ON DETERMINABLE CLASSES OF SIGNALS AND LINEAR CHANNELS

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ABSTRACT

In a recent paper, Root [1] inaugurated a study of the measurement and processing problems arising when a signal passes through an unknown linear channel. Subsequently Prosser and Root [2] characterized bounded determinable classes of signals and channels which are Hilbert-Schmidt operators on $L^2(-\infty, \infty)$. In this paper we consider the signal space to be an arbitrary Hilbert space H and a channel to be any continuous endomorphism on H. We obtain a characterization of ε-determinable convex classes of signals by relating this property to the concept of n-dimensional diameters introduced by Tikhomirov [3] and thus demonstrating the relevance of our results to the theory of best approximations. We next generalize the results of Prosser and Root dealing with bounded determinable classes of channels, and also obtain some properties of various classes of unbounded sets of channels. The motivation of dealing with abstract space of signals and channels is the applicability of our results to various problems in control-system identification and the theory of approximations.

I. INTRODUCTION

In a recent paper, Root [1] has inaugurated a study of measurement and processing problems arising when a signal passes through an unknown linear channel. He develops a terminology and shows that it is useful for formulating a large class of problems which involve channel identification. Some of the questions provoked by this paper have been subsequently answered by Prosser and Root [2]. Specifically, they show that if it is assumed that the unknown signal belongs to a fixed bounded subset of $L^2(-\infty, \infty)$ or the unknown channel belongs to a fixed bounded subset of the Hilbert-Schmidt operators on $L^2(-\infty, \infty)$, then that subset is determinable if and only if it is conditionally compact. This characterization is exploited to yield a number of useful, interesting results.

In this paper we remove the condition of boundedness and determinability on classes of signals and impose convexity, i.e., we study the properties of ε-determinable convex classes of signals contained in an arbitrary Hilbert space. Since the convex closure of compact sets are compact, this condition does not impose restrictions for bounded, determinable sets. Furthermore, almost all the classes of signals appearing in the literature are convex. The main result of this paper is the relationship between ε-determinable convex classes of signals and their n-dimensional diameters. The concept of the n-dimensional diameter of a set has been extensively studied by Russian mathematicians, notably Tikhomirov [3]. As a corollary of this relationship we show that a convex set of signals is determinable if and only it is contained in the vector sum of a finite-dimensional subspace and a compact set. Furthermore, for a convex set of signals linear determinations are "almost as good" as nonlinear determinations. These results are given in Section II.

In Section III we study the problem of channel determination. We allow the channel to be any linear continuous transformation of the signal space H into itself. We generalize the results of Root and Prosser for bounded determinable classes of channels. We also study some special cases of unbounded determinable classes.

II. DETERMINABLE CLASSES OF SIGNALS

By a signal we mean an element x of a real or complex, infinite-dimensional Hilbert space H. By an n-measurement we mean a fixed n-tuple of vectors $(y_1, \cdots, y_n)$ from H. An n-estimator function is a continuous mapping f from E^n (the n-dimensional vector space over the real or complex field depending on H), into H. The estimator function is said to be linear if f is affine. By a (linear) n-experiment $(y_1, \cdots, y_n; f)$ we shall mean an n-measurement $(y_1, \cdots, y_n)$ and a (linear) n-estimator function f.

Let C be a subset of signals. C is said to be ε-determinable if there is an n-experiment $(y_1, \cdots, y_n; f)$ such that

$$|x - f(\langle x, y_1\rangle, \cdots, \langle x, y_n\rangle| \leq \varepsilon, \quad \text{for } x \text{ in } C$$

where $|z|$ is the norm of z in H. C is said to be determinable if it is ε-determinable for each $\varepsilon > 0$.

An n-variety $L = (a, N)$ of H is the set $L = a + N$ where $a \in H$ is a fixed vector and N is a fixed n-dimensional subspace of H. By the distance of C from $L = (a, N)$ we mean the number (possibly $+\infty$)

$$d(C, L) = \sup_{x \in C} \inf_{y \in L} |x-y|$$

$$= \sup_{x \in C} |(x-a) - P_N(x-a)|$$

where P_N is the orthogonal projection of H onto N. By the n-dimensional diameter of C we mean the number (possibly $+\infty$)

$$d_n(C) = \inf\{d(C, L) \mid L \text{ is an n-variety of } H\}.$$

For elaboration on the above definitions the reader is referred to Root [1] and Tikhomirov [3].

Lemma 2.1. If the n-dimensional diameter of C is equal to ε, then for each $\delta > 0$, C is $(\varepsilon+\delta)$-determinable by a linear n-experiment.

Proof. Let $L = (a, N)$ be an n-variety such that

$$\sup_{x\in C} |(x-a) - P_N(x-a)| \leq \varepsilon + \delta .$$

Let $y_1, \cdots, y_n$ be an orthonormal basis for N and let $f(\lambda_1, \cdots, \lambda_n) = (a + \lambda_1 y_1 + \cdots + \lambda_n y_n - P_N a)$. Then for x in C,

$$|x - f(\langle x, y_1\rangle, \cdots, \langle x, y_n\rangle)|$$

$$= \left| x - \sum_{i=1}^{n} \langle x, y_i\rangle y_i + a - P_N a \right| = |(x-a) - P_N(x-a)| \leq \varepsilon + \delta .$$

Q.E.D.

A set C of H is said to be symmetric about a point c_0 if $c_0 + \lambda(c-c_0)$ is in C for $c \in C$ and $|\lambda| \leq 1$. C is said to be symmetric if it is symmetric about 0. As a partial converse to Lemma 2.1 we have Lemma 2.2.

