ON THE MAXIMUM INTERNALLY STABLE SETS OF A GRAPH

by

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ABSTRACT

The location of an internally stable set with a maximum number of elements is closely related to the solution of several important problems in the theory of graphs. In this paper, a number of strong sufficient conditions are developed to test whether a given internally stable set is maximum. The sufficient conditions depend purely on the topological properties of the graph. Furthermore, an algorithm to obtain a maximum internally stable set is proposed. It is shown that in many cases the algorithm always terminates in a maximum set. However, a completely general proof is not yet available.

INTRODUCTION

The concept of an internally stable set is of fundamental interest in the theory of graphs [1] [2]. Applications range from game theory [1] [3] to information theory [4]. Basically, an internally stable set of a graph is a collection of vertices such that no two vertices in the collection are adjacent. A set S of vertices is a maximum internally stable set if no other internally stable set of the graph contains more elements.

The importance of maximum internally stable sets is illustrated by the following two problems.

(1) Shannon's Problem [4]: Given a set of q symbols to be transmitted through a communication channel. Because of noise in the channel, some of the symbols may be confused with others at the receiver. If word is composed of n symbols, find a maximum set of words which can be transmitted through the channel without confusion. The solution of this problem can be shown to correspond to a maximum internally stable set of an appropriately defined graph (for example, see Fig. 8).

(2) Vulnerability of a Communication Network [5] [6]: A set of communication stations are interconnected by various data links. An enemy would like to disrupt communications by bombing "enough" stations to completely isolate each station. Naturally, the enemy would like to perform this act in some "optimal" fashion. One definition of optimal is that a minimum number of stations are bombed. It is easily shown that the complement of such an optimal set of stations is

a maximum internally stable set of a graph whose vertices are the communication stations and whose branches are the data links.

To find a maximum internally stable set, one could consider all possible subsets of vertices of the graph. Naturally, such a procedure is highly impractical for a large graph. Maghout [7] proposed an algorithm, based on Boolian functions, to generate all possible internally stable sets.[1] The main difficulty with this algorithm is that it is extremely inefficient and computations become time consuming for even relatively small graphs. A more promising approach could be based on linear integer programming [9] . Again, the computational requirements for such a procedure rapidly become impractical. Furthermore, since both of the above methods are analytical in nature, they yield no direct relationships between topological structure and the maximum internally stable sets.

Berge [1] , has shown that in some cases finding a maximum internally stable set of a graph is equivalent to finding a maximum matching of the graph. However, in most cases, there is no clear procedure to obtain a maximum internally stable set from a maximum matching. Other partial results have been obtained by Matthys [10] [11] . Also, some interesting results on discrete optimization theory have been obtained recently by Reiter and Sherman [12] .

In this paper, a number of sufficient conditions are developed to determine if a given internally stable set is maximum. It will be

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[1] Later generalized by Hakimi [8] , to generate all matchings,all factors and all possible subgraphs with given degrees.

seen that these conditions can be applied to relatively large classes of graphs. The conditions presented here suggest an efficient algorithm to find a maximum set. Unfortunately, although the authors believe the algorithm will always result in a maximum set, they were unable to prove this optimality for all cases.

PRELIMINARY CONSIDERATIONS[2]

A graph $G = (X, \Gamma)$ is a pair consisting of a set X and a relation Γ on X. In the discussion to follow, the set X may be represented by points (vertices) in the plane and the relation between elements of X by continuous directed lines (branches), such that if $y \in \Gamma x$, the points x and y are joined by a directed branch from x to y. The graph G is said to be non-directed (symmetric) if and only if $y \in \Gamma x$ implies $x \in \Gamma y$ for all x in X. Let A be a set and let $|A|$ denote the number of elements in A. The graph $G = (X, \Gamma)$ is said to be finite if $|X| < \infty$. Only finite, non-directed graphs will be considered in the sequel.

A set $S \subset X$ is said to be internally stable if no two vertices in S are adjacent; that is, if

$$|\Gamma S \cap S| = 0$$

Let $\mathcal{L}(G)$ denote the family of internally stable sets. Then the coefficient of internal stability of G is defined to be

2 The notation employed here is standard and may be found in [1].

$$\alpha(G) \triangleq \max_{S \in \mathcal{L}(G)} |S|$$

A set S_o is said to be a maximum internally stable set if S_o is internally stable and $|S_o| = \alpha(G)$.

Let $G = (X, \Gamma)$ be a graph. $G_1 = (X_1, \Gamma^1)$ is a subgraph of G if $X_1 \subset X$ and $\Gamma^1 x = \Gamma x \cap X_1$ of all $x \in X_1$ (we will adopt the notation that $\Gamma^1 = \Gamma_{X_1}$). $G_2 = (X, \Gamma^2)$ is a partial graph if $\Gamma^2 x \subset \Gamma x$. $G_3 = (X_3, \Gamma^3)$ is a partial subgraph of G if $X_3 \subset X$ and $\Gamma^3 x \subset \Gamma x \cap X_3$. $G = (X, \Gamma)$ is connected if for any two vertices x and y of G, and for some integer r there exist vertices $x_{i_1}, x_{i_2}, \ldots, x_{i_r} \in X$ such that $x_{i_j} \in \Gamma x_{i_{j-1}}$ for $j = 2, 3, \ldots, r$ and $x_{i_1} \in \Gamma x$, $y \in \Gamma x$. In this paper, only connected graphs will be considered. For graphs that are not connected, obvious modifications of our statements are possible.

If $G = (X, \Gamma)$ is connected, a tree $T = (X, \Gamma^t)$ is a minimally connected partial graph of G.[3] A pendant vertex $x_p \in X$ is a vertex such that $|\Gamma x_p| = 1$. A chain is a sequence $x_{i_1} x_{i_2} \ldots x_{i_r}$ of distinct vertices such that $x_{i_j} \in \Gamma x_{i_{j-1}}$ $j = 2 \ldots, r$, r an integer. An isolated, vertex x_o is a vertex such that $|\Gamma x_o| = 0$.

In the sequel, we will call vertices in a given internally stable set S dark vertices, and vertices not in S light vertices. An internally stable set thus defines a coloring of the vertices of G such that dark vertices are not adjacent. A chain $x_{i_1} x_{i_2} \ldots x_{i_r}$ is said to be alternating (with respect to an internally stable set S) if every light vertex is only adjacent to dark vertices. Finally, if $G = (X, \Gamma)$, the graph

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[3] In contrast to usual procedure, we will allow an isolated vertex to be called a partial tree of a graph (i.e., a tree of partial subgraph containing only one vertex).

$G^i = (X - \{x_i\}, \hat{\Gamma})$ is the graph obtained from G by deleting the vertex x_i and its incident branches (i.e., $\hat{\Gamma}x = \Gamma x \cap (X - \{x_i\})$).

Some Basic Lemmas and Sufficient Conditions

Lemma 1. (Berge) Let G_p be any partial graph of a graph G. Then, $\alpha(G) \leq \alpha(G_p)$.

