COFFEE MACHINES AND A QUADRATIC EQUATION

by

E. R. Berlekamp

Memorandum ERL-154
10 January 1966

ELECTRONICS RESEARCH LABORATORY
College of Engineering
University of California, Berkeley
94720

Manuscript submitted 10 January 1966

Coffee Machines and a Quadratic Equation*

by

Elwyn R. Berlekamp

The late John L. Kelly, Jr., of Bell Telephone Laboratories was a member of that set of mathematicians who could no more work without cigarettes and coffee than without paper and pencil. These essential supplies are commonly purchased from vending machines, those primitive computers which accept coins as inputs and dole out coins and essentials as output. Coffee is sold for ten cents per cup. The coffee machine accepts only nickels, dimes and quarters, and gives change only in nickels. If the machine has less than three nickels, then it is unable to make change for a quarter. The more advanced machines turn on a light to notify the customer of this situation.

The light was on one day in the summer of 1962 when Dr. Kelly and I approached. We both wanted coffee, but we could muster only two nickels and one quarter between us. "What odds will you wager," he asked, "that inserting these two nickels will turn the light out?" To that question we devote this paper.

---

* The research herein was supported in part by the Air Force Office of Scientific Research under Grant AF-AFOSR-639-65.

We must first introduce some appropriate assumptions in order to make the problem more precise. Since the insertion of dimes does not affect the machine's nickel supply, we may consider customers who use dimes to be inconsequential. We assume that, if the light is out, each consequential customer will insert a quarter with probability q or two nickels with probability n. If the light is on, however, we assume that each potentially consequential customer will insert two nickels with probability n', or he will go away (or get a dime) with probability q'. Since $n' \geq n$, we may set $n' = n + q(b)$; $q' = qa$. If we consider the unfortunate customer who finds the light on when he arrives with a quarter in hand, we may interpret a as the probability that he will go away (or use a dime) and $b = 1 - a$ as the probability that he will use two nickels. We further assume that different customers behave independently of each other. We also assume that the machine is capable of holding an infinite number of nickels. We wish to determine s_k, the probability that the machine has k nickels. Finally, we assume that this probability distribution has reached a steady state. This assumption will be valid if the machine is allowed to operate for a long time without internal tampering with the number of nickels. If the vendor inserts or removes large numbers of nickels frequently, then we cannot expect this formulation of the problem to provide a reasonable answer to Dr. Kelly's question. However, the only vendor with whom I have discussed the situation stated that he usually confines his activities to refilling the coffee and removing the dimes and quarters, leaving the nickel supply unchanged. Under these circumstances our assumptions are not unreasonable.

Let us first list all of the notation:

s_k is the probability that the coffee machine contains k nickels.

We require that $s_k \geq 0$ and that $\sum_{k=0}^{\infty} s_k = 1$.

P_L is the probability that the light is on. $P_L = s_0 + s_1 + s_2$.

P_0, P_1, and P_2 are the conditional probabilities of finding the machine with 0, 1, or 2 nickels, given that the light is on.

$P_0 = s_0/P_L$; $P_1 = s_1/P_L$; $P_2 = s_2/P_L$.

q is the probability that a consequential customer uses a quarter when the light is off.

n is the probability that a consequential customer uses two nickels when the light is off. $n = 1 - q$.

q' is the probability that a potentially consequential customer does not put in two nickels when the light is on.

n' is the probability that a potentially consequential customer uses two nickels when the light is on. $n' = 1 - q'$.

a is the probability that a potential customer who arrives with quarter in hand and finds the light on goes away (or uses a dime). $a = q'/q$.

b is the probability that a potential customer who arrives with quarter in hand and finds the light on uses two nickels. $b = 1 - a = (q - q')/q =$
$= (n' - n)/q$.

$S(z) = \sum_{k=0}^{\infty} s_k z^k$ is the generating function for the state probabilities.

$m = n/q$ is another parameter useful in subsequent calculations.

$h = 1/m = q/n$ is likewise subsequently useful.

