BOUNDED-INPUT BOUNDED-OUTPUT STABILITY OF NONLINEAR TIME-VARYING DIFFERENTIAL SYSTEMS

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1. INTRODUCTION AND NOTATION. The problem of boundedness of solutions of the differential equation

$$\dot{x} = f(x, t) \tag{1}$$

has been studied by Yoshizawa [1]. He gets necessary and sufficient conditions for various kinds of stability of (1) using the techniques of the Liapunov direct method. We have extended the definitions and the methods of Yoshizawa to the study of the bounded-input bounded-output stability of the differential system

$$\dot{x} = f(x, u, t) \tag{S}$$

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Here $x \in R^n$ is the state of (S), $u \in R^n$ is the input or control and $t \in I = [0,\infty)$ is the time. $f: R^n \times R^m \times I \rightarrow R^n$ is the instantaneous velocity function which satisfies the following conditions. For fixed $t \in I$, f is continuous in the pair (x, u), whereas for fixed (x, u) it is measurable in t. Moreover, for bounded sets $X \subseteq R^n$ and $U \subseteq R^m$ there exist measurable functions L(t) and M(t) (dependent on X, U) which are summable over every finite interval and such that,

$$|f(x, u, t)| \leq M(t) \tag{2}$$

and

$$|f(x, u, t) - f(x', u, t)| \leq L(t)|x - x'| \tag{3}$$

for every x, x' in X and u in U. In general, $|x|$ and $|u|$ denotes the Euclidean norm of x and u respectively. Also if x(t) (u(t)) are measurable functions of time, then

$$||x(t)|| \triangleq \sup_t |x(t)| \quad (||u(t)|| \triangleq \sup_t |u(t)|)$$

where the supremum is taken in each case, over those values of t for which the function is defined. The solutions of (S) are to be interpreted in the sense of Carathéodory [2, 3]. Thus let u(t), $t \in I$ be any bounded measurable function and let (x_0, t_0) be any initial condition. Then a function

$$x(\tau) = x_u(\tau; x_0, t_0) \tag{4}$$

is a solution of (S) if it is absolutely continuous in τ, satisfies the initial condition

$$x(t_0) = x_u(t_0; x_0, t_0) = x_0$$

and satisfies (S) almost everywhere in the domain of definition of (4). Because of the conditions (2), (3) imposed on f, $x(\tau)$ is defined on a nonvanishing interval containing t_0 and furthermore it is unique [2, 3].

For each $r \geq 0$ we define the set

$$\Delta_r = \{x \mid x \in R^n, \ |x| \geq r\} \tag{5}$$

Following Yoshizawa [1] we shall need to consider Liapunov functions $V(t, x)$ defined continuously on $\Delta_r \times I$ (for some r) and such that $V \in C_0(x)$. That is to say, for each $\alpha \geq 0$, there is a continuous function $L(t) = L_\alpha(t)$ such that

$$|V(t,x) - V(t,x')| \leq L(t)|x - x'| \tag{6}$$

for every x, x' with norm less than α. We also say that $V(t, x)$ is absolutely continuous in t uniformly at a point (x_0, t_0) if there is a positive number ρ (depending on x_0, t_0) such that for each $\epsilon > 0$ there is a number $\delta = \delta(\epsilon)$ such that for every m,

$$\sum_{k=1}^{m} |V(t_k', x_k) - V(t_k, x_k)| < \epsilon$$

whenever

$$\sum_{k=1}^{m} |t_k' - t_k| < \delta ; \ t_0 - \rho \leq t_1' \leq t_1 \leq \cdots \leq t_m' \leq t_m \leq t_0 + \rho$$

and

$$|x_k - x_0| < \delta \quad \text{for each } k.$$

We will always suppose that the Liapunov functions have this property so that if $x(t)$ is an absolutely continuous function, $V(t, x(t))$ is also absolutely continuous in a neighborhood of t. Then corresponding to each bounded, measurable function $u(t)$, $t \in I$ we can define

$$V'_u(t,x) = \overline{\lim_{h \to 0+}} \frac{1}{h} \{V(t+h, x+hf(x, u(t), t) - V(t,x)\}$$

2. DEFINITION. The system (S) is bounded-input bounded-output stable (BIBO) if for every $\alpha \geq 0$, for every $a \geq 0$ there is a number $\beta = \beta(\alpha, a)$ such that

$$|x_u(\tau; x_0, t_0)| \leq \beta \quad \text{for all} \quad \tau \geq t_0 \tag{7}$$

for every initial condition (x_0, t_0) with $|x_0| \leq \alpha$ and every measurable function $u(t)$, $t \in I$ with $||u|| \leq a$.