Lemma 2. Let $G_n = (X_n, \Gamma)$ be a graph such that $|X_n| = n$. Let $G_{n-1} = (X_{n-1}, \Gamma^1)$ where $X_n = X_{n-1} \cup \{x_o\}$ and for all $x \in X_{n-1}$, $\Gamma^1 x = \Gamma x - \{x_o\} \cap \Gamma x$.[4] Then

$$\alpha(G_{n-1}) \leq \alpha(G_n) \leq \alpha(G_{n-1}) + 1$$

Proof: The graph (X_n, Γ^2) where for all $x \in X_{n-1}$, $\Gamma^2 x = \Gamma^1 x$ and $|\Gamma^2 x_o| = 0$ has coefficient of internal stability $\alpha(G_{n-1}) + 1$ and is a partial graph of G_n. By Lemma 1, $\alpha(G_n) \leq (G_{n-1}) + 1$. Furthermore, if $x, y \in X_{n-1}$ such that $y \notin \Gamma^1 x$, then by definition of Γ^1, $y \notin \Gamma x$. Hence, $\alpha(G_n) \geq \alpha(G_{n-1})$.

Definition 1. Let S be an internally stable set of $G = (X, \Gamma)$. $T_a(S) = (X_1 \Gamma^1)$ is an improper alternating partial tree[5] of G (with respect to S) if $T_a(S)$ is a tree of some partial subgraph (X_1, Γ^2) such that

(a) $x \in X_1 - S \cap X_1$ implies $\Gamma^1 x \subset S$ (all chains in tree are alternating).

(b) For any x and $y \in X_1 - S \cap X_1$, $y \notin \Gamma x$ (no pair of light vertices in tree are adjacent in G).

(c) For any $x \in X_1 - S \cap X_1$, $\nexists y \in S - S \cap X_1$ such that $y \in \Gamma x$ (no light vertex in tree is adjacent to a dark vertex not in tree).

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[4] If A and B are two sets, by $A - B \cap A$, we mean $\overline{A \cap B \cap A}$.

[5] In the sequel, often called an improper tree.

(d) $T_A(S)$ contains at least one connected subgraph $\hat{T} = (\hat{X}, \Gamma^*)$ such that $\hat{T}$ satisfies (a), (b) and (c) and $|S \cap \hat{X}| < |\hat{X} - S \cap \hat{X}|$.

An example of the above definition is shown in Fig. 1.

Lemma 3. Let S be an internally stable set of $G = (X, \Gamma)$. Let $T_a(S) = (X_1, \Gamma^1)$ be an improper alternating partial tree of G (if one exists). Then the set $S' = (S - S \cap X_1) \cup (X_1 - S \cap X_1)$ is an internally stable set of G such that $|S'| \geq |S| + 1$, and $T_a(S')$ is a proper alternating partial tree.

Proof: Since $T_a(S)$ is an improper tree, clearly $|X_1 - S \cap X_1| \geq |S \cap X_1| + 1$. Hence, $|S'| \geq |S| + 1$. Vertices of $S - S \cap X_1$ are not adjacent by assumption. Any two vertices of $X_1 - S \cap X_1$ are not adjacent by condition (b) of Definition 1. No vertex of $X_1 - S \cap X_1$ is adjacent to a vertex of $S - S \cap X_1$ by condition (c) of Definition 1. Consequently S' is internally stable.

Definition 2. Let S be an internally stable set of $G = (X, \Gamma)$. $\tilde{T}_a(S) = (X_2, \Gamma^2)$ is a proper alternating partial tree[6] of G if

(1) conditions (a), (b) and (c) of Definition 1 are satisfied.

(2) $|S \cap X_2| \geq |X_2 - S \cap X_2|$ and $\tilde{T}_a(S)$ does not contain any subgraph which is an improper alternating partial tree.

Theorem 1. Let S be an internally stable set of $G = (X, \Gamma)$. Let $T_a(S) = (X_2, \Gamma^2)$ be a proper alternating partial tree of G. Then $S' = S \cap X_2$ is a maximum internally stable set of $\tilde{T}_a(S)$.

Proof: We will use induction of $|X_2|$. If $|X_2| = 1$ or 2, the theorem is trivially true since $\alpha(\tilde{T}_a(S) = 1$. Assume the theorem is true for all proper $\tilde{T}_a(S)$ such that $|X_2| < i$. Consider $\tilde{T}_a(S)$ such that $|X_2| = i$.

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6 Hereafter often called a proper tree.

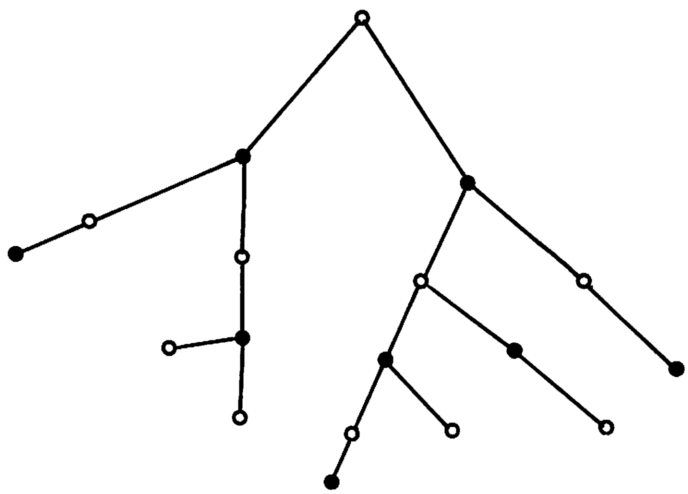


Fig. 1. An improper alternating partial tree.

From Definition 2, it is easily seen that there exists an $x_p \in S \cap X_2$ such that $|\Gamma^2 x_p| = 1$. Let $\tilde{T}'_a(S) = (X_2 - \{x_p\}, \Gamma^3)$ where $\Gamma^3 x = \Gamma^2 x - \{x_p\} \cap \Gamma^2 x$.

Case I: $\tilde{T}'_a(S)$ is a proper tree. Since $|X_2 - \{x_p\}| < i$, by the induction assumption, $|S \cap X_2 - \{x_p\}| = \alpha(\tilde{T}'_a(S))$. By Lemma 2, $\alpha(\tilde{T}_a(S)) \leq \alpha(T'_a(S)) + 1 = |S \cap X_2| \leq \alpha(\tilde{T}_a(S))$. Hence, $\alpha(\tilde{T}_a(S)) = |S \cap X_2|$.

Case II: $T'_a(S)$ is an improper tree. Since $\tilde{T}_a(S)$ is proper, then $|X_2 - S \cap X_2| = |S \cap X_2|$. Let $S'' = (S - S \cap X_2) \cup (X_2 - S \cap X_2)$. $\tilde{T}'_a(S'')$ is a proper alternating tree as is $\tilde{T}_a(S'')$. Hence, we now have a Case I situation and $\alpha(T_a(S'')) = |S'' \cap X_2|$. But $|S'' \cap X_2| = |S \cap X_2|$. Hence, the theorem.

