ON THE RECIPROCITY RELATIONS IN A COMPRESSIBLE PLASMA

by

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In a recent paper[1] H. C. Chen and D. K. Cheng proved a reciprocity relation for electromagnetic fields in a compressible plasma with high frequency sources. It was shown that the Lorentz reciprocity relation holds if $\nabla \cdot \overline{J}_e = 0$. From the well-known decomposition of the fields in a compressible plasma into electromagnetic and plasma modes, it is easy to see that such restriction on $\overline{J}_e$ results in no plasma mode, thus the fields are purely electromagnetic.[2] Therefore, the plasma only affects the dielectric constant in the Maxwell equations, the reciprocity relation of which is well known.

In this communication we wish to show that Lorentz reciprocity relation of electromagnetic fields holds in a compressible plasma in a much more general condition, namely,

(a) $\nabla \cdot \overline{J}_e \neq 0$,

(b) the ion motion is not negligible,

provided both the electromagnetic and plasma modes of the fields satisfy either the radiation condition at infinity or homogeneous boundary conditions on a closed surface.

The linearized hydrodynamic equation of motion and the equation of continuity for plasma are given by[3]

$$i\omega M_e N_o \overline{V}_e = N_o e \overline{E} + \nabla P_e, \tag{1}$$

$$i\omega M_i N_o \overline{V}_i = -N_o e \overline{E} + \nabla P_i \tag{2}$$

$$KTN_o \nabla \cdot \overline{V}_e = i\omega P_e, \tag{3}$$

$$KTN_o \nabla \cdot V_i = i\omega P_i, \tag{4}$$

$$KT = M_i U_i^2 = M_e U_e^2,$$

where the subscripts e and i denote the association of the fields with electrons and ions, respectively.

Maxwell equations are given by

$$\nabla \times \overline{E} = i\omega\mu_o \overline{H} - \overline{J}_m, \tag{5}$$

$$\nabla \times \overline{H} = -i\omega\epsilon_o \overline{E} + N_o e(\overline{V}_i - \overline{V}_e) + \overline{J}_e \tag{6}$$

From (1) and (3), and (2) and (4), we obtain

$$\overline{E} = i\omega \frac{M_e}{e} \overline{V}_e - \frac{KT}{i\omega e} \nabla \nabla \cdot \overline{V}_e, \tag{7}$$

$$\overline{E} = -i\omega \frac{M_i}{e} \overline{V}_i + \frac{KT}{i\omega e} \nabla \nabla \cdot \overline{V}_i. \tag{8}$$

The reciprocity relations can be derived by considering two sets of sources $\overline{J}_{e1}, \overline{J}_{m1}$ and $\overline{J}_{e2}, \overline{J}_{m2}$. Through two sets of curl equations similar to (5) and (6) we have the following:

$$\iint_s (\overline{E}_1 \times \overline{H}_2 - \overline{E}_2 \times \overline{H}_1) \cdot \overline{n} ds = \iiint_v (\overline{H}_1 \cdot \overline{J}_{m2} - \overline{H}_2 \cdot \overline{J}_{m1} + \overline{E}_2 \cdot \overline{J}_{e1} - \overline{E}_1 \cdot$$

$$\overline{J}_{e2}) dv + N_o e \iiint_v (\overline{E}_2 \cdot \overline{V}_{e1} - \overline{E}_1 \cdot \overline{V}_{e2} - \overline{E}_2 \cdot \overline{V}_{i1} + \overline{E}_1 \cdot \overline{V}_{i2}) dv. \quad (9)$$

Let R_1 be the second integral on the right hand side of (9). Substituting (7) or (8) into R_1, we can transform it into a surface integral.

$$R_1 = \frac{iKTN_o}{o\omega} \iint_s (\overline{V}_{e1} \quad \nabla \cdot \overline{V}_{e2} - \overline{V}_{e2} \quad \nabla \cdot \overline{V}_{e1} + \overline{V}_{i1} \quad \nabla \cdot \overline{V}_{i2}$$

$$- \overline{V}_{i2} \quad \nabla \cdot \overline{V}_{i1}) \cdot \overline{n} ds, \quad (10)$$

For the unbounded plasma, we decompose the fields into E-M mode and plasma mode[4] with

$$\overline{E} = \overline{E}_o + \overline{E}_p, \qquad \overline{H} = \overline{H}_o,$$

$$\overline{V}_e = \overline{V}_{oe} + \overline{V}_{pe}, \qquad \overline{V}_i = \overline{V}_{oi} + \overline{V}_{pi}$$

and satisfy the following sets of equations.