Remarks: 1. Since the definition (7) depend on the solution (4) to the system (S), it is useful for a large class of dynamical systems. Of course the nature of the results is such as to be particularly useful for differential systems.

2. Various weaker notions of boundedness can also be introduced. In some cases analogous results can be obtained. The reader is refered to Yoshizawa [1] for a thorough discussion of the behavior of (1).

3. CONDITIONS FOR STABILITY.

Definition. We say that the Liapunov function $V(t,x)$ has property A if there is a positive continuously increasing function $a(r)$ such that $V(t,x) \leq a(|x|)$. It has property B if there is a nonnegative continuously increasing function $b(r)$ with $b(r) \to \infty$ as $r \to \infty$ and such that $b(|x|) \leq V(t,x)$.

Theorem 1. Suppose for each $a \geq 0$, there is a positive Liapunov function $V(t,x) = V_a(t,x)$, defined in $\Delta = \Delta_{r(a)}$, and possessing property A and B. Then if

$$V_u'(t,x) \leq 0 \tag{8}$$

for (t,x) in Δ and for each measurable function $u(t)$, $t \in I$ with $||u|| \leq a$, the system (S) is BIBO.

Proof. Let u be any measurable function on I with $||u|| \leq a$ and let (x_0, t_0) be any initial condition with $|x_0| > r(a)$. Since V has property A, $V(t_0, x_0) \leq a(\alpha)$ for all t_0 and all x_0 with $|x_0| = \alpha$. By property B since $b(r) \to \infty$ as $r \to \infty$ there is a $\beta = \beta(\alpha)$ such that $b(\beta) = a(\alpha)$. Therefore,

$$V(t_0, x_0) \leq b(\beta) \quad \text{for all} \quad t_0 \in I, \quad \text{for all } x_0 \text{ with } |x_0| = \alpha .$$

Hence by (8),

$$b(\beta) \geq V(t_0, x_0) \geq V(t, x_u(t; x_0, t_0)) \quad t \geq t_0 .$$

But,

$$V(t, x_u(t; x_0, t_0)) \geq b(|x_u(t; x_0, t_0)|)$$

and by property B. Since b is increasing we have,

$$\beta = \beta(\alpha) \geq |x_u(t; x_0, t_0)|, \quad t \geq t_0$$

Q. E. D.

The following lemma will be very useful to prove the converse of Theorem 1.

<u>Lemma</u>. Let (x_0, t_0) and (x_1, t_1) be two initial conditions with $t_0 \leq t_1$, let $u(t)$ be an arbitrary measurable function on I with $||u|| \leq a$. Suppose that the two solutions,

$$x_u(t; x_0, t_0) \text{ and } x_u(t; x_1, t_1)$$

can be defined to the left over the interval $t^* \leq t \leq t_0$, $t^* \leq t \leq t_1$. (We assume that $0 \leq t^*$). Also suppose that $|x_u(t; x_0, t_0)|$ and $|x_u(t; x_1, t_1)|$ are less than α over these intervals. Then

$$|x_u(t^*; x_0, t_0) - x_u(t^*; x_1, t_1)| \leq$$

$$\left[|x_1 - x_0| + \int_{t_0}^{t_1} M(\tau) d\tau \right] \left[\exp \int_{t^*}^{t_0} L(\tau) dt \right] \tag{9}$$

where the functions L and M are the same as those in (2) and (3).

The proof of this lemma is very similar to the proof of the (generalized) Gronwall's lemma given in [2].

Theorem 2. If (S) is BIBO, for each a $a \geq 0$ there is a Liapunov function $V(t,x) = V_a(t,x)$ defined on $\Delta = \Delta_{r(a)}$ such that V has properties A and B and $V'(t,x) \leq 0$.

Proof. Fix $a \geq 0$. Since (S) is BIBO, for each u, with $||u|| \leq a$, for each (x_0, t_0) with $|x_0| = \alpha$,

$$|x_u(\tau; x_0, t_0)| \leq \beta(\alpha) = \beta(a, \alpha) \tag{10}$$

for all $\tau \geq t_0$. We can assume that β is a continuous strictly monotonically increasing function and $\beta(\alpha) \to \infty$ as $\alpha \to \infty$. Hence the inverse function $\alpha = \alpha(\beta)$ is defined for $\alpha \geq \beta(0)$ and has the same properties as β.