Corollary 1. Let S be an internally stable set of the graph $G = (X, \Gamma)$. Suppose $\tilde{T}_a(S) = (X, \Gamma^2)$ is a proper alternating tree. Then S is a maximum internally stable set of G. Furthermore, if $\tilde{T}_a(S)$ is an improper alternating tree of G, then $X - S$ is a maximum internally stable set of G.

Proof: By Theorem 1, if $\tilde{T}_a(S)$ is a proper alternating tree, $|S| = \alpha(\tilde{T}_a(S))$. Lemma 1 implies that $\alpha(G) \leq \alpha(\tilde{T}_a(S))$. Hence, the first part of the corollary follows. If $\tilde{T}_a(S)$ is improper interchanging the vertex colors results in a proper alternating tree. The above argument then holds.

Corollary 2. Let S be an internally stable set of $G = (X, \Gamma)$. Let $\mathcal{T}(S) = \{\tilde{T}_{a_i}(S), i = 1, \ldots, q\}$ such that the $\tilde{T}_{a_i}(S) = (X_i, \Gamma^i)$ are proper alternating partial trees and $\bigcup X_i = X$. Then S is a maximum internally stable set of G.

Proof: Similar to proof of Corollary 1.

Remark 1. Corollaries 1 and 2 give sufficient conditions for an internally stable set to be maximum. If G can be decomposed into a forest of proper alternating partial trees, then S is maximum. Furthermore, if G can be decomposed into a forest of proper trees and/or improper trees, then interchanging the colors of the vertices in the improper trees will result in a maximum internally stable set. As an example of these conditions, consider the following variation of Gauss' Problem of the Queens; find the maximum number of knights that may be placed on a chess board so that no knight may capture any other knight. The solution is shown in Fig. 2.

A MORE GENERAL SUFFICIENT CONDITION

Definition 3. Let S be an internally stable set of $G = (X, \Gamma)$. A connected subgraph $G_o(S) = (X_o, \Gamma^o)$ is said to be strongly structured if

(1) $G_o(S)$ contains no improper alternating partial trees.

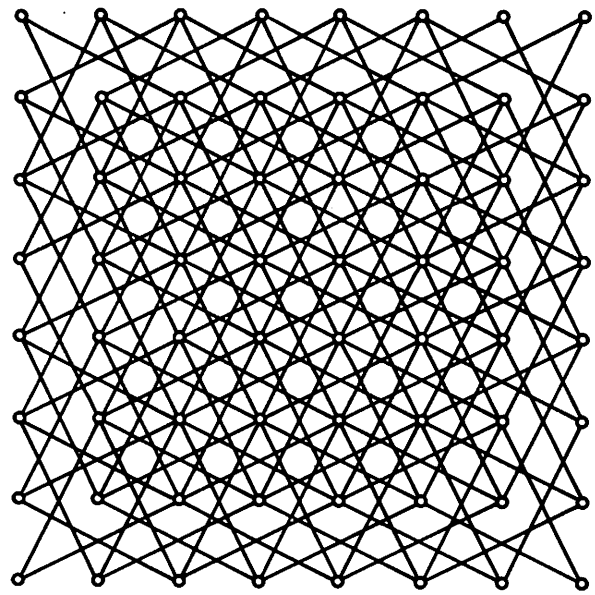
(2) $G_o(S) = (X_1 \cup X_2, \Gamma^o)$ such that

(a) $|X_1 \cap X_2| = 0$; $S \subset X_1$

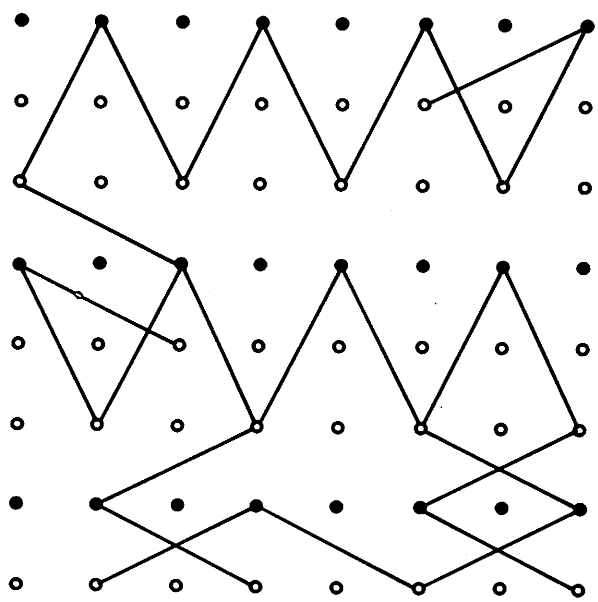
(b) (X_1, Γ^1) where $\Gamma^1 x = \Gamma^o x - \Gamma^o x \cap X_2$ is a proper alternating tree.

(c) For all $x_i \subset X_2$, any partial subgraph $(X_1 \cup \{x_i\}, \Gamma^2)$, such that $\Gamma^2 x_i = y \in S$ and for all $x \neq x_i$, $x \neq y$, $\Gamma^2 x = \Gamma^o x \cap X_1$, is an improper alternating tree with respect to the graph $(X_1 \cup \{x_n\}, \Gamma^2)$.

Each element of X_2 is called an excess vertex. The definition is illustrated in Fig. 3.

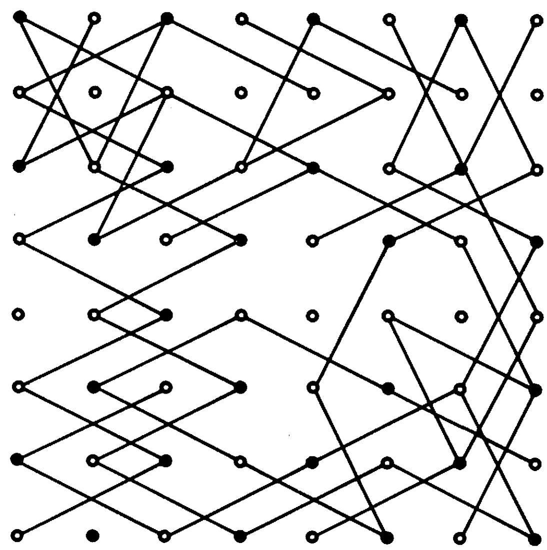


(a)

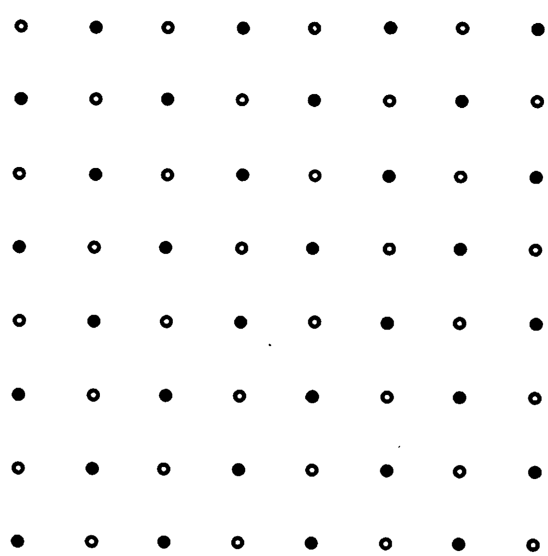


(b)

Fig. 2. (a) The graph for the movement of a knight on a chessboard. (b) An internally stable set with 24 elements and an improper alternating partial tree.