According to ous assumptions, the generating function S(z) must satisfy the equilibrium equation

$$S(z) = [S(z) - (s_0 + s_1 z + s_2 z^2)][nz^2 + qz^{-3}] + [s_0 + s_1 z + s_2 z^2][n'z^2 + q']$$

Solving for S(z) gives

$$S(z) = \frac{(s_2 z^2 + s_1 z + s_0)(-qz^{-3} + qbz^2 + q')}{1 - nz^2 - qz^{-3}}$$

Multiplying by z^3, substituting $s_2 z^2 + s_1 z + s_0 = P_L(P_2 z^2 + P_{1z} + P_0)$ and cancelling the common factor of (z - 1) from both numerator and denominator, we obtain

$$S(z) = \frac{P_L[P_2 z^2 + P_1 z + P_0][b(z^4 + z^3) + (z^2 + z + 1)]}{[-m(z^4 + z^3) + (z^2 + z + 1)]}$$

Since the probabilities must sum to one, we require

$$\begin{aligned} S(1) &= 1 \\ &= \frac{P_L(2b+3)}{(3 - 2m)} \end{aligned}$$

From this we deduce that the probability of the light being on must be

$$P_L = \frac{(3 - 2m)}{(2b + 3)} = \frac{(5q - 2)}{q(2b+3)} = \frac{(3 - 5n)}{(1 - n)(2b+3)}$$

This expression is valid only when $m \leq 3/2$, or $n \leq 3/5$. If $n \geq 3/5$, then we expect the number of nickels in the machine to increase indefinitely, and the steady-state distribution for which we are searching will not exist. As n approaches 3/5, then P_L approaches zero, corresponding to the fact that if nickels are inserted almost as often as they are needed for change, then the machine should rarely run out. As n approaches zero, P_L approaches $1/(1+2b/3)$. Again, this agrees with our intuitive expectations. If b is 0, then nickels are never inserted and once the light goes on it will remain on forever. On the other hand, if $n=0$, $b=1$, then the machine states will follow the cycle: 0, 2, 4, 1, 3, 0, 2, 4, 1, 3 ... . Since there are five states in this cycle, and three of them have the light on, P_L must be 3/5 in agreement with the formula. We also note that the conditional state probabilities when the light is on must be given by $P_0 = P_1 = P_2$ in this limiting situation. For other values of n and b, however, our problem is far from solved. We have a formula for S(z) in terms of the parameters P_L, P_0, P_1, P_2, m, and b. m and b are presumed known; L and we have just derived an equation for P_L. But the conditional probabilities P_0, P_1, and P_2 remain unknown, except for the obvious restriction that they sum to one.

How can they be determined? Taking additional moments of the generating function (i.e., S'(1) = the average number of nickels, etc.) will accomplish nothing since no information about the moments of the distribution are known. The original equilibrium equation already embodies all relations between the state probabilities. It appears that we have already used all the available information, and yet our answer still contains two undetermined parameters. Is the problem inadequately specified? The reader is invited to stop and ponder our predicament before continuing.

As the experienced reader familiar with queing theory will readily deduce, the key to the resolution of our difficulties lies hidden away in a condition that we have not yet exploited, namely, that $s_k \geq 0$ for all k. Because of this requirement and the already-exploited condition that

$$\sum_{k=0}^{\infty} s_k = 1,$$ we conclude that $S(z)$ must be analytic within the unit circle in the complex plane. For, if $|z| \leq 1$, then $|S(z)| \leq \sum_k |s_k||z|^k \leq \sum s_k = 1.$

Thus, any poles of $S(z)$ must have magnitude greater than one. The complex roots of the polynomial $m(z^4 + z^3) - (z^2 + z + 1)$ for real m are plotted in Figure 1.

The plot of Figure 1 shows the location of possible roots of both the denominator and the latter numerator factor $[b(z^4 + z^3) + (z^2 + z + 1)]$. For positive m, the denominator has one real positive root, one real negative root, and a pair of complex conjugate roots which have negative real part and lie within the unit circle. For b between 0 and 1, all four roots of the latter numerator factor are complex and lie outside the unit circle. Hence this latter numerator factor cannot have any roots in common with the denominator.

