt{\omega^2 \mu \epsilon_2 - k_z^2} \qquad \text{Re } k_{x_2} > 0, \tag{18}$$

$\epsilon_1 = \epsilon_0$ free space dielectric constant,

$$\epsilon_2 = \epsilon_s - j \frac{\sigma_s}{\omega},$$

$$\delta_s = \sqrt{\frac{2}{\omega \mu \sigma_s}}.$$

ϵ_s and σ_s are the dielectric constant and conductivity of the lossy medium, respectively. It is seen from Eq. (16) that $|\mathcal{H}_a|^2$ is negligible for $k_z >> \frac{1}{L}$. If the aperture size, L, is on the order δ_s, the skin depth in the lossy medium, and $\frac{\sigma_s}{\omega} >> \epsilon_s$,

$$k_{x_2} = \sqrt{-j\frac{4}{\delta_s^2} - k_z^2} \cong \sqrt{-j\frac{4}{\delta_s^2}} = (1 - j)\sqrt{\frac{\omega \mu \sigma_s}{2}};\ k_z \lesssim \frac{1}{L} \approx \frac{1}{\delta_s}. \tag{19}$$

Thus k_{x_2} is roughly independent of k_z for this range of k_z. If $d > \delta_s$ then $\exp(-2jk_{x_2} d) << 1$ and since $|q| < 1$

$$h(k_z) \cong \frac{-\sigma_s}{(j+1)\sqrt{\frac{\omega\mu\sigma_s}{2}}} \tag{20}$$

and is independent of k_z as assumed in Eq. (13). From Eq. (10),

$$P_{in} = \frac{1}{2}\int_{-\infty}^{\infty} dk_z \, \frac{2}{\pi} \, \frac{\sin^2 k_z L}{k_z^2 L^2} \sqrt{\frac{\sigma_s}{2\omega\mu}} = \sqrt{\frac{\sigma_s}{8\omega\mu}} \int_{-\infty}^{\infty} dk_z \, |\mathcal{K}_a|^2 . \tag{21}$$

Again by Parseval's formula:

$$P_{in} = \sqrt{\frac{\sigma_s}{8\omega\mu}} \int_{-L/2}^{L/2} |K_a(z)|^2 \, dz = \sqrt{\frac{\sigma_s}{8\omega\mu}} \int_{-L/2}^{L/2} \frac{1}{L^2} \, dz$$

$$= \frac{1}{L}\sqrt{\frac{\sigma_s}{8\omega\mu}} \tag{22}$$

Note that the Area of the strip ~ L.

Equation (22) is seen to agree with the postulate that P_{in} is inversely proportional to the area, so that the relation (13) is accurate if the medium is lossy (i.e., $\sigma >> \omega\epsilon_s$), if the aperture is not large compared to the skin depth and if the depth of the antenna is greater than one skin depth.

III. CONSTRUCTION OF AN APERTURE ANTENNA

Equation (10) along with the constraint (4) can be used to determine the optimum aperture field distribution for a given application.

As is well-known, at high frequencies (> 500 megacycles), it is very difficult to obtain a desired aperture distribution. However, at the low frequencies involved in submerged or buried antenna applications ($< 10^6$ cps), lumped element circuit components are very effective in controlling voltage and current distributions [4]. For example, suppose one desires an aperture field distribution of

$$\vec{E}_a = \hat{z} \cos \frac{\pi y}{b} \tag{23}$$

over an aperture extending from $-\frac{a}{2} < z < \frac{a}{2}$ and $-\frac{b}{2} < y < \frac{b}{2}$. The process of establishing this field can be summarized as follows (refer to Fig. 4 below).