(c)



(d)

Fig. 2. (c) An internally stable set with 26 elements and an improper partial tree. (d) An internally stable set with 32 elements. It is easily seen that the graph contains a proper tree, hence $\alpha(G) = 32$.

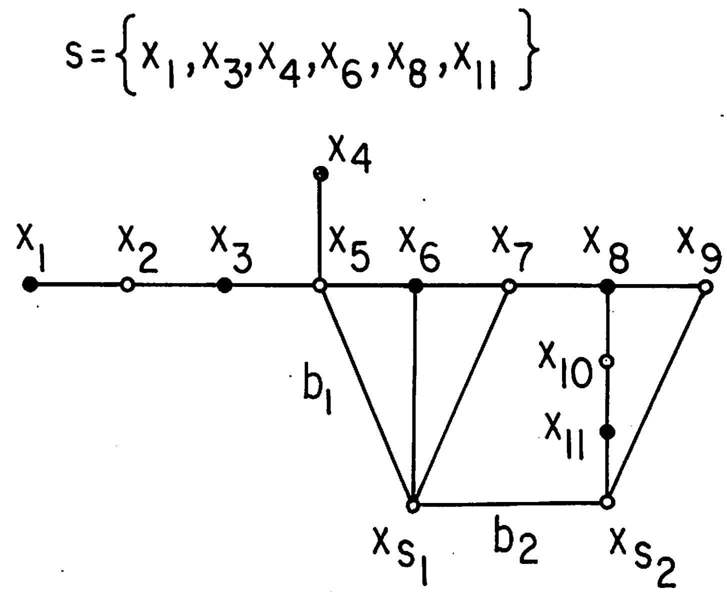


Fig. 3. A strongly structured graph G(S).

Note: Often, in what follows, it is not necessary to consider branches between excess vertices or branches between elements of X_2 and X_3 which may be deleted from the graph without destroying the strongly structured property of the graph. Thus, the branches labeled b_1 and b_2 may be deleted from the graph shown in Fig. 3.

Remark 2. Let $G(S) = (X, \Gamma)$ be a strongly structured graph with respect to the internally stable set S. Let $X = X_1 \cup X_2$ where X_2 is the set of excess vertices of G. Then, if G_s is the graph obtained from G by deleting all elements of X_1, $\alpha(G) \leq |S| + \alpha(G_s)$. Furthermore, let $A_1, \ldots, A_r$ be the family of internally stable sets of G_s. Let G_j be the graph obtained from G by deleting all elements of $X_2 - A_j$. Then, $\alpha(G) = \max\{(G_1), \ldots \alpha(G_r)\}$. $\alpha(G) = \max\{\alpha(G_1), \ldots, \alpha(G_r)\}$.

Lemma 5. Let G(S) be a strongly structured graph with internally stable set S. Let G_s and G_j be the graphs defined in Remark 2. Furthermore, let G^i_j be the graph obtained from G_j by deleting the vertex x_i. Then, if there exists an element x_{s_i} in A_j such that $\alpha(G_j^{s_i}) = |S|$, it follows that $\alpha(G_j) = |S|$.

Proof: Let $A_j = \{x_{s_1} \ldots, x_{s_q}\}$. We will use induction on q. If q = 1, by Corollary 1 of Theorem 1, the strongly structured graph satisfies the hypothesis; namely, $\alpha(G_j^{s1}) = |S|$. Let S' be a maximum internally stable set of G_j. Clearly, if $x_{s_1} \notin S'$, then $|S'| = |S|$. Suppose $x_{s_1} \in S'$. Then, there exist at least two vertices $x_\mu \in S$ and $x_\nu \in X_1 - S$ that are adjacent to x_{s_1} (it is only necessary to consider the case where there are exactly two such vertices). Since $x_{s_1} \in S'$, x_μ and x_ν cannot be in S'. Let us delete x_μ and x_ν from G_j. This results in a graph $G_j^{\mu,\nu}$ which is a forest of alternating trees and the isolated vertex x_{s_1}. By Corollary 2 of Theorem 1, $\alpha(G_j^{\mu,\nu,s1}) = |S - \{x_\mu\}| = |S| - 1$. Therefore,

$(S - \{x_\mu\}) \cup \{x_{s_1}\}$ is a maximum set of internally stable vertices of G which does not contain x_μ and x_ν. Consequently, $|S'| = |S| - 1 + 1 = |S|$, and the theorem is true for q = 1. Figure 4 illustrates this situation.

Assume that the theorem is true for q = m - 1 and let q = m. We know[7] that there exists an excess vertex x_{s_i} (in fact, any excess vertex) such that $\alpha(G_j^{s_i}) = |S|$. Suppose the theorem is not true. Then, there exists an S' such that $|S'| = |S| + 1$ and $x_{s_i} \in S'$. <u>All</u> excess vertices $x_{s_1}, \ldots, x_{s_m}$ are in S'; if not, we can delete all of the x_{s_k} not in S' to obtain the graph $G_j^{s_{k_1}, \ldots, s_{k_v}}$ with the same coefficient of internal stability as G_j. Now, if we delete x_{s_i}, we find that $\alpha(G_j^{s_{k_1}, \ldots, s_{k_v} s_i}) = |S|$ (since $\alpha(G_j^{s_i}) = |S|$). But, since the number of excess vertices of $G_j^{s_{k_1}, \ldots, s_{k_v}}$ is less than m, this implies that this graph has coefficient of internal stability $|S|$ which is a contradiction.

In the last paragraph, we have shown that if the theorem is true for q = m - 1 but not true for q = m, a maximum set S' must contain all of the excess vertices of G_j. We will now show that given an internally stable set S', such that $A_j \subset S'$, we can always find an internally stable set S'' such that $|S''| = |S'|$ and there is at least one excess vertex x_{s_k} not in S''. This contradicts the conclusion reached above and implies that the theorem is true for q = m.

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[7] If not, we can delete any x_{s_i} to obtain a stable set with m - 1 vertices. If $\alpha(G_j^{s_i}) \neq |S|$, we can delete another vertex, say x_{s_k}; if $\alpha(G_j^{s_i, s_k}) \neq |S|$, we can continue deleting vertices until $\alpha(G_j^{s_i, s_k, \ldots\ldots s_r, s_t}) = |S|$. Then by our induction assumption, $\alpha(G_j^{s_i, \ldots s_r}) = |S|$ which is a contradiction.