For positive m greater than $3/2$, the real positive root of the denominator as well as the pair of complex roots lies within the unit circle. Since the former numerator factor, $(P_2 z^2 + P_1 z + P_0)$ is only a quadratic, it can cancel out at most two of these three roots. Thus, for $m > 3/2$, the requirement that $S(z)$ be analytic within the unit circle cannot be satisfied. This verifies our earlier observation that if $m > 3/2$

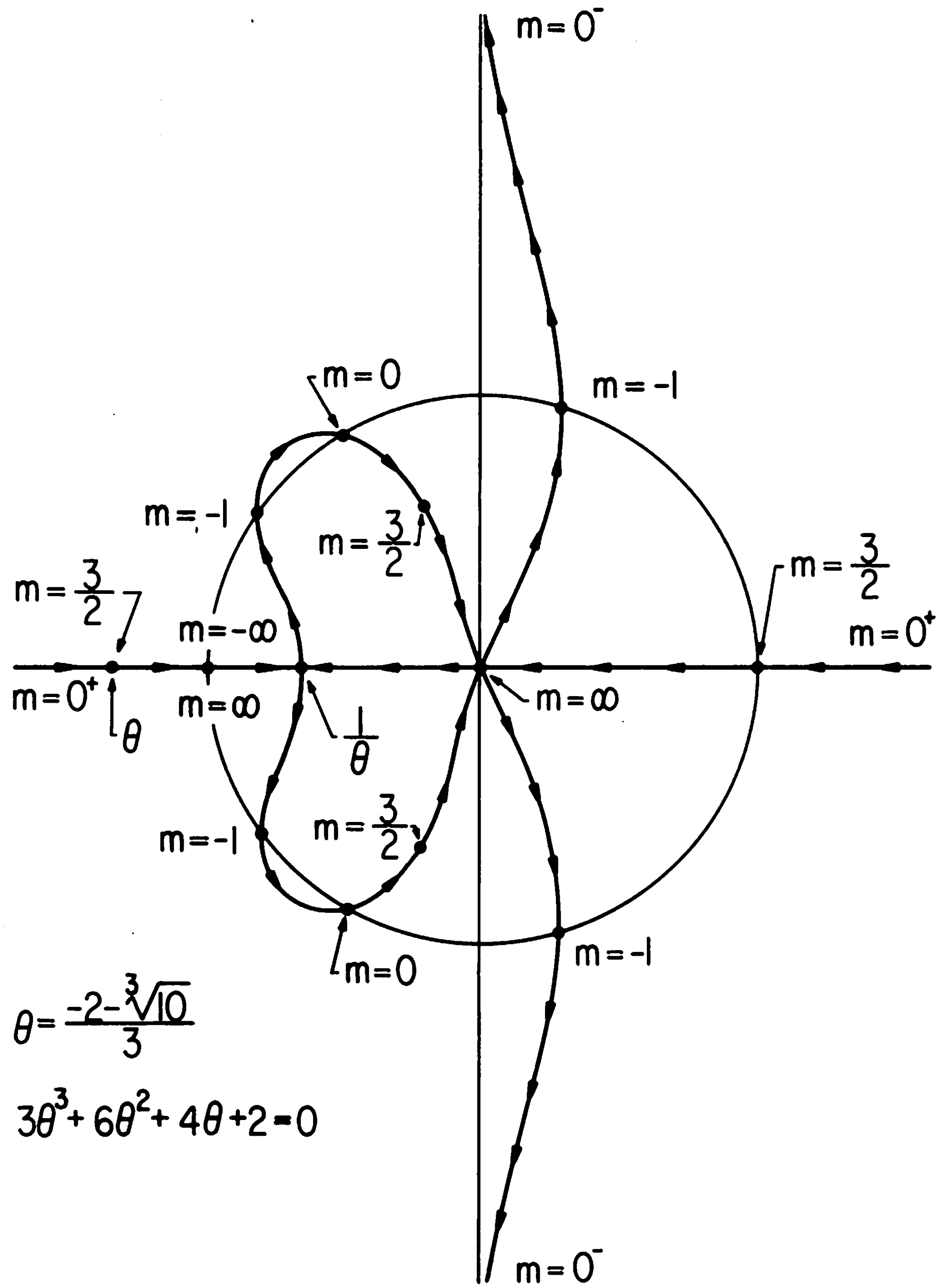

Fig. 1. Roots of the polynomial $m(z^4 + z^3) - (z^2 + z + 1)$ for real m.

the nickel supply will increase indefinitely and no stationary state can exist.

On the other hand, if $m < 3/2$, then $S(z)$ will be analytic within the unit circle iff the coefficients P_0, P_1, and P_2 are chosen so that the two roots of the former numerator factor cancel out the pair of complex conjugate roots of the denominator. This requirement will give us two additional equations, which will then enable us to determine P_0, P_1, and P_2.

