ON BOUNDED-INPUT - BOUNDED-OUTPUT STABILITY OF NONLINEAR FEEDBACK SYSTEMS

by

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ABSTRACT

It is proved that the V. M. Popov theorem also establishes absolute stability in the bounded-input - bounded-output sense, i.e., if the Popov theorem establishes absolute stability of the autonomous system S, $(r(t) \equiv 0)$, then the system is also absolutely b.i.b.o. stable.

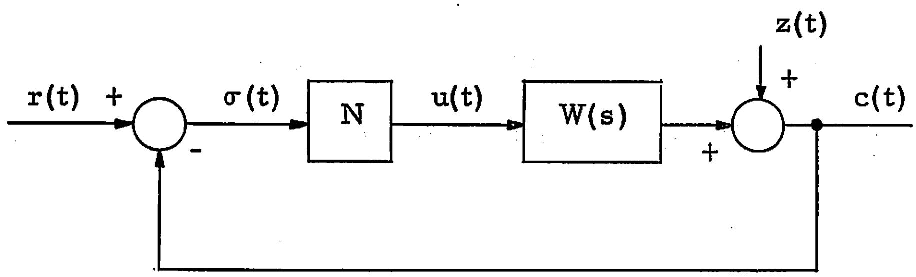

Fig. 1. System S.

I. Introduction

The stability of nonlinear deterministic systems has been the object of intensive research in the past few years. However, the main concern has been for autonomous systems. Popov,[1] Jury and Lee,[2] Desoer,[3] and others obtained frequency domain criteria for absolute stability. Sandberg[4] gave some very general results for L_2 - stability.

Another important and very practical type of stability is absolute stability in the bounded-input-bounded-output sense (b. i. b. o). The object of this paper is to prove that the V. M. Popov theorem also establishes absolute stability in the b. i. b. o. sense for the system S given in Fig. 1. Recently, Sandberg[5] proved a similar result. However, the model he considered is different from the one in Fig. 1. The proof presented in this paper is also simpler than Sandberg's and is closely connected to the proof of the autonomous case.

The notation and terminology in this paper follow those used by Aizerman and Gantmacher in Ref. 6.

II. Description of System

The system S under consideration is the single input - single output unity feedback system shown in Fig. 1. The nonlinear element is memoryless and time-invariant, the linear plant is nonanticipative time-invariant, and completely controllable and observable.

Assumption 1. The nonlinear element N is characterized by a piecewise continuous function $\varphi(\cdot)$ defined on $(-\infty, +\infty)$ such that $0 \leq \frac{\varphi(\sigma)}{\sigma} \leq k < \infty$ $\forall \quad \sigma \neq 0$ and $\varphi(0) = 0$. For ease of notation, let $\varphi(\sigma(t)) = u(t)$.

Assumption 2. The linear plant is characterized by its transfer function W(s). W(s) is a rational fraction in s with its numerator polynomial of lower degree than the denominator. W(s) has poles only in the left half s-plane (principal case), or has some poles on the $j\omega$ axis (particular cases). z(t) is the zero input response of the linear plant.

Assumption 3. The input signal to the system is such that r(t) and $\dot{r}(t)$ are bounded for all $t \geq 0$.

III. Main Results

Theorem. For the system S satisfying the previous assumptions to be absolutely b.i.b.o stable in the sector $[0, k]$ for the principal case and in the sector $[\epsilon, k]$ for the particular cases ($\epsilon > 0$ arbitrarily small), it is sufficient that there exist a real number q such that for all $\omega \geq 0$ the following inequality is satisfied

$$\mathrm{Re}\left\{ (1 + j\omega q)\, W(j\omega) \right\} + \frac{1}{k} \geq \delta > 0 \tag{P}$$

In addition, for particular cases, the conditions for stability-in-the-limit must be satisfied.

Remarks: without loss of generality the Theorem need only be proved for

(i) principal cases of $W(s)$

(ii) $0 \leq q < \infty$,

(iii) the nonlinearity $\varphi(\sigma)$ in the reduced sector $[\epsilon, k - \epsilon]$, i.e.,

$$\epsilon \leq \frac{\varphi(\sigma)}{\sigma} \leq k - \epsilon \quad \forall \quad \sigma \neq 0$$

where $\epsilon > 0$ is arbitrarily small.