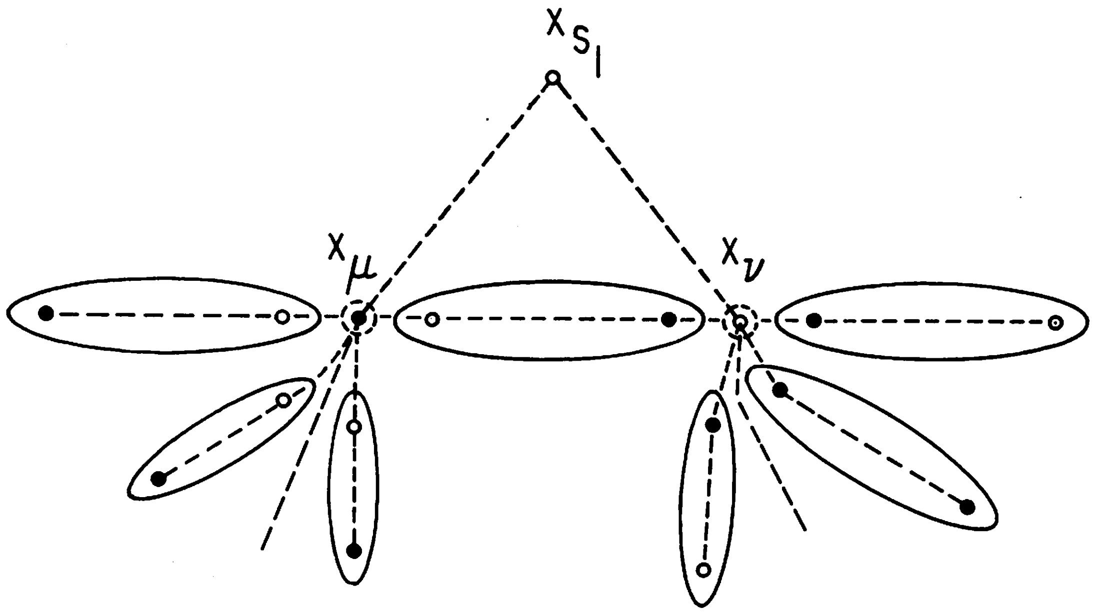


Fig. 4. Graph used in proof of Lemma 5.

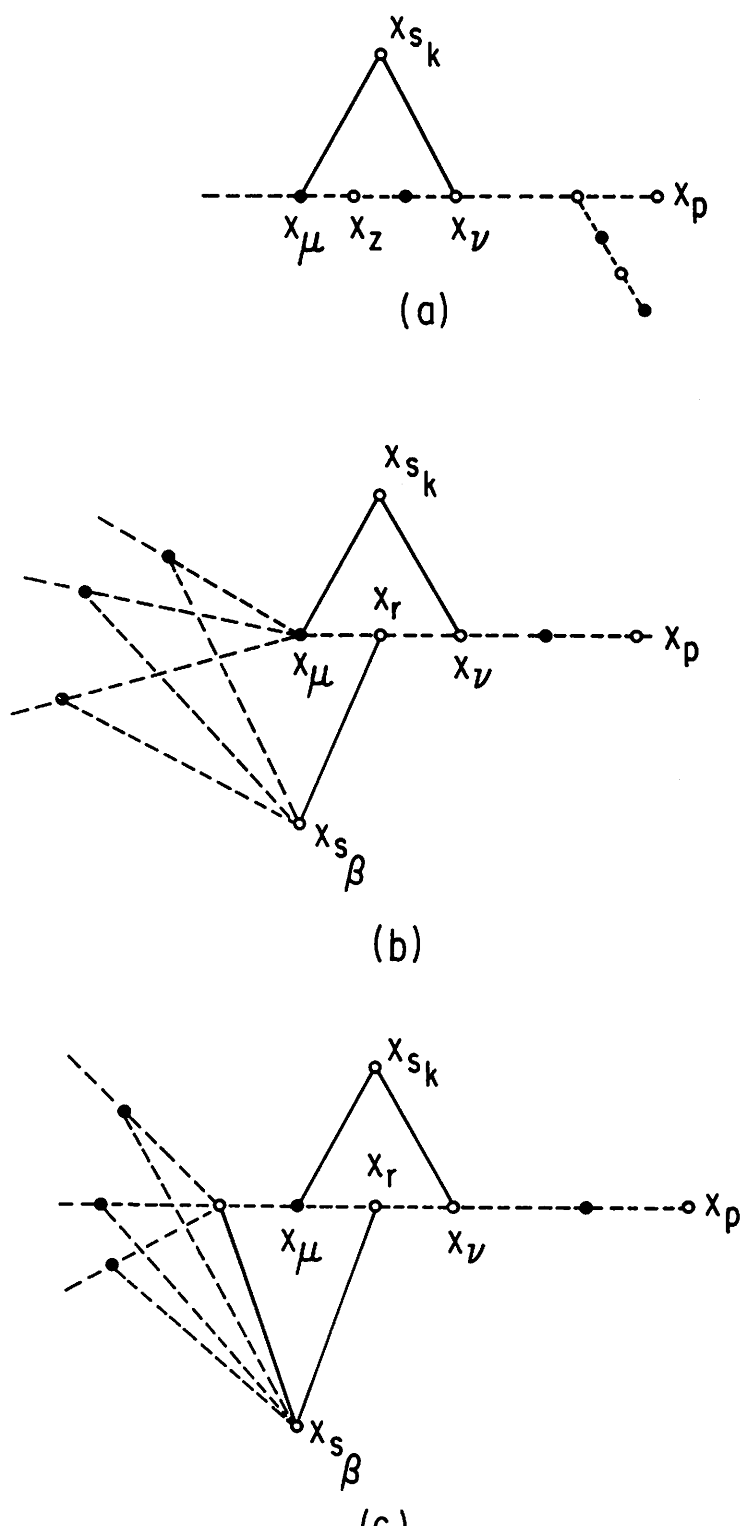


Fig. 5. Graphs used in proof of Lemma 5.

G_j is strongly structured. Hence, there must be a vertex $x_p \in X_1 - S$ which is either pendant or connected to only excess vertices. Let $x_p \ldots x_{\nu-1} x_\nu$ be an alternating chain of vertices of X_1 such that x_ν is in $X_1 - S$ and is adjacent to some excess vertex x_{s_k} (such that x_p is in the improper partial tree of condition (c) of Def. 3). Let there be no vertex on the chain $x_p \ldots x_{\nu-1}$ adjacent to any excess vertex. Let $x_p \ldots x_\nu x_{\nu+1} \ldots x_\xi x_\mu$ be an alternating chain such that $x_\mu \in S$, $x_\mu \in \Gamma^1 x_{s_k}$ and no vertex in S on the chain $x_{\nu+1} \ldots x_\xi$ is adjacent to x_{s_k}. Suppose that x_{s_k} is the only vertex adjacent to vertices on the chain $x_p \ldots x_\xi$. Then an internally stable set S'' can be obtained from S' such that $x_{s_k} \notin S''$ and $|S''| = |S'|$. This is done by letting x_p, x_ν and x_ξ be in S'' and making the chain alternate (since the number of vertices on the chains $x_p \ldots x_{\nu-1}$, and $x_{\nu+1}, x_\xi$ is even). Hence, we arrive at the contradiction noted above.