E-M mode:

$$\nabla \times \overline{H}_o = - i\omega\epsilon_o \epsilon \overline{E}_o + \overline{J}_e, \qquad \epsilon = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2} \quad (11)$$

$$\nabla \times \overline{E}_o = i\omega\mu_o \overline{H}_o - \overline{J}_m, \quad (12)$$

$$i\omega M_e \overline{V}_{oe} = e\overline{E}_o, \quad (13)$$

$$i\omega M_i \overline{V}_{oi} = - e\overline{E}_o. \tag{14}$$

Plasma mode:

$$[\overline{V}] = \begin{bmatrix} \overline{V}_{pe} \\ \overline{V}_{pi} \end{bmatrix} = - \frac{i\omega}{KTN_o} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \nabla \begin{bmatrix} P_e \\ P_i \end{bmatrix} = - i \frac{\omega}{KTN_o} A \nabla [p], \tag{15}$$

where

$$A_{11} = \frac{\omega^2 - \omega^2_{pi}}{\omega^4 \epsilon} U^2_e, \qquad A_{22} = \frac{(\omega^2 - \omega^2_{pe}) U_i^2}{\omega^4 \epsilon},$$

$$A_{12} = A_{21} = - \frac{\omega^2_{pi} U_e^2}{\omega^4 \epsilon} .$$

$$\overline{E}_p = \frac{1}{\omega^2 \epsilon N_o e} [\omega^2_{pe} \nabla P_e - \omega^2_{pi} \nabla P_i] , \tag{16}$$

$$A \nabla^2 [p] + [p] = \frac{i U^2_e N_o e \nabla \cdot \overline{J}_e}{\omega^3 \epsilon_o \epsilon} \begin{bmatrix} 1 \\ M_e/M_i \end{bmatrix}. \tag{17}$$

Take the divergence of (15) and combine with (17) to eliminate the Laplacian term. Since $\overline{J}_e$ is confined in a finite region, we obtain at infinity,

$$\nabla \cdot [\overline{V}] = - \frac{i\omega}{KTN_o} A \nabla^2 [p] = \frac{i\omega}{KTN_o} [p] . \tag{18}$$

Equation (17) is a coupled wave equation of P_e and P_i through a real symmetrical matrix A, which can be diagonalized by an orthonormal matrix T composed of the eigen vectors of A^{-1}.

$$T^{-1} A^{-1} T = \begin{pmatrix} k^2_1 & o \\ o & k^2_2 \end{pmatrix}, \tag{19}$$

where k_1 and k_2 are the eigen values of A^{-1}.

We define a new vector $[\hat{p}]$ by

$$[p] = T[\hat{p}] = T\begin{bmatrix} P_a \\ P_b \end{bmatrix}. \tag{20}$$

Thus, (17) can be written in the following form:

$$\nabla^2[\hat{p}] + T^{-1}A^{-1}T[\hat{p}] = \frac{iU^2_e N_o e \nabla\cdot \overline{J}_e}{\omega^3 \epsilon_o \epsilon} T^{-1}A^{-1} \begin{bmatrix} 1 \\ -M_\epsilon/M_i \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} \nabla^2 P_a + k^2_1 P_a \\ \nabla^2 P_b + k^2_2 P_b \end{bmatrix} = \frac{i U^2_e N_o e \nabla\cdot \overline{J}_e}{\omega^3 \epsilon_o \epsilon} T^{-1}A^{-1} \begin{bmatrix} 1 \\ -M_e/M_i \end{bmatrix}. \tag{21}$$

Equation (21) gives the uncoupled equations for P_a and P_b with excitation sources of the form $\nabla\cdot\overline{J}_e$ on the right. If $\nabla\cdot\overline{J}_e = 0$, Equation (17) is a coupled wave equation of P_e and P_i through a real symmetrical matrix A, which can be diagonalized by an ortho-normal matrix T composed of the eigen vectors of A^{-1}.

$$T^{-1}A^{-1}T = \begin{pmatrix} K_1^2 & 0 \\ 0 & K_2^2 \end{pmatrix}, \tag{19}$$

where K_1 and K_2 are the eigen values of A^{-1}.