These remarks are justified in Ref. 6 for the zero-input stability of system S. However, the same arguments that Aizerman and Gantmacher use in Ref. 6, can be applied for the non-zero input case.

Auxiliary Lemmas

The proof of the Theorem uses two lemmas. One of them is a well-known lemma concerning the frequency domain analysis in the V. M. Popov Theorem.[6] The second one is the main contribution of this paper.

Lemma 1

If the three real functions $f_1(t)$, $f_2(t)$, $f_3(t)$ belong to $L_2(0,\infty)$, and if their Fourier transforms are related by the equation $F_1(j\omega) = H(j\omega)F_3(j\omega) + F_2(j\omega)$ where $\mathrm{Re}\, H(j\omega) \geq \beta > 0 \;\forall\; \omega \geq 0$, then

$$-\int_0^\infty f_1(t)\, f_3(t)\, dt \leq \frac{1}{4\beta}\int_0^\infty [f_2(t)]^2\, dt. \qquad \text{(For proof see Ref. 6.)}$$

Main Lemma

If the system S satisfies all the conditions of the Theorem, then the following inequality holds for sufficiently small $\alpha > 0$

$$\left(\int_0^t e^{2\alpha\tau} u^2(\tau)\, d\tau\right)^{1/2} \leq \left[\frac{1}{\delta^2}\int_0^t e^{2\alpha\tau}\,[r(\tau) - z(\tau) + q(\dot{r}(\tau) - \dot{z}(\tau))]^2\, d\tau + \frac{2q}{\delta}\int_0^{\sigma(0)} \varphi(\sigma)\, d\sigma\right]^{1/2} \quad \forall\; t \geq 0 \qquad \text{(L)}$$

Proof of Theorem

Referring to the remarks, the Theorem need only be proved for principal cases of $W(s)$ and $0 \leq q < \infty$. It may also be assumed that $\varphi(\sigma)$ in the sector $[0, k]$ satisfies the reduced sector condition $\epsilon \leq \frac{\varphi(\sigma)}{\sigma} \leq k - \epsilon \quad \forall \ \sigma \neq 0, \ \varphi(0) = 0$ where $\epsilon > 0$ is arbitrarily small. Let $w(t)$ be the impulse response corresponding to $W(s)$. System S is described by

$$\sigma(t) = r(t) - z(t) - \int_0^t w(t-\tau)\, u(\tau)\, d\tau \tag{1}$$

or equivalently

$$\sigma(t) = r(t) - z(t) - \int_0^t e^{\alpha(t-\tau)}\, w(t-\tau)\, e^{-\alpha(t-\tau)}\, u(\tau)\, d\tau$$

By the triangle inequality and the Schwarz inequality

$$|\sigma(t)| \leq |r(t) - z(t)| + \left(\int_0^\infty e^{2\alpha x}\, w^2(x)\, dx \right)^{1/2} e^{-\alpha t} \left(\int_0^t e^{2\alpha\tau} u^2(\tau) d\tau \right)^{1/2} \tag{2}$$

Using inequality (L) of the Main Lemma yields

$$|\sigma(t)| \leq |r(t) - z(t)| + \left(\int_0^{\infty} e^{2\alpha x} w^2(x)\,dx\right)^{1/2} \cdot$$

$$\cdot \left(\frac{1}{\delta^2}\int_0^t e^{-2\alpha(t-\tau)}\left[r(\tau) - z(\tau) + q(\dot{r}(\tau) - \dot{z}(\tau))\right]^2 d\tau + \frac{2q}{\delta}\, e^{-2\alpha t}\int_0^{\sigma(0)} \varphi(\sigma)\,d\sigma\right)^{1/2} \tag{3}$$

Since $W(s)$ is a principal case, there exist positive constants K_1, K_2, such that $|w(t)| \leq K_1 e^{-K_2 t}$. Therefore there exists an α, $0 < \alpha < K_2$ such that $\int_0^{\infty} e^{2\alpha x} w^2(x)\,dx \leq A < \infty$. The second integral is bounded since it is a convolution of a strictly stable linear system with a bounded input. (Note that $z(t)$ and $\dot{z}(t)$ for principal cases are bounded). Thus, the right hand side of inequality (3) is bounded for all $t \geq 0$. Therefore

$$|\sigma(t)| \leq B < \infty \quad \forall\ t \geq 0 \tag{4}$$

which implies that the output $c(t)$ is bounded. This completes the proof of the Theorem.