Notice that P_0, P_1, and P_2 depend only on $m = n/q$ (and not on a and b).

Since the general quartic equation is solvable, we may find these conditional probabilities explicitly. We start by reducing the quartic to a cubic. If $z^4 + z^3 = hz^2 + hz + h$, then, for any value of y,

$$((8z^2 + 4z) - (4h + y))^2 = 16(1-y)z^2 + (32h - 8y)z + 64h - 16h^2 - 8hy - y^2$$

We may now choose y in such a way that the right hand side of this equation is a perfect square. This will happen iff y satisfies

$$(32h - 8y)^2 = 4 \cdot 16(1-y)(64h + 16h^2 + 8hy + y^2)$$

This equation can be simplified to

$$y^3 + 8hy^2 + (16h^2 + 48h)y - 64h = 0$$

or equivalently,

$$m^2y^3 + 8my^2 + (16 + 48m)y - 64m = 0$$

It may readily be verified that this equation has a real positive root and that

$$y \leq 4m \quad \text{with equality as} \quad m \to 0$$

and

$$y \leq 4m^{-1/3} \quad \text{with equality as} \quad m \to \infty$$

and

$$y \leq 1 \quad \text{with equality when} \quad m = 4$$

If y is chosen to satisfy this cubic, then we have

$$((8z^2 + 4z) - (4h + y))^2 = \left(4\sqrt{1 - y}\; z + [\operatorname{sign}(4h - y)] \cdot \sqrt{64h + (4h + y)^2}\right)^2$$

Taking square roots of boths sides reveals the two quadratic factors of the original quartic, with $h = 1/m = q/n$.

$$z^2 + \frac{1}{2}\left(1 + [\operatorname{sign}(4h - y)]\sqrt{1 - y}\right) z + \frac{\sqrt{(4h + y)^2 + 64h} - (4h + y)}{8}$$

and

$$z^2 + \frac{1}{2}\left(1 - [\operatorname{sign}(4h - y)]\sqrt{1 - y}\right) z - \frac{\sqrt{(4h + y)^2 + 64h} + (4h + y)}{8}$$

For positive h, the first of these two quadratics has complex conjugate roots within the unit circle; the second quadratic has real roots. Since the first of these two quadratics must be cancelled by the former numerator factor, we have

$$\frac{P_0}{P_2} = \frac{\sqrt{(4h+y)^2 + 64h} - (4h+y)}{8} = \frac{8}{\sqrt{(4 + ym)^2 + 64m} + (4+ym)}$$

$$\frac{P_1}{P_2} = \frac{1 + [\mathrm{sign}(4h-y)]\sqrt{1-y}}{2} = \frac{1 + [\mathrm{sign}(4-m)]\sqrt{1-y}}{2}$$

The cubic may be solved for y by setting $x = 3y/4 + 2h$. In terms of x, the cubic becomes

$$x^3 - 3Ax - 2B = 0$$

where $A = h(h - 9)$

$B = H(H^2 + 27h + 27/2)$

Setting $D = B^2 - A^3 = 27h^2(3h^3 + 19h^2 + \Sigma 4h + 27/4)$ and introducing u and v defined by $u+v = x$, $uv = A$, one obtains quadratic equations for u^3 and v^3 whose solutions are $B \pm \sqrt{D}$. Hence

$$y = \frac{4}{3}\left[-2h + \sqrt[3]{B + \sqrt{D}} + \sqrt[3]{B - \sqrt{D}}\right]$$

This equation is too cumbersome to be of much value. For numerical calculations, it is much easier to deal directly with the cubic

$$\left[\left(\frac{y}{4}\right)^3 - 2\left(\frac{y}{4}\right)^2 - 2\left(\frac{y}{4}\right) + 1\right] n^2 + \left[2\left(\frac{y}{4}\right)^2 + \left(\frac{y}{4}\right) - 1\right] n + \left(\frac{y}{4}\right) = 0$$

Although cubic in y, it is only quadratic in n. This makes it fairly easy to obtain plots of P_0, P_1, and P_2 vs n, using y as a parameter. The resulting graphs are shown in Figure 2. The graph has been continued for n between 3/5 and 1, even though the problem makes sense only if $0 < n < 3/5$. It is seen that P_0, P_1 and P_2 all start from 1/3 when n = 0, in accord with our earlier observations. P_0 and P_1 both have an initial slope of -1/9; P_2 starts with an initial slope of 2/9 and remains very nearly linear for $0 < n < 3/5$.