Consider the case where there are vertices other than x_{s_k} adjacent to vertices on the chain $x_p \ldots x_\xi$. The cases of interest are shown in Fig. 6. Case (a) is not possible since this implies that x_{s_k} is not excess. Case (b) is not possible, since then the graph contains an improper tree. Case (c) is possible, but the graph $\hat{G}_j$ obtained by deleting the branch between x_{s_β} and x_r must remain strongly structured. Furthermore, $\alpha(G_j^{'s_\beta}) = \alpha(\hat{G}_j^{s_\beta}) = |S|$ and consequently, $\alpha(\hat{G}_j) = \alpha(G_j)$. If we consider $\hat{G}_j$ we now arrive at the same contradiction as above.

<u>Definition 4</u>. Let S be an internally stable set of $G = (X, \Gamma)$. A connected subgraph $G_o(S) = (X_o, \Gamma^o)$ is said to be weakly structured if

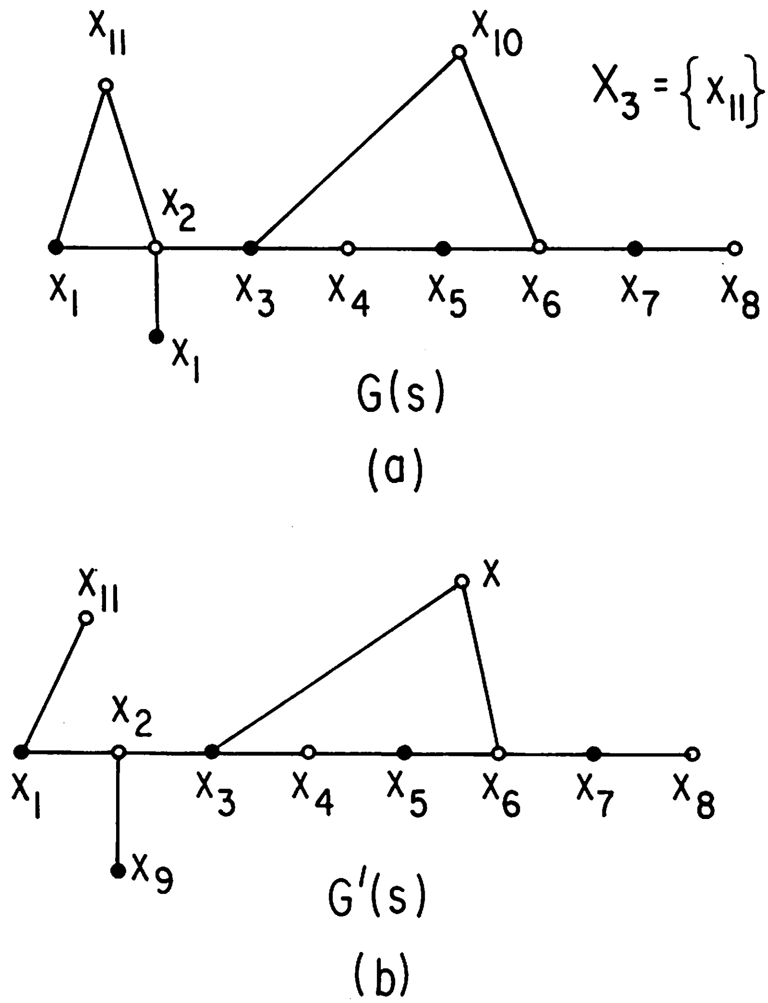


Fig. 6. (a) A weakly structured graph G(S).
(b) Strongly structured partial graph G(S) of G(S).

(1) $G_o(S)$ contains no improper alternating partial trees.

(2) $G_o(S) = (X_1 \cup X_2 \cup X_3, \Gamma^o)$ such that

(a) X_1, X_2, and X_3 are pairwise disjoint and $S \subset X_1$.

(b) $(X_1, \Gamma^o_{X_1})$ is a proper alternating tree.

(c) The partial subgraphs $(X_1 \cup X_3, \Gamma^*)$, such that for all $x_i \subset X_3$ $\Gamma^* x_i \subset S$, $|\Gamma^* x_i| = 1$, and for all $x_j \in X_1$. $\Gamma^* x_j = \Gamma^o_{x_1 x_j} \cup z$ where $z \in X_3 \ni \Gamma^* z = x_j$, is a proper alternating tree of the graph $\hat{G}(S)$, where $\hat{G}(S)$ is obtained from $G(S)$ by deleting the connections from elements of X_3 to elements of $X_1 - X_1 \cap S$.

(d) Any element of X_2 is an excess vertex of $\hat{G}(S)$.

Definition 4 is illustrated in Fig. 6.

Remark 3. It is easy to show that Lemma 5 applies to weakly structured graphs, if elements of X_2 are called excess vertices. To do this, it is only necessary to consider the graph $\hat{G}(S)$ defined above.

Theorem 2. Let S be an internally stable set of the graph $G = (X, \Gamma)$. Let $G_o(S) = (X_o, \Gamma^o)$ be a weakly structured subgraph of G. Then, $S' = S \cap X_o$ is a maximum internally stable set of G_o.

Proof: Let $X_o = X_1 \cup X_2 \cup X_3$ where the X_i are given in Definition 4. Obtain the graph $\hat{G}_o$ from G_o by deleting the branches between elements of X_3 and $X_1 - X_1 \cap S$. $\hat{G}_o$ is strongly structured with respect to the stable set $S \cap X_1$. Let $\hat{G}_j$ be the graph obtained from $\hat{G}_o$ by deleting all excess vertices not in the set A_j (see Remark 2). From Lemma 5, $\alpha(\hat{G}_j) = |S'|$ if there exists an elements x_{s_i} of A_j such that $\alpha(\hat{G}_j^{s_i}) = |S'|$. However, there must exist such an element. If not, we can form the graph $\hat{G}_j^{s_{k_1}, \ldots, s_{k_t}, s_{k_v}}$ such that $\alpha(\hat{G}_j^{s_{k_1}, \ldots, s_{k_t}}) \neq |S'|$ and $\alpha(G_j^{s_{k_1}, \ldots, s_{k_t}}) = |S'|$. Then, by Lemma 5, $\alpha(G_j^{s_{k_1}, \ldots, s_{k_t}}) = |S'|$ which is a contradiction.

Consequently, $\alpha(\hat{G}_j) = |S'|$ and furthermore, since A_j is an arbitrary internally stable set of G_s, $\max_j \alpha(\hat{G}_j) = |S'|$. Thus, $\alpha(\hat{G}_o) = |S'|$ and since by Lemma 1, $\alpha(G_o) \leq \alpha(\hat{G}_o)$, we have that $\alpha(G_o) = |S'|$ (since S' is an internally stable set of G_o).