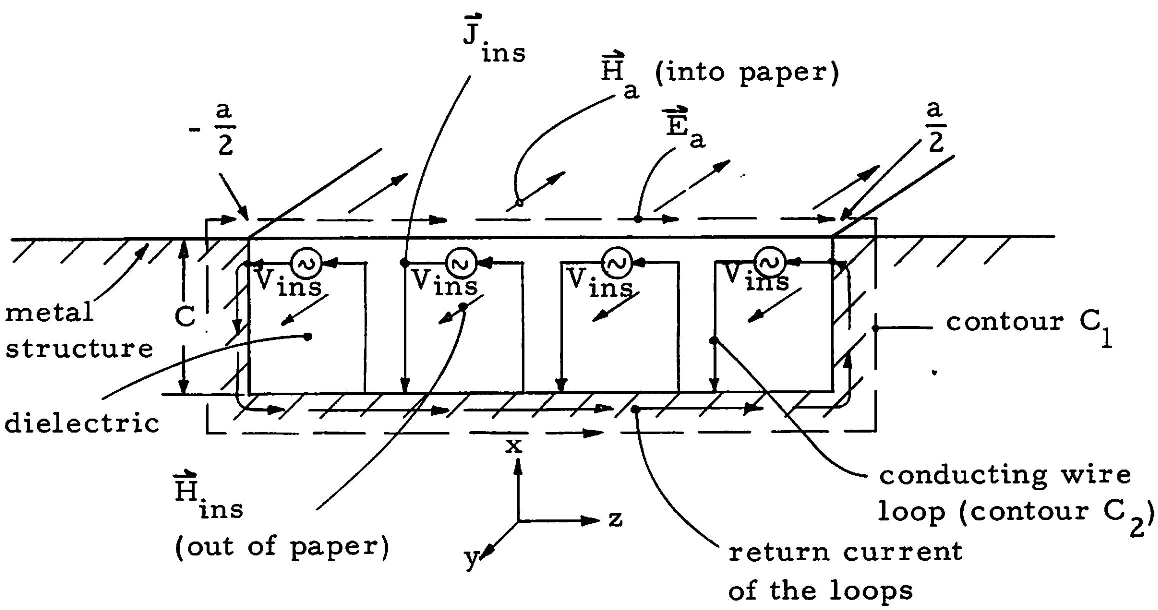

Fig. 4. Cross section of an aperture antenna.

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[4] M. Gans, "An analysis of techniques for improving the gain of submerged or buried antennas," MB Associates Tech. Memo., MB-TM-64/31, December 1964.

Electric current loops, $\vec{J}_{ins}$, are driven inside the dielectric to give rise to a changing magnetic field, $\vec{H}_{ins}$, inside the dielectric. The changing magnetic field inside the dielectric gives rise to the aperture electric field, $\vec{E}_a$, which in turn determines the aperture magnetic field, $\vec{H}_a$. Thus, given $\vec{E}_a$, one can calculate $\vec{H}_a$ and $\vec{H}_{in}$. From the difference between $\vec{H}_a$ and $\vec{H}_{in}$ one knows $\vec{J}_{ins}$, and from $\vec{H}_{ins}$ one knowns the voltage around the loop, V_{ins}. From J_{ins} and V_{ins}, one knows the input impedance of the current loop and can then calculate a circuit network using lumped circuit elements so that one voltage source can feed all the current loops with the proper current.

From the $\vec{E}_a$ given in Eq. (23) one uses Eqs. (7) and (9) to find $\vec{H}_a$. From Maxwell's equations,

$$\oint_{C_1} \vec{E}_a \cdot \vec{d\ell} = \int_{-a/2}^{a/2} \vec{E}_a \cdot \hat{z}\, dz = a \cos \frac{\pi y}{b}$$

$$= - \iint_{\substack{\text{cross}\\\text{section}}} j\omega\, \mu_0\, \vec{H}_{ins} \cdot \hat{y}\, dx\, dz\ ,$$

$$\text{so} \qquad \vec{H}_{ins} = + \hat{y}\, \frac{\cos \pi y/b}{c}\ . \tag{24}$$

Since the electric current is given by the discontinuity in magnetic field,

$$\vec{J}_{ins} = \hat{z}\,(H_{a_y} = H_{ins_y})\ . \tag{25}$$

Again from Maxwell's equations,

$$V_{ins} = -\oint_{C_2} \vec{E} \cdot \vec{d\ell} = j\omega\mu_0 \iint_{\substack{\text{area of} \\ \text{loop}}} \vec{H}_{ins} \cdot \hat{y}\, dx\, dz = j\omega\mu_0 H_{ins_y} A_\ell , \tag{26}$$

where A_ℓ is the area of the loop. The loop impedances are then given by

$$Z_{loop} = \frac{V_{ins}}{J_{ins}} , \tag{27}$$

and this gives the necessary information to construct the network which feeds the loops, so as to obtain the desired aperture distribution, $\vec{E}_a$.

IV. CONCLUSIONS

The foregoing discussion has pointed out two electrical advantages of the aperture-type submerged or buried antenna over its conventional linear counterpart: (1) a "free" gain of 2 is obtained because the aperture antenna radiates only towards the surface and (2) an improvement in efficiency proportional to the ratio of the area of the aperture antenna to the area of the linear antenna.

In order to specify the actual improvement that can be obtained with the aperture antenna one would have to make a comparison with a specific operational linear antenna. However, advantage (2) listed above could improve reception (or transmission by reciprocity) by several orders of magnitude, depending on the area limitations involved.