Proof of Main Lemma

From system equation (1), one obtains

$$\dot{\sigma}(t) = [\dot{r}(t) - \dot{z}(t)] - \int_0^t \dot{w}(t-\tau)\,u(\tau)\,d\tau - w(0)\,u(t) \tag{5}$$

The variables $r(t)$, $z(t)$, $\dot{r}(t)$, $\dot{z}(t)$ and $u(t)$ will be truncated at T and then denoted $r_T(t)$, $z_T(t)$, $\dot{r}_T(t)$, $\dot{z}_T(t)$ and $u_T(t)$. By truncation, it is meant that the function is identically zero for $t > T$. Then, define $\sigma_T(t)$ and $\dot{\sigma}_T(t)$ by the following equations.

$$\sigma_T(t) = r_T(t) - z_T(t) - \int_0^t w(t-\tau)\, u_T(\tau)\, d\tau \tag{6}$$

$$\dot{\sigma}_T(t) = \dot{r}_T(t) - \dot{z}_T(t) - \int_0^t \dot{w}(t-\tau)\, u_T(\tau)\, d\tau - w(0)\, u_T(t) \tag{7}$$

Note that $\sigma_T(t)$ and $\dot{\sigma}_T(t)$ are not identically zero for $t > T$ but satisfy the following inequalities

$$|\sigma_T(t)| \leq K_3\, e^{-K_2 t}, \forall\ t > T, \quad |\dot{\sigma}_T(t)| \leq K_4\, e^{-K_2 t}, \quad \forall\ t > T$$

where K_3 and K_4 are positive constants and K_2 was defined in $|w(t)| \leq K_1\, e^{-K_2 t}$. Equations (6) and (7) yield

$$-\sigma_T(t) - q\dot{\sigma}_T(t) = -[r_T(t) - z_T(t) + q(\dot{r}_T(t) - \dot{z}_T(t))]$$

$$+ \int_0^t [w(t-\tau) + q\,\dot{w}(t-\tau)]\, u_T(\tau)\, d\tau + q\, w(0)\, u_T(t) \tag{8}$$

Adding $\left(\frac{1}{k} - \gamma\right) u_T(t)$ to both sides and multiplying by $e^{\alpha t}$, $0 < \alpha < K_2$, yields

$$\left\{ - \sigma_T(t) - q\,\dot{\sigma}_T(t) + \left(\frac{1}{k} - \gamma \right) u_T(t) \right\} e^{\alpha t}$$

$$= - e^{\alpha t} \left[r_T(t) - z_T(t) + q(\dot{r}_T(t) - \dot{z}_T(t)) \right]$$

$$+ \int_0^t e^{\alpha(t-\tau)} [w(t-\tau) + q\,\dot{w}(t-\tau)]\, e^{\alpha\tau} u_T(\tau)\, d\tau$$

$$+ q\,w(0)\, e^{\alpha t} u_T(t) + \left[\frac{1}{k} - \gamma \right] e^{\alpha t} u_T(t) \tag{9}$$

Identify

$$f_1(t) = \left\{ - \sigma_T(t) - q\,\dot{\sigma}_T(t) + \left[\frac{1}{k} - \gamma \right] u_T(t) \right\} e^{\alpha t}$$

$$f_2(t) = - e^{\alpha t} \left[r_T(t) - z_T(t) + q(\dot{r}_T(t) - \dot{z}_T(t)) \right]$$

Then (9) is rewritten as

$$f_1(t) = f_2(t) + \int_0^t e^{\alpha(t-\tau)} [w(t-\tau) + q\,\dot{w}(t-\tau)]\, e^{\alpha\tau} u_T(\tau)\, d\tau$$

$$+ q\,w(0)\, e^{\alpha t} u_T(t) + \left[\frac{1}{k} - \gamma \right] e^{\alpha t} u_T(t) \tag{10}$$

Since all terms in (10) belong to $L_2\,(0,\infty)$ because of the truncation at T, one can take the Fourier transform of (10).