Let $r(a) = \beta(0)$. Let $\Delta = \Delta_{r(a)}$. Then, for each (x_0, t_0) in $\Delta \times I$ and each u with $||u|| \leq a$, define

$$K_u(t_0, x_0) = \min\{|x_u(\tau; x_0, t_0)| \mid 0 \leq \tau \leq t_0\} \tag{11}$$

where the region of τ is that for which the solution $x_u(\tau; x_0, t_0)$ exists. The required Liapunov function is

$$V(t_0, x_0) = V_a(t_0, x_0) \triangleq \operatorname{Inf}\{K_u(t_0, x_0) \mid ||u|| \leq a\} \tag{12}$$

Clearly $0 \leq K_u(t_0, x_0) \leq |x_0|$ so that

$$V(t_0, x_0) \leq |x_0| \tag{13}$$

Hence V has the property A. We also claim that,

$$0 < \alpha(|x_0|) \leq V(t_0, x_0) \tag{14}$$

If this is not the case, then there is a u, $||u|| \leq a$ such that for some x_0

$$K_u(t_0, x_0) < \alpha(|x_0|)$$

Hence for some τ, $0 \leq \tau \leq t_0$ we must have

$$|x_u(\tau; x_0, t_0)| < \alpha(|x_0|)$$

$$\therefore \quad \beta(|x_u(\tau; x_0, t_0)|) < \beta(\alpha(|x_0|) = |x_0|$$

But $\beta(|x_u(\tau; x_0, t_0)|) \geq |x_0|$ which is a contradiction. Hence (14) is true, so that V has property B. It remains to show that V has the required smoothness properties.

Let (x_0, t_0) be any element of $(\Delta \times I)$ and let $u(t)$, $t \in I$ be any measurable function with $||u|| \leq a$, and consider the solution $x_u(\tau; x_0, t_0)$ for $0 \leq \tau \leq t$. If this solution is continuable to $\tau = 0$ then there is a δ-neighborhood N of (x_0, t_0) in $R^n \times I$ such that all solutions of

$$\dot{x} = f(x, u(t), t) \tag{15}$$

starting from N and going to the left lie in an ϵ-neighborhood of $x_u(\tau; x_0, t_0)$, $0 \leq \tau \leq t$. This can be easily seen from the lemma. Also δ just depends on the bound on $||u||$ and $|x_0|$. In this case, for every (x, t) in N we consider in the definition (11) of $K_u(t, x)$ the interval $0 \leq \tau \leq t$. On the other hand suppose that there is an $\alpha \geq 0$ such that

$$|x_u(\tau; x_0, t_0)| \to \infty \quad \text{as} \quad \tau \to \alpha + \tag{16}$$

In this case we define,

$$\beta(\eta) \triangleq \sup\{|x_u(\tau;x_0,t_0)| \,\big|\, |x_0| \leq \eta, \quad t_0 \in I, \quad \tau \geq t_0\} \tag{17}$$

This is a finite number since (S) is BIBO. Now let t^* be the largest time smaller than t_0 at which we have

$$|x_u(\tau;x_0,t_0)| = 2\beta(2|x_0|) \tag{18}$$

for the first time. This is possible because of (16). Again there is a δ-neighborhood N of (x_0,t_0) such that every solution of (15) starting from N and going to the left lies in an ϵ-neighborhood of $x_u(\tau;x_0,t_0)$ on the interval $t^* \leq \tau \leq t_0$. Let (x,t) be a member of N and consider $x_u(\tau;x,t)$. We claim that

$$|x_u(\tau;x,t)| \geq 2|x_0| \quad \text{for} \quad \tau < t^* \tag{19}$$

Suppose this is not the case. Then for some $\tau < t^*$,

$$|x_u(\tau;x,t) < 2|x_0| \tag{20}$$

But then,

$$\beta(|x_u(\tau;x,t)|) \geq 2\beta(2|x_0|) - \epsilon \geq \beta(2|x_0|)$$

for ϵ sufficiently small. This follows from (18). But because of (20) this contradicts the definition (17) of β. Hence (19) must be true. Therefore for every (x,t) in N we can again consider in the definition (11) of $K_u(t,x)$ the interval $t^* \leq \tau \leq t$ for some $0 \leq t^* \leq t$.