Another advantage of the aperture antenna, particularly in submarine applications, is its flush mounting capability. In contrast, linear antennas must be mounted well away from metal structures, otherwise the lossy medium would short them to the metal structure.

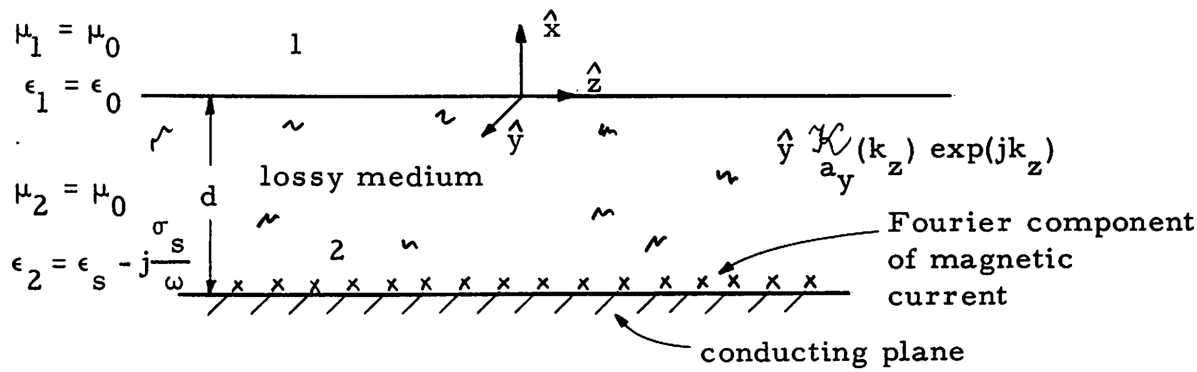
As far as we are aware, the advantages of the aperture antenna have not been utilized [5]. The next step is to analyze a specific application, such as a specific submarine antenna. From the space requirements, Eq. (10) along with constraint (4) would be used to determine the optimum aperture field through the use of a computer. Its efficiency could then be compared theoretically with the presently used linear antenna. Then the feeding network for the aperture antenna could be calculated as described in Sec. III of this report, and the aperture antenna could be constructed. Its performance would then be compared experimentally to the presently used linear antenna.

[5] An excellent review of work on submerged and buried antennas is contained in, "Electromagnetic waves in the earth," Special Issue IEEE Trans. Antennas and Propagation, Vol. AP-11, No. 3, May 1963.

APPENDIX

FIELD PRODUCED BY A FOURIER COMPONENT

This appendix gives the fields produced by the Fourier component, $\vec{\mathcal{K}}_a(k_z)\exp(jk_z z)$, of the magnetic current. (See Eqs. (15) and (16.)



The fields produced by $\vec{\mathcal{K}}_a(k_z)\exp(jk_z z)$ are:

$$\mathcal{E}_{y_1} = \mathcal{E}_{y_2} = 0,$$

$$\mathcal{H}_{x_1} = \mathcal{H}_{x_2} = \mathcal{H}_{z_1} = \mathcal{H}_{z_2} = 0,$$

$$\mathcal{E}_{x_2} = \frac{k_z}{k_{x_2}}\left(\frac{\exp(-jk_{x_2}x) - q[\exp(jk_{x_2}x)]}{\exp(jk_{x_2}d) + q[\exp(-jk_{x_2}d)]}\right)\mathcal{K}_{a_y}(k_z)\exp(jk_z z),$$

$$\mathcal{E}_{z_2} = \left(\frac{q[\exp(jk_{x_2}x)] + \exp(-jk_{x_2}x)}{\exp(jk_{x_2}d) + q[\exp(-jk_{x_2}d)]}\right)\mathcal{H}_{a_y}(k_z)\exp(jk_z z) ,$$

$$\mathcal{H}_{y_2} = \frac{\omega\epsilon_2}{k_{x_2}}\left(\frac{q[\exp(jk_{x_2}x)] - \exp(-jk_{x_2}x)}{\exp(jk_{x_2}d) + q[\exp(-jk_{x_2}d)]}\right)\mathcal{H}_{a_y}(k_z)\exp(jk_z z),$$

$$\mathcal{E}_{x_1} = \frac{\epsilon_2 k_z}{\epsilon_1 k_{x_2}}\left(\frac{1-q}{\exp(jk_{x_2}d) + q[\exp(-jk_{x_2}d)]}\right)\mathcal{H}_{a_y}(k_z)$$