Remark 4. Theorem 2 gives a strong sufficient condition for an internally stable set to be maximum. Given a graph G, and an internally stable set S, if the graph can be decomposed into a collection of weakly structured partial subgraphs such that any vertex of G is included in some subgraph, then the set S is amximum. Remark 4 is illustrated in Figs. 7 and 8. Figure 7(a) is an example taken from Berge [1], in which the maximum internally stable set S cannot be obtained directly from a maximum matching. However, if $S = \{x_2, x_4, x_6, x_7\}$, we immediately find that G(S) is a strongly structured graph (Fig. 7b). Figure 8 is an example of Shannon's problem, for the case where 4 symbols are to be transmitted through a noisy communication channel, and each word is of length 2.

FURTHER RESULTS, A CONJECTURE, AND AN ALGORITHM

Assertion. Let S be an internally stable set of the graph $G = (X, \Gamma)$. If G has no improper alternating partial trees, then G can be decomposed into a set of vertex disjoint, weakly structured partial subgraphs, and a subgraph $G_e = (X_e, \Gamma^e)$ such that $X_e \cap S$ is empty.

Proof. G can be decomposed into a set of proper alternating partial trees such that any element of S is in one such tree. A subset of the vertices not in any of these partial trees may be judiciously affixed to

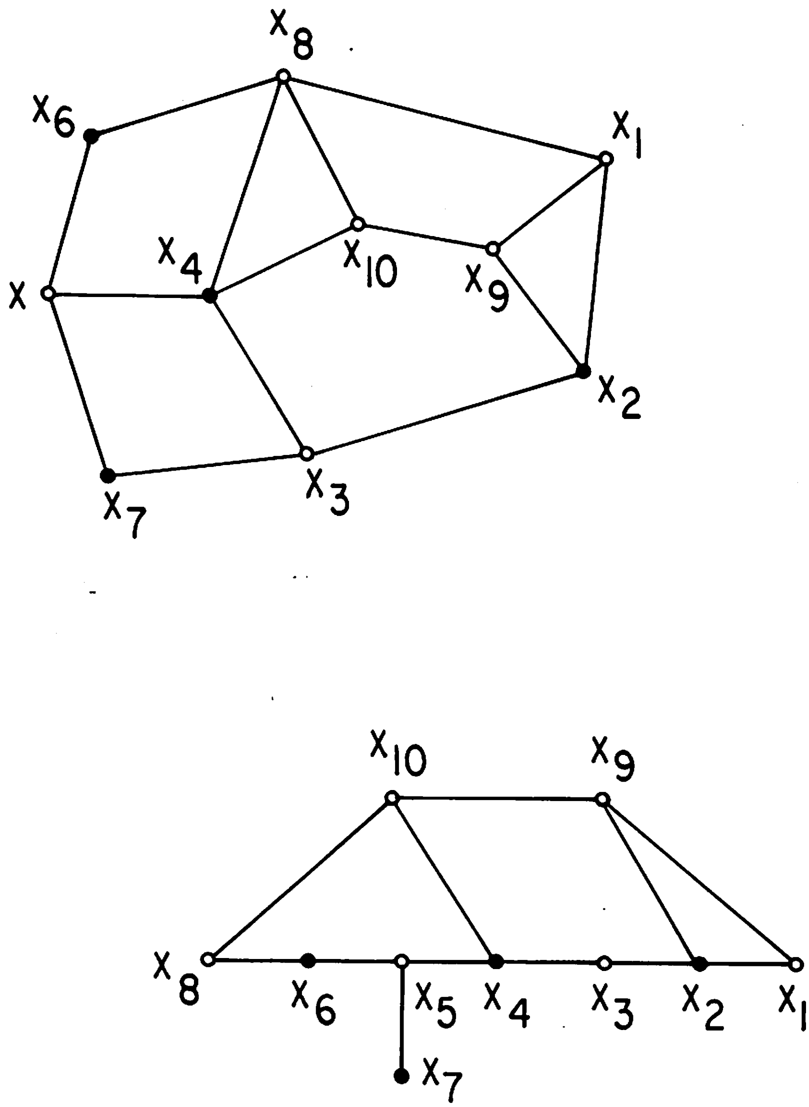
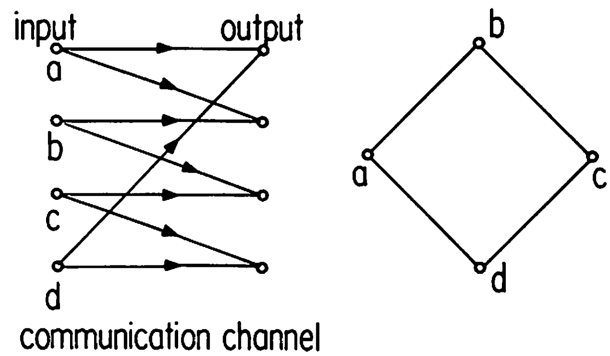


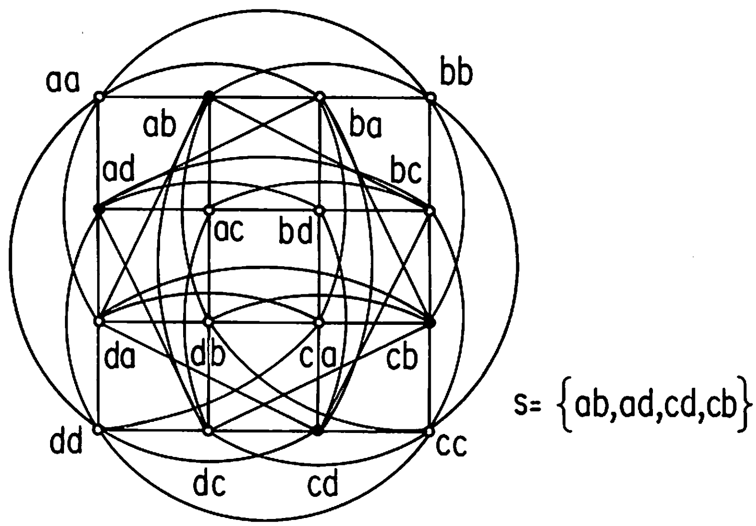
Fig. 7. Example to illustrate Remark 4.

(a)

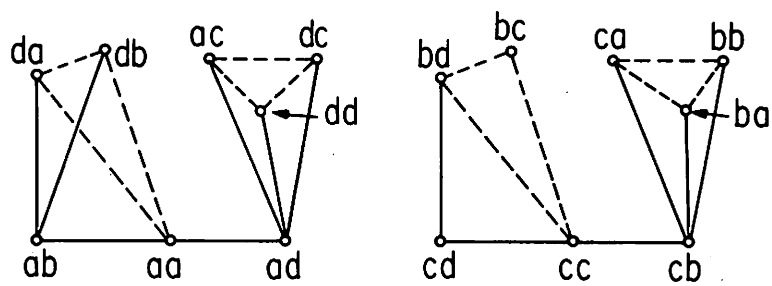


Confusion graph for symbols a, b, c, d.