The answer to Kelly's query is given by P_0, which represents the probability that his two nickels will not turn the light out.

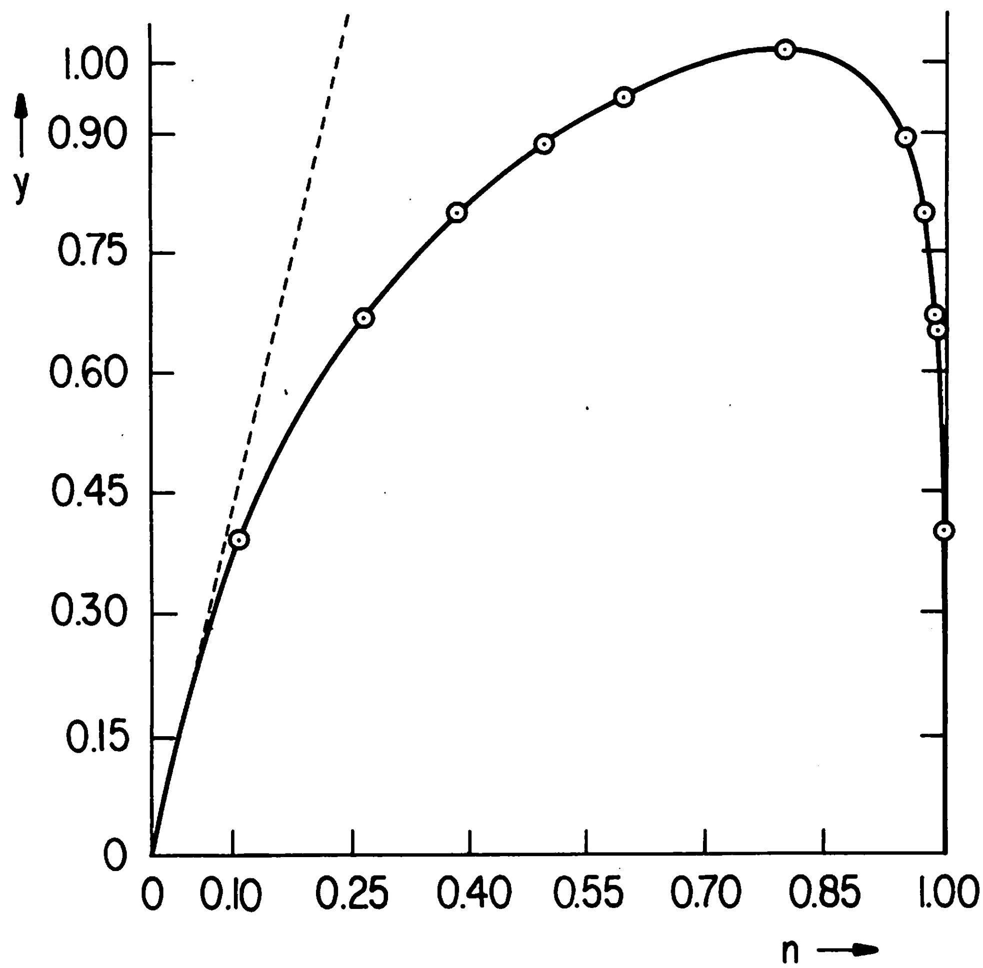

Fig. 2. y vs n.