$$\cdot \exp(-jk_{x_1}x + jk_z z) ,$$

$$\mathcal{E}_{z_1} = \frac{k_{x_1}\epsilon_2}{k_{x_2}\epsilon_1}\left(\frac{1-q}{\exp(jk_{x_2}d) + q[\exp(-jk_{x_2}d)]}\right)\mathcal{H}_{a_y}(k_z)$$

$$\cdot \exp(-jk_{x_1}x + jk_z z) ,$$

$$\mathcal{H}_{y_1} = \frac{\omega\epsilon_2}{k_{x_2}}\left(\frac{q-1}{\exp(jk_{x_2}d) + q[\exp(-jk_{x_2}d)]}\right)\mathcal{H}_{a_y}(k_z)$$

$$\cdot \exp(-jk_{x_1}x + jk_z z) .$$

It is readily confirmed that these fields satisfy Maxwell's equations if we let

$$k_{x_2} = \sqrt{\omega^2 \mu \epsilon_2 - k_z^2}; \ \mathrm{Re}\, k_{x_2} > 0,$$

$$\mathcal{E}_{z_2} = \left(\frac{q[\exp(jk_{x_2} x)] + \exp(-jk_{x_2} x)}{\exp(jk_{x_2} d) + q[\exp(-jk_{x_2} d)]} \right) \mathcal{H}_{a_y}(k_z) \exp(jk_z z) ,$$

$$\mathcal{H}_{y_2} = \frac{\omega \epsilon_2}{k_{x_2}} \left(\frac{q[\exp(jk_{x_2} x)] - \exp(-jk_{x_2} x)}{\exp(jk_{x_2} d) + q[\exp(-jk_{x_2} d)]} \right) \mathcal{H}_{a_y}(k_z) \exp(jk_z z),$$

$$\mathcal{E}_{x_1} = \frac{\epsilon_2 k_z}{\epsilon_1 k_{x_2}} \left(\frac{1 - q}{\exp(jk_{x_2} d) + q[\exp(-jk_{x_2} d)]} \right) \mathcal{H}_{a_y}(k_z)$$

$$\cdot \exp(-jk_{x_1} x + jk_z z) ,$$

$$\mathcal{E}_{z_1} = \frac{k_{x_1} \epsilon_2}{k_{x_2} \epsilon_1} \left(\frac{1 - q}{\exp(jk_{x_2} d) + q[\exp(-jk_{x_2} d)]} \right) \mathcal{H}_{a_y}(k_z)$$

$$\cdot \exp(-jk_{x_1} x + jk_z z) ,$$

$$\mathcal{H}_{y_1} = \frac{\omega \epsilon_2}{k_{x_2}} \left(\frac{q - 1}{\exp(jk_{x_2} d) + q[\exp(-jk_{x_2} d)]} \right) \mathcal{H}_{a_y}(k_z)$$

$$\cdot \exp(-jk_{x_1} x + jk_z z) .$$

It is readily confirmed that these fields satisfy Maxwell's equations if we let

$$k_{x_2} = \sqrt{\omega^2 \mu \epsilon_2 - k_z^2} ; \ \mathrm{Re}\, k_{x_2} > 0,$$

$$k_{x_1} = \sqrt{\omega^2 \mu \epsilon_1 - k_z^2} \; ; \; \mathrm{Re}\, k_{x_1} > 0, \quad \mathrm{Im}\, k_{x_1} > 0,$$

$$q = (k_{x_1} - k_{x_2} \epsilon_1/\epsilon_2)/(k_{x_1} + k_{x_2} \epsilon_1/\epsilon_2) \; .$$

And it is seen that the above fields satisfy the continuity conditions at $x = 0$,

$$\mathcal{E}_{z_1} = \mathcal{E}_{z_2} ; \; \mathcal{H}_{y_1} = \mathcal{H}_{y_2} ; \quad \epsilon_1 \mathcal{E}_{x_1} = \epsilon_2 \mathcal{E}_{x_2} ,$$

the boundary condition at $x \to \infty$,

$\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ outgoing or attenuated,

and the boundary condition at $x = -d$

$$\mathcal{E}_{z_2}(x = -d) = \mathcal{K}_{ay} \; .$$

Therefore the field $\mathcal{H}_y$ at $x = -d$ produced by $\mathcal{K}_{ay}$ is:

$$\mathcal{H}_y(x = -d) = h(k_z)\, \mathcal{K}_{ay}(k_z) \; ,$$

where

$$h(k_z) = \frac{\omega \epsilon_2}{k_{x_2}} \, \frac{q[\exp(-2jk_{x_2} d)] - 1}{q[\exp(-2jk_{x_2} d)] + 1} \; .$$