(b)



Confusion graph for words of length 2 with internally stable set S.



(c)

Decomposition into two weakly structured subgraphs proving S is a maximum internally stable set.

Fig. 8.

to the partial trees to form weakly structured partial subgraphs. Any vertex not in a weakly structured partial subgraph is assigned to X_e.

Assertion. Let S be an internally stable set of the graph $G = (X, \Gamma)$. Let G have no improper alternating partial trees. Then, if G can be decomposed into a set of vertex disjoint weakly structured partial subgraphs, and a subgraph $G_e = (X_e, \Gamma^e)$ such that $\alpha(G_e) \leq 1$, then S is a maximum internally stable set of G.

Proof: If $\alpha(G_e) = 0$, the Assertion trivially follows from Remark 4. If $\alpha(G_e) = 1$, the Assertion may be proven by resorting to essentially "brute force" methods; i.e., enumerating the various possible topological structures of G and showing its validity for each case. An example of this Assertion is given in Fig. 9.

Conjecture 1. A necessary and sufficient condition for an internally stable set of a graph G to be maximum is that G contain no improper alternating partial tree.

The necessity of the condition follows from Lemma 3. We have shown above that in a large number of cases the condition is also sufficient. The authors were not able to obtain a general proof of the above conjecture. However, it is possible to show that if the following weaker conjecture is true, then Conjecture 1 is also true.

Conjecture 2. Let S be a maximum internally stable set of a graph $G = (X, \Gamma)$. Let x_1 and x_2 be elements of $X - S$ such that $x_2 \in \Gamma x_1$. Then, if $\hat{G}$, the graph obtained from G by deleting the branch between x_1 and x_2, contains no improper alternating partial tree, it follows that $\alpha(\hat{G}) = |S|$.

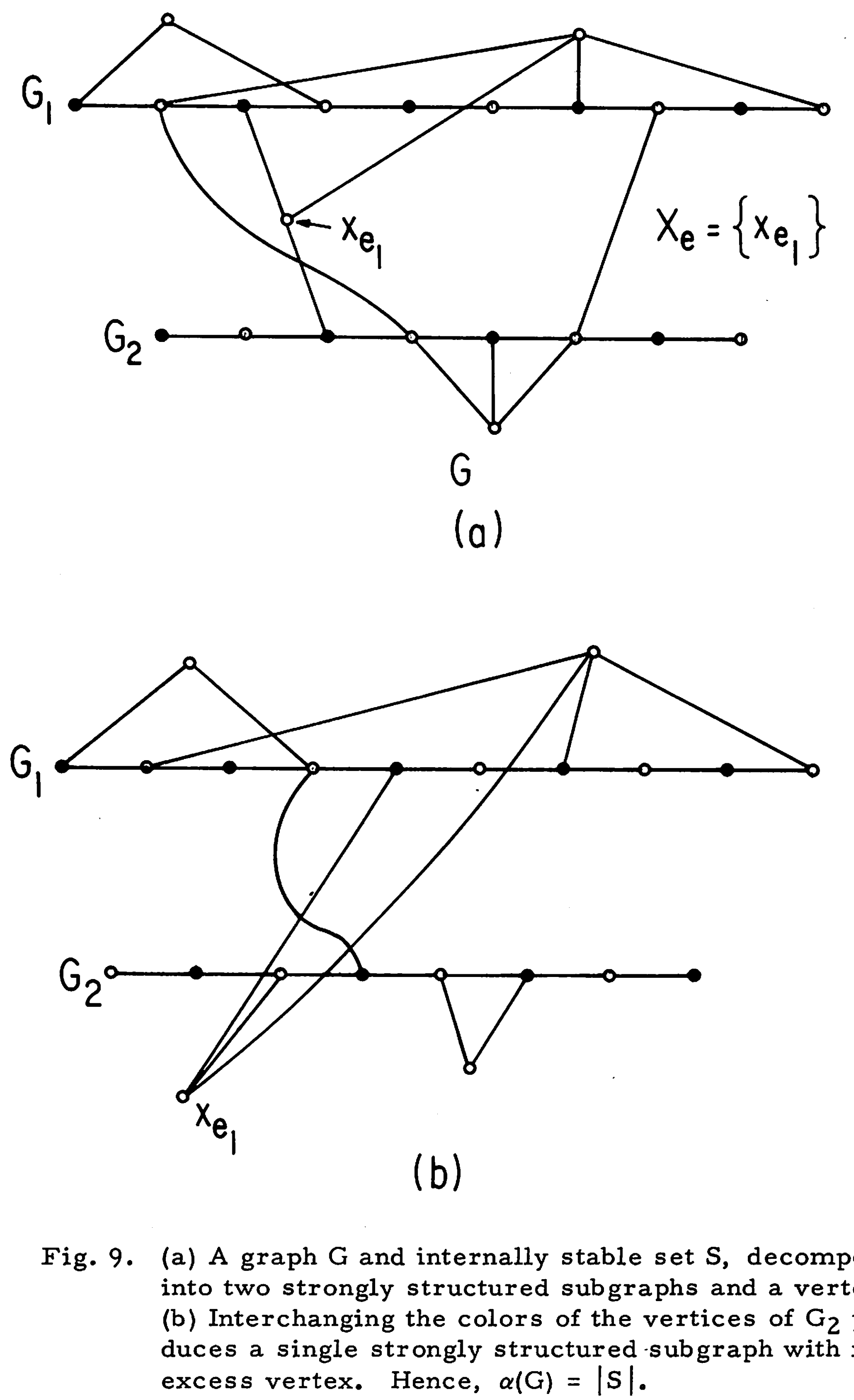


Fig. 9. (a) A graph G and internally stable set S, decomposed into two strongly structured subgraphs and a vertex x_{e_1}. (b) Interchanging the colors of the vertices of G_2 produces a single strongly structured subgraph with x_{e_1} an excess vertex. Hence, $\alpha(G) = |S|$.

Algorithm for finding maximum internally stable set (based on Conjecture 1)

I. Construct a feasible internally stable set S_o on $G = (X, \Gamma)$ by selecting non-adjacent vertices until any vertex in X is in S_o or adjacent to an element of S_o.

II. Select a vertex x_{i_1} in $X - S_o$ and search for an improper alternating partial tree.

III. If an improper alternating tree exists, interchange the colors of the vertices on the tree and form the stable set S_1. Return to step II.

IV. If an improper alternating tree cannot be found which contains x_{i_1}, select a vertex x_{i_2} not in the stable set and return to step II.

V. If S_j is an internally stable set obtained after j iterations of step III $(j = 0, 1, 2 \ldots)$ such that $G(S_j)$ contains no improper alternating trees, then $\alpha(G) = |S_j|$.

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