We define a new vector $\{\hat{P}\}$ by

$$\{P\} = T\{\hat{P}\} = T\begin{Bmatrix} P_a \\ P_b \end{Bmatrix}. \tag{20}$$

Thus, (17) can be written in the following form:

$$\nabla^2\{\hat{P}\} + T^{-1}A^{-1}T\{\hat{P}\} = \frac{iU_o N_o e \nabla\cdot\vec{J}_e}{\varepsilon_o \omega^2}\, T^{-1}A^{-1}\begin{bmatrix} 1 \\ -M_e/M_i \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} \nabla^2 P_a + K_1^2 P_a \\ \nabla^2 P_b + K_2^2 P_b \end{bmatrix} = \frac{iU_o N_o e \nabla\cdot\vec{J}_e}{\varepsilon_o \omega^2}\, T^{-1}A^{-1}\begin{bmatrix} 1 \\ -M_e/M_i \end{bmatrix}. \tag{21}$$

Equation (21) gives the uncoupled equations for P_a and P_b with excitation sources of the form $\nabla\cdot\vec{J}_e$ on the right. If $\nabla\cdot\vec{J}_e = 0$,

P_a and P_b are identically zero. It follows from (20), (15), and (16) that no plasma mode is excited. The $\overline{E}$ and $\overline{H}$ fields are completely described by (11) and (12). Reciprocity relations of the form (32) is then obvious.

If $\nabla\cdot\overline{J}_e \neq 0$, we have the following restrictions on P_a and P_b:

$$\lim_{r\to\infty} r\, P_a \to \text{finite}, \qquad \lim_{r\to\infty} r\, P_b \to \text{finite}, \tag{22}$$

$$\lim_{r\to\infty} r \left[\frac{\partial P_a}{\partial n} - ik_1 P_a\right] \to 0 \qquad \lim_{r\to\infty} r \left[\frac{\partial P_b}{\partial n} - ik_2 P_b\right] \to 0\,. \tag{23}$$

The $\nabla\cdot\overline{V}_{pe,i}$ terms can be shown by (18) and (22) to behave as $\frac{1}{r}$ at infinity. Also, (16) shows that $\overline{E}_p$ is longitudal at infinity. This implies that the transversal components diminish at infinity faster than $\frac{1}{r}$ and can be neglected in the surface integral on the left hand side of (9). We have then

$$\iint_{s\to\infty} (\overline{E}_1 \times \overline{H}_2 - \overline{E}_2 \times \overline{H}_1)\cdot \overline{n}ds = \iint_{s\to\infty} (\overline{E}_{o1} \times \overline{H}_{o2} - \overline{E}_{o2} \times \overline{H}_{o1})\cdot \overline{n}ds = 0 \tag{24}$$

From (11) we get $\nabla\cdot\overline{E}_o = \dfrac{\nabla \overline{J}_e}{i\omega\,\epsilon_o\,\epsilon}$. Take the divergence of (13) and (14) and notice that the source is confined in a finite region, we have at infinity

$$\nabla\cdot\overline{V}_{oe} = \nabla\cdot\overline{V}_{oi} = 0.$$

Equation (10) can then be rewritten as

$$R_1 = \frac{iKTN_o}{\omega} \iint_s [(\overline{V}_{oe1} + \overline{V}_{pe1}) \nabla\cdot\overline{V}_{pe2} - (\overline{V}_{oe2} + \overline{V}_{pe2}) \nabla\cdot\overline{V}_{pe1}$$

$$+ (\overline{V}_{oi1} + \overline{V}_{pi1}) \nabla\cdot \overline{V}_{pi2} - (\overline{V}_{oi2} + \overline{V}_{pi2}) \nabla\cdot \overline{V}_{pi1}]\cdot nds. \tag{25}$$