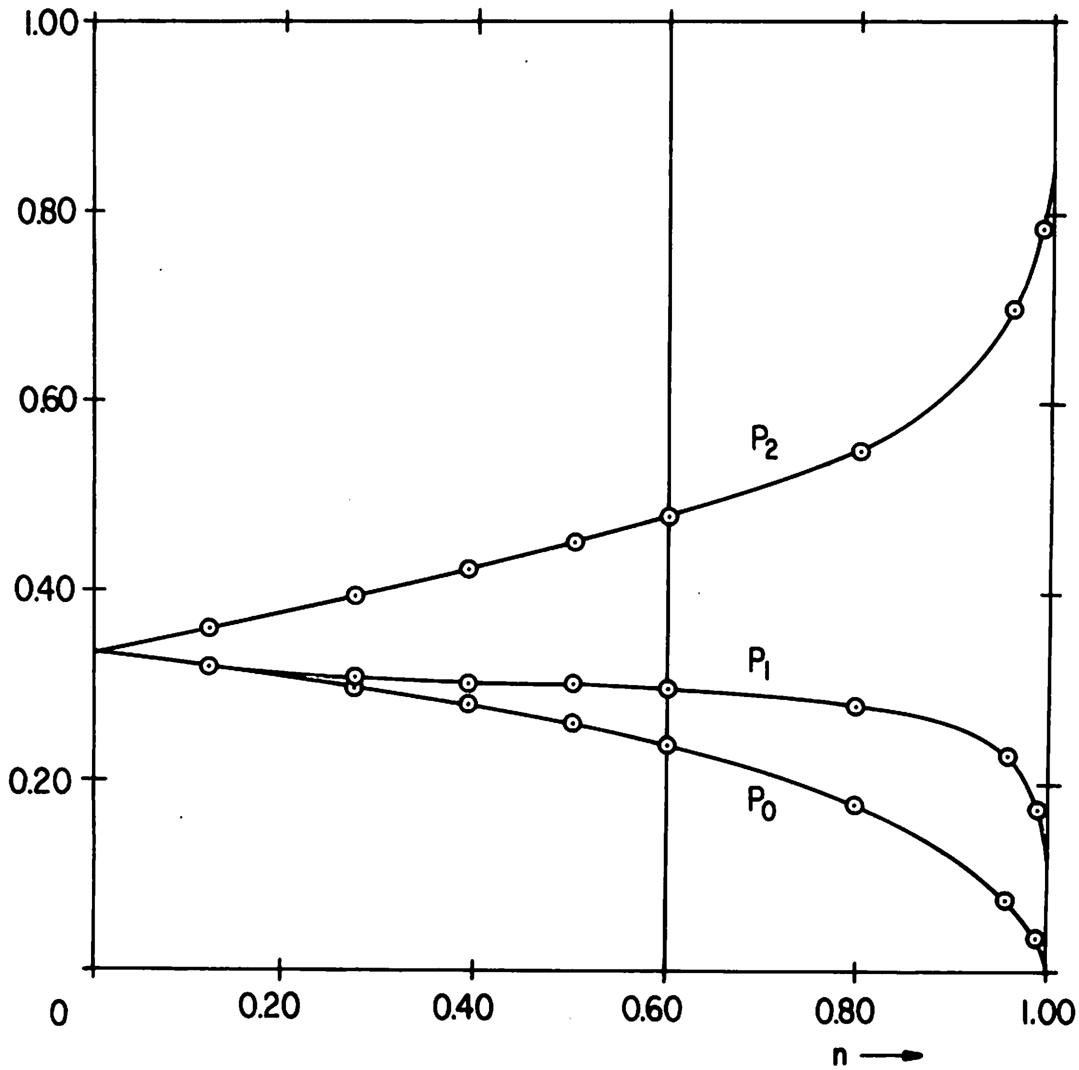

Fig. 3. The conditional state probabilities when the light is one versus n, the consequential customers probability of using two nickels.

APPENDIX 1

Table of q, P_0, P_1, and P_2 for certain values of y.

y	q	P_2	P_1	P_0
$4\epsilon^{1/3}$	ϵ	$1-\epsilon^{1/3}$	$\epsilon^{1/3}$	$\epsilon^{2/3}$
4/10	0.0015	0.89	0.10	0.01
2/3	0.0103	0.79	0.17	0.04
4/5	0.024	0.74	0.20	0.06
8/9	0.046	0.69	0.23	0.08
1	1/5	0.55	0.28	0.17
0.948	2/5	0.48	0.29^+	0.23
8/9	0.502	0.44	0.30^-	0.26
4/5	0.612	0.42	0.30^+	0.28
2/3	0.726	0.395	0.31	0.295
4/10	0.881	0.36	0.32	0.32
4ϵ	$1-\epsilon$	$\frac{1}{3}+\frac{2\epsilon}{9}$	$\frac{1}{3}-\frac{\epsilon}{9}$	$\frac{1}{3}-\frac{\epsilon}{9}$