Now let u be an arbitrary measurable function with $||u|| \leq a$. Let (x, t) and (x', t') be any two points in the neighborhood N of (x_0, t_0). Suppose $t \leq t'$. Then

$$K_u(t,x) - K_u(t',x') = K_u(t,x) - |x_u(\tau';x',t')| \tag{21}$$

$$\leq |x_u(\tau';x,t)| - |x_u(\tau';x',t')|$$

$$\leq |x_u(\tau';x,t) - x_u(\tau';x',t')|$$

$$\leq (|x - x'| + \int_t^{t'} M(\tau)d\tau)(\exp \int_{t'}^{t} L(\tau)d\tau)$$

$$\leq (|x - x'| + \int_t^{t'} M(\tau)d\tau)(\exp \int_0^{t} L(\tau)dt)$$

$$\leq A(|x - x'| + \int_t^{t'} M(\tau)d\tau) \text{ by the lemma.}$$

Here in (21), τ' is the time at which the minimum in the definition of $K_u(t', x')$ is achieved. In a similar manner we can prove that

$$K_u(t,x) - K_u(t',x') \geq -A(|x - x'| + \int_t^{t'} M(\tau)d\tau)$$

Combining the two estimates we get,

$$|K_u(t,x) - K_u(t',x')| \leq A(|x - x'| + \int_t^{t'} M(\tau)d\tau)$$

for every u, $||u|| \leq a$ and every $(x, t), (x', t')$ in N.

Therefore by the definition (12) of V_a we have,

$$|V_a(t,x) - V_a(t',x')| \leq A(|x-x'| + \int_t^{t'} M(\tau)dt) \tag{22}$$

in a δ-neighborhood N of (t_0, x_0). Trivially from (22) $V \epsilon C_0(x)$. Also for every m, and

$$t_1' \leq t_1 \leq \cdots \leq t_m' \leq t_m$$

and every $x_1, x_2, \ldots, x_m$ with (x_k, t_k) and (x_k, t_k') in N we have from (22)

$$\sum_{k=1}^{m} |V_a(t_k', x_k) - V_a(t_k, x_k)| \leq A \sum_{k=1}^{m} \int_{t_k}^{t_k'} M(\tau)dt)$$

Since $M(\tau)$ is an integrable function, its indefinite integral is absolutely continuous. Hence $V(t,x)$ is absolutely continuous in t uniformly at each point. V therefore has the required smoothness properties. Also by the definition (11) of $K_u(t,x)$ we see that $K_u(\tau; x_u(\tau; x, t))$ is non-increasing in τ. Hence $V_a(t,x)$ is nonincreasing along every solution of (S) for each u with $||u|| \leq a$. Therefore $V_a'(t,x) \leq 0$. The theorem is proved.

Q. E. D.

4. A SIMPLE APPLICATION.

We close this paper by a simple application of Theorem 1. Let the differential system [4,5] be given by

$$
\begin{aligned}
\dot{x} &= Ax + bf(\sigma) \\
\dot{\sigma} &= d^T x - rf(\sigma) + u
\end{aligned}
\qquad (23)
$$

where A is an $n \times n$ matrix with all its eigenvalues having negative real parts. b, d and x are n-vectors whereas σ and u are scalaars. f is a locally integrable function of σ such that

$$f(\sigma) \to \pm \infty \quad \text{as} \quad \sigma \to \pm \infty$$

Consider the function,

$$V(x, \sigma) \triangleq x^T Q x + \int_0^\sigma f(\sigma')d\sigma'$$

where $Q > 0$. Clearly $V(x, \sigma)$ is positive for $|x| + |\sigma|$ sufficiently large and V enjoys properties A and B. Then if $y = (x, f(\sigma))$ we have,

$$\dot{V} = -y^T F y + f(\sigma)u$$

where F is an $(n+1) \times (n+1)$ matrix with

$$F = \left[\begin{array}{c|c} G & g \\ \hline g^T & r \end{array}\right]$$

where $-G = A^T Q + QA$, $-g = Qb + \frac{1}{2} d$

The following result is a straightforward application of Theorem 1.

<u>Theorem 3.</u> If $F > 0$ then the system (23) is BIBO.

REFERENCES

1. T. Yoshizawa, "Liapunov's Function and Boundedness of Solutions," Funkcialaj Ekvacioj 2 (1959), pp 95-142.

2. G. Sansone and R. Conti, Nonlinear Differential Equations. New York: The Mac Millan Company, 1964.

3. C. Carathéodory, Vorlesungen ueber Reele Funkionen New York: Chelsea, 1948, pp. 665.

4. J. P. LaSalle, "Stability and Control," J. SIAM Ser. A: Control 1 (1962).

5. V. A. Yakubovich, "The Method of Matrix Inequalities in the Stability Theory of Nonlinear Control Systems II." Automation and Remote Control Vol. 26 No. 4 April 1962, pp. 577-592.