$$F_1(j\omega) = F_2(j\omega) + \left\{ [1 + q(j\omega - \alpha)]\, W(j\omega - \alpha) + \frac{1}{k} - \gamma \right\} U_T(j\omega - \alpha) \tag{11}$$

Equation (11) satisfies the condition of Lemma 1 if

$$\text{Re}\left\{[1+q(j\omega-\alpha)]W(j\omega-\alpha)\right\} + \frac{1}{k} - \gamma \geq \delta - \gamma > 0 \qquad (P')$$

is satisfied. It is proved in the Appendix that satisfaction of (P) implies (P'). Then

$$-\int_0^\infty f_1(t)\, u_T(t)\, e^{\alpha t} dt \leq \frac{1}{4(\delta-\gamma)} \int_0^\infty [f_2(t)]^2\, dt \qquad (12)$$

Substituting for $f_1(t)$ and $f_2(t)$ into (12) and using the fact that the functions were truncated, yields

$$\int_0^T \left(\sigma(t) - \frac{u(t)}{k}\right) u(t)\, e^{2\alpha t} dt + q \int_0^T \dot{\sigma}(t)\, u(t)\, e^{2\alpha t} dt$$

$$+ \gamma \int_0^T e^{2\alpha t}\, u^2(t)\, dt \leq \frac{1}{4(\delta-\gamma)} \int_0^T e^{2\alpha t} [r(t) - z(t) + q(\dot{r}(t) - \dot{z}(t))]^2\, dt$$

The right hand side of the inequality will be denoted C(T). Note that $u(t) = \varphi(\sigma(t))$, integrate $\int_0^T \dot{\sigma}(t)\, u(t)\, e^{2\alpha t} dt$ by parts, and add $q \int_0^{\sigma(0)} \varphi(\sigma) d\sigma$ to both sides.

$$\int_0^T \left(\sigma - \frac{\varphi(\sigma)}{k}\right) \varphi(\sigma)\, e^{2\alpha t}\, dt + q\, e^{2\alpha T} \int_0^{\sigma(T)} \varphi(\sigma)\, d\sigma$$

$$- 2q\alpha \int_0^T e^{2\alpha t} \left[\int_0^{\sigma(t)} \varphi(\sigma)\, d\sigma\right] dt$$

$$+ \gamma \int_0^T e^{2\alpha t} u^2(t)\, dt \leq C(T) + q \int_0^{\sigma(0)} \varphi(\sigma)\, d\sigma \tag{13}$$

Since $\varphi(\sigma)$ lies in a reduced sector $[\epsilon, k-\epsilon]$, $\epsilon > 0$ arbitrarily small, it is noted that

(a) $\displaystyle\int_0^{\sigma(t)} \varphi(\sigma)\, d\sigma \leq \frac{k}{2}\, \sigma^2(t)$

(b) $\displaystyle\frac{\epsilon^2}{k}\, \sigma^2 \leq \left(\sigma - \frac{\varphi(\sigma)}{k}\right) \varphi(\sigma)$

Inequality (13) may then be strengthened by using (a) and (b) and deleting the positive quantity $q\, e^{2\alpha T} \displaystyle\int_0^{\sigma(T)} \varphi(\sigma)\, d\sigma$ on the left hand side.

$$\frac{1}{\gamma} \int_0^T e^{2\alpha t} \left[\frac{\epsilon^2}{k} - kq\alpha\right] \sigma^2(t)\, dt$$

$$+ \int_0^T e^{2\alpha t} u^2(t)\, dt \leq \frac{1}{4\gamma(\delta - \gamma)} \int_0^T e^{2\alpha t} [r(t) - z(t) + q(\dot{r}(t) - \dot{z}(t))]^2\, dt$$

$$+ \frac{q}{\gamma} \int_0^{\sigma(0)} \varphi(\sigma)\, d\sigma \tag{14}$$

Set $\gamma = \frac{\delta}{2}$, since γ is arbitrary as long as $0 < \gamma < \delta$. $\gamma = \frac{\delta}{2}$ minimizes the right hand side as far as the choice of γ is concerned. Then

$$\frac{2}{\delta} \int_0^T e^{2\alpha t} \left[\frac{\epsilon^2}{k} - kq\alpha \right] \sigma^2(t)\, dt$$

$$+ \int_0^T e^{2\alpha t} u^2(t)\, dt \leq \frac{1}{\delta^2} \int_0^T e^{2\alpha t} \left[r(t) - z(t) + q(\dot{r}(t) - \dot{z}(t)) \right]^2 dt$$

$$+ \frac{2q}{\delta} \int_0^{\sigma(0)} \varphi(\sigma)\, d\sigma \tag{15}$$

Denote $I_1 = \int_0^T e^{2\alpha t} \left[\frac{\epsilon^2}{k} - kq\alpha \right] \sigma^2(t)\, dt$

