SIMPLIFIED CONDITIONS FOR CONTROLLABILITY AND OBSERVABILITY OF LINEAR TIME-INVARIANT SYSTEMS

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Let the linear time-invariant systems under consideration be described either by

$$\begin{cases} \dot{x} = A x + B u \\ y = C x + D u \end{cases} \quad \text{or} \quad \begin{cases} x_{i+1} = A x_i + B u_i \\ y_{i+1} = C x_{i+1} + D u_{i+1} \end{cases} \tag{1}$$

(where A, B, C and D are constant $n \times n$, $n \times p$, $r \times n$ and $r \times p$ matrices, respectively). Then it is well known that any state is reachable from the origin in a finite time if and only if the rank of the matrix

$$P_n \triangleq [B \vdots AB \vdots \ldots \vdots A^{n-1} B]$$

is equal to n [1, 2]. Observe that this statement does not assume that A is nonsingular. In the following we shall not assume that A is nonsingular.

We use the following notation: if P is a matrix, $\rho(P)$ denotes the rank of P and $\mathcal{R}(P)$ the range of P. We also recall the fact that the rank of a matrix is equal to the dimension of its range [3].

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To check controllability in many instances one need not calculate $\underset{\sim}{P}_n$ but only a matrix with a smaller number of columns. This is based on the following.

Assertion. If k is the least integer such that

$$\rho(\underset{\sim}{P}_k) = \rho(\underset{\sim}{P}_{k+1}) \triangleq f \leq n$$

then, for all integers $s \geq k$,

$$\rho(\underset{\sim}{P}_s) = \rho(\underset{\sim}{P}_k)$$

and

$$\mathcal{R}(\underset{\sim}{P}_s) = \mathcal{R}(\underset{\sim}{P}_k) .$$

Thus in order to check controllability one needs only calculate $\underset{\sim}{P}_k$ and check its rank.

Proof. By assumption, $\mathcal{R}(\underset{\sim}{P}_k)$ and $\mathcal{R}(\underset{\sim}{P}_{k+1})$ have the same dimension f. Observing that the columns of any matrix span its range and that all the columns of $\underset{\sim}{P}_k$ are in $\underset{\sim}{P}_{k+1}$, we conclude that the subspaces $\mathcal{R}(\underset{\sim}{P}_k)$ and $\mathcal{R}(\underset{\sim}{P}_{k+1})$ are identical. For simplicity call $\mathcal{R}$ this subspace.

We now assert $\underset{\sim}{A}\,\mathcal{R} \subset \mathcal{R}$, where by $\underset{\sim}{A}\,\mathcal{R}$ we denote the image of $\mathcal{R}$ under the linear transformation $\underset{\sim}{A}$. Indeed, if $\underset{\sim}{x} \in \mathcal{R}$ then $\underset{\sim}{x}$ is some linear combination of columns of $\underset{\sim}{P}_k$; multiplying on the left by $\underset{\sim}{A}$, we see that $\underset{\sim}{A}\underset{\sim}{x}$ is a linear combination of columns of $\underset{\sim}{P}_{k+1}$ and, hence, $\underset{\sim}{A}\underset{\sim}{x} \in \mathcal{R}$.

Clearly then for any integer $s \geq k$, the columns of $\underset{\sim}{A}^s \underset{\sim}{B}$ are vectors in $\mathcal{R}$ and since all the columns of $\underset{\sim}{P}_k$ are columns of $\underset{\sim}{P}_s$ it follows then that $\mathcal{R}(\underset{\sim}{P}_k) = \mathcal{R}(\underset{\sim}{P}_s)$.

Corollary. Let r be the rank of $\underset{\sim}{B}$ and k be defined as in the previous assertion. Let n_o be the degree of the minimal polynomial of $\underset{\sim}{A}$, then

$$k \leq \min(n - r + 1, n_o) \tag{2}$$

Proof. $k \leq n_o$ because $\underset{\sim}{A}^{n_o}$ is a linear combination of $\underset{\sim}{I}, \underset{\sim}{A}, \ldots, \underset{\sim}{A}^{n_o-1}$. $k \leq n - r + 1$ because each time a block of the form $\underset{\sim}{A}^m \underset{\sim}{B}$ is added to $\underset{\sim}{P}_{m-1}$ $(m < k)$, the dimension of the range is increased by at least one.

The inequality (2) is not the best possible as is shown by the following example.

$$\underset{\sim}{A} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \underset{\sim}{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Here $n = 4$, $n_o = 2$, $r = 1$, hence $\min(4, 2) = 2$. However $k = 1$ and $f = 1$.

Observability. Invoking the duality theorem of Kalman, we conclude that either system described by (1) is completely observable if and only if $\rho(\underset{\sim}{O}_k) = n$ where

$$\underset{\sim}{O}_m \triangleq \left[\underset{\sim}{C}^* \vdots \underset{\sim}{A}^* \underset{\sim}{C}^* \vdots \ldots \vdots \underset{\sim}{A}^{*(m-1)} \underset{\sim}{C}^*\right] \quad m = 1, 2, \ldots$$

and k is the least integer such that

$$\rho(\underset{\sim}{O}_k) = \rho(\underset{\sim}{O}_{k+1}) .$$

References

1. R. E. Kalman, "On the General Theory of Control," Proc. First I.F.A.C. Conference, Butterworths, London, 1961.

2. L. A. Zadeh, C. A. Desoer, "Linear System Theory," McGraw-Hill, New York, 1963.

3. F. R. Gantmacher, "The Theory of Matrices," Chelsea Publishing Company, New York, N. Y., 1959 (see p. 64).