The $\overline{E}_o$ field is transversal at infinity which implies that $\overline{n}\cdot\overline{E}_o$ diminishes faster than $\frac{1}{r}$. As a consequence of (13) and (14), we conclude that $\overline{n}\cdot\overline{V}_{oe}$ and $\overline{n}\cdot\overline{V}_{oi}$ also behave in the same way at infinity. The product of $\overline{n}\cdot\overline{V}_o$ with $\nabla\cdot\overline{V}_p$ then diminishes faster than $\frac{1}{r^2}$, hence its surface integral vanishes as $r\to\infty$. Then R_1 reduces to the following form:

$$R_1 = \frac{iKTN_o}{\omega} \iint_s (\overline{V}_{pe1} \nabla\cdot\overline{V}_{pe2} - \overline{V}_{pe2} \nabla\cdot\overline{V}_{pe1} + \overline{V}_{pi1} \nabla\cdot\overline{V}_{pi2} -$$

$$\overline{V}_{pi2} \nabla\cdot\overline{V}_{pi1})\overline{n}ds. \tag{26}$$

The integrand in (26) can be written in matrix form

$$R_{IN} = (V_{pe1} V_{pi1}) \nabla \begin{pmatrix} V_{pe2} \\ V_{pi2} \end{pmatrix} - (V_{pi2} V_{pi2}) \nabla \begin{pmatrix} V_{pe1} \\ V_{pi1} \end{pmatrix} =$$

$$V_1^{-t} \nabla\cdot V_2 - V_2^{-t} \nabla\cdot V_1 . \tag{27}$$

Substituting (15) and (18) into (27), we have

$$R_{IN} = -\left(\frac{i\omega}{KTN_o}\right)^2 [\nabla[p_1]^t A[p_2] - \nabla[p_2]^t A[p_1]] .$$

In terms of $[\tilde{p}]$, R_{IN} can be expressed as

$$R_{IN} = -\left(\frac{i\omega}{KTN_o}\right)^2 [\nabla [\tilde{p}_1]^t T^t A T[\tilde{p}_2] - \nabla[\tilde{p}_2]^t T^t[p_1]] . \tag{28}$$

Notice that $T^t = T^{-1}$ and $T^{-1}AT = \begin{pmatrix} k_1^{-2} & 0 \\ 0 & k_2^{-2} \end{pmatrix}$. We obtain

$$R_{IN} = \frac{\omega^2}{(KTN_o)^2} (k_1^{-2} P_{2a} \nabla P_{1a} - k_1^{-2} P_{1a} \nabla P_{2a} + k_2^{-2} P_{2b} \nabla P_{1b} - k_2^{-2} P_{1b} \nabla P_{2b} . \tag{29}$$

Substituting (29) back into (26), we have

$$R_1 = \frac{i\omega}{KTN_o} \iint_s (k_1^{-2} P_{2a} \frac{\partial P_{1a}}{\partial n} - k_1^{-2} P_{1a} \frac{\partial P_{2a}}{\partial n} + k_2^{-2} P_{2b} \frac{\partial P_{1b}}{\partial n} - k_2^{-2} P_{1b} \frac{\partial P_{2b}}{\partial n}) \, ds . \tag{30}$$

Relations (22) and (23) imply

$$\frac{\partial P_a}{\partial n} = ik_1 P_a + 0 (\frac{1}{r^{1+\alpha}}) , \ \alpha > 0, \tag{31}$$

P_a diminishes as $\frac{1}{r}$ at infinity.

Substitute (31) into (30). The integration involving a part then becomes

$$\iint_s (P_{2a} \frac{\partial P_{1a}}{\partial n} - P_{1a} \frac{\partial P_{2a}}{\partial n}) \, ds = \iint_s (P_{2a} \, 0 (\frac{1}{r^{1+\alpha}}) - P_{1a} \, 0 (\frac{1}{r^{1+\alpha}})) \, ds.$$

This integration is zero because S is of the order r^2, but $P_a \, 0 (\frac{1}{r^{1+\alpha}})$

is of the order $\frac{1}{r^{1+\alpha}}$. It follows that the integration of the b part vanishes likewise. The reciprocity relation

$$\iiint_v (\overline{E}_1 \cdot \overline{J}_{e2} - \overline{E}_2 \cdot \overline{J}_{e1})\, dv = \iiint_v (\overline{H}_2 \cdot \overline{J}_{m1} - \overline{H}_1 \cdot \overline{J}_{m2})\, dv \tag{32}$$

holds.

If the volume is bounded by a nonpenetrable perfect conductor with homogeneous boundary conditions $\overline{n} \cdot \overline{V}_e = \overline{n} \cdot \overline{V}_i = 0$ and $\overline{n} \times \overline{E} = 0$ at the surface, it is easy to show through (9) and (10) that (32) is valid.

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