If for $\alpha > 0$, $\frac{\epsilon^2}{k} - kq\alpha \geq 0$ then $I_1 \geq 0$ and it may be deleted from the left hand side of inequality (15). For any $\epsilon > 0$, $q < \infty$, $k < \infty$ one can always find an α small enough such that

$$0 < \alpha \leq \frac{\epsilon^2}{q k^2}$$

Hence inequality (15) becomes

$$\left(\int_0^T e^{2\alpha t} u^2(t)\, dt \right)^{1/2} \leq \left[\frac{1}{\delta^2} \int_0^T e^{2\alpha t} \left[r(t) - z(t) + q(\dot{r}(t) - \dot{z}(t)) \right]^2 dt + \frac{2q}{\delta} \int_0^{\sigma(0)} \varphi(\sigma)\, d\sigma \right]^{1/2} \quad \forall\ T \geq 0 \tag{16}$$

which completes the proof of the Main Lemma.

IV. Extensions

Note 1

The case where $\varphi(\sigma)$ satisfies the inequality $a \leq \frac{\varphi(\sigma)}{\sigma} \leq b$ can be treated by making the change of variables $\varphi(\sigma) = \tilde{\varphi}(\sigma) + a\sigma$. Then $\tilde{\varphi}(\sigma)$ is contained in the sector $[0, b-a]$ and the Theorem can be applied. For principal cases of $W(s)$ the parameter a may also assume negative values. For the case $q = 0$ this reduces to the familiar circle criterion for autonomous systems.[7]

Note 2

It can easily be shown that with $q = 0$ the Theorem proves b.i.b.o. stability when N in the system S is a time-varying nonlinearity described by $u(t) = \varphi[\sigma(t), t]$. The function $\varphi(\sigma, t)$ satisfies

$$0 \leq \frac{\varphi(\sigma, t)}{\sigma} \leq k < \infty, \quad \forall\, \sigma \neq 0, \quad \forall\, t \geq 0$$

and

$$\varphi(0, t) = 0 \quad \forall t \geq 0$$

Note 3

The results of the Theorem also apply when the linear plant is not described by a rational transfer function in s, provided that

(a) for arbitrarily small $\alpha > 0$ the impulse response $w(t)$ satisfies

$$\int_0^\infty e^{2\alpha t} w^2(t)\, dt \leq A < \infty, \qquad \text{A some positive number.}$$

(b) the zero input response $z(t)$ and its derivative $\dot{z}(t)$ are bounded for all $t \geq 0$ by decaying exponentials.

(c) $W(s) = \int_0^\infty w(t)\, e^{-st}\, dt$ is analytic in the domain $\mathrm{Re}\ s \geq -\alpha$.

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APPENDIX

Proof That Satisfaction of Inequality (P) Implies (P').

In the expression of (P') replace δ by δ_α. It will be shown that this has no consequences and that if there exists a $\delta > 0$ satisfying (P), then there also exists a δ_α, $0 < \delta_\alpha < \delta$, satisfying (P').

(P') is rewritten as

$$\mathrm{Re}\left\{ [1+q(j\omega-\alpha)]W(j\omega-\alpha) \right\} + \frac{1}{k} - \gamma \geq \delta_\alpha - \gamma > 0 \qquad \text{(P')}$$

Given any principal case W(s), there exists an arbitrarily small $\alpha > 0$ such that W(s) is analytic in the domain Re $s \geq -\alpha$. It follows that $|W(j\omega-\alpha) - W(j\omega)|$ and $|(j\omega-\alpha)W(j\omega-\alpha) - j\omega\, W(j\omega)|$ approach zero as α becomes arbitrarily small. Then, there exists δ_α such that $0 < \delta_\alpha < \delta$ and $(\delta - \delta_\alpha) \to 0$.