Lemma 2.2. Let C be a symmetric, convex set of signals. If C is ε-determinable by an n-experiment, then the n-dimensional diameter of C is $d_n(C) \leq \varepsilon$.

Lemma 2.2 is a corollary of Theorem 2.1.

Theorem 2.1. Let C be a symmetric, convex set. Let N be any n-dimensional subspace of H. Then there exists an n-dimensional subspace N_1 of H (dependent on N) such that

$$P_{N_1^{\perp}}(C) \subseteq P_{N_1^{\perp}}\left(C \cap N^{\perp}\right) \tag{2.1}$$

where $N^{\perp}(N_1^{\perp})$ is the orthogonal complement of $N(N_1)$ in H.

Before proving (2.1) let us prove Lemma 2.2 (assuming Theorem 2.1).

Proof of Lemma 2.2. Let C be ε-determinable by the n-experiment $(y_1, \cdots, y_n; f)$. Let N be the subspace generated by $\{y_1, \cdots, y_n\}$. Then for x_1 and x_2 in C and $P_N(x_1-x_2) = 0$ we must have $|x_1-x_2| \leq 2\varepsilon$. Since C is symmetric this means that for x in C and $P_N x = 0$ we have $|x| \leq \varepsilon$. Now let N_1 be an n-dimensional subspace which satisfies (2.1). Then

$$d(C, N_1) = \sup_{x \epsilon C} \left| x - P_{N_1} x \right|$$

$$= \sup_{x \epsilon C} \left| P_{N_1^\perp} x \right|$$

$$= \sup_{x \epsilon C \cap N^\perp} \left| P_{N_1^\perp} x \right| \quad \text{by } (2.1).$$

But $x \epsilon C \bigcap N^\perp$ means that $x \epsilon C$ and $P_N x = 0$, so that $|x| \leq \varepsilon$. Hence $d(C, N_1) \leq \varepsilon$ so that $d_n(C) \leq \varepsilon$. Q.E.D.

Proof of Theorem 2.1. The proof proceeds by induction on the dimension n of N. The case $n = 1$ is treated in the Appendix. Let us assume that the theorem is true for subspaces of dimension $m \leq n \geq 1$ and let N be any $(n+1)$-dimensional subspace. We factor N arbitrarily into $N = L \oplus \{x\}$ where L is an n-dimensional subspace and $\{x\}$ is a 1-dimensional subspace generated by a vector x in H. By the induction hypothesis there is a subspace L_1 of dimension n which corresponds to L. It is therefore enough to show that there exists a vector y in H such that,

$$P_{(L_1 \oplus \{y\})^\perp}(C) \subseteq P_{(L_1 \oplus \{y\})^\perp} C \bigcap (L \oplus \{x\})^\perp .$$

Now $(L \oplus \{x\})^\perp = L^\perp \bigcap x^\perp$ and $(L_1 \oplus \{y\})^\perp = L_1 \bigcap y^\perp$, so that we must show, for some y in H, that

$$P_{L_1^{\perp} \cap y^{\perp}}(C) \subseteq P_{L_1^{\perp} \cap y^{\perp}}\left(C \cap L^{\perp} \cap x^{\perp}\right), \quad \text{or}$$

$$P_{y^{\perp}} P_{L_1^{\perp} \cap y^{\perp}} P_{L_1^{\perp}}(C) \subseteq P_{y^{\perp}} P_{L_1^{\perp} \cap y^{\perp}} P_{L_1^{\perp}}\left(C \cap L^{\perp} \cap x^{\perp}\right). \tag{2.2}$$

By the induction hypothesis, $P_{L^{\perp}}(C) \subseteq P_{L_1^{\perp}}(C \cap L^{\perp})$. Let $Q = C \cap L^{\perp}$. Then (2.2) is equivalent to

$$P_{y^{\perp}} P_{L_1^{\perp} \cap y^{\perp}} P_{L_1^{\perp}}(Q) \subseteq P_{y^{\perp}} P_{L_1^{\perp} \cap y^{\perp}} P_{L_1^{\perp}}\left(Q \cap x^{\perp}\right),$$

or

$$P_{y^{\perp}} P_{L_1^{\perp} \cap y^{\perp}} P_{L_1^{\perp}}(Q + L_1) \subseteq P_{y^{\perp}} P_{L_1^{\perp} \cap y^{\perp}} P_{L_1^{\perp}}\left((Q \cap x^{\perp}) + L_1\right),$$

or

$$P_{y^{\perp}} P_{L_1^{\perp} \cap y^{\perp}}(Q + L_1) \subseteq P_{y^{\perp}} P_{L_1^{\perp} \cap y^{\perp}}\left((Q \cap x^{\perp}) + L_1\right),$$

or

$$P_{L_1^{\perp} \cap y^{\perp}} P_{y^{\perp}}(Q + L_1) \subseteq P_{L_1^{\perp} \cap y^{\perp}}\left((Q \cap x^{\perp}) + L_1\right).$$

Hence, it is enough to show that there is a y such that

$$P_{y^{\perp}}(Q + L_1) \subseteq P_{y^{\perp}}\left((Q \cap x^{\perp}) + L_1\right), \quad \text{or}$$

$$Q + L_1 + \{y\} \subseteq (Q \cap x^{\perp}) + L_1 + \{y\}. \tag{2.3}$$

Now $Q = C \cap L^{\perp}$ is a convex symmetric set so that by the induction hypothesis, for $n = 1$, there exists a vector y (depending on x) such that,

$$P_{y^{\perp}}(Q) \subseteq P_{y^{\perp}}(Q \cap x^{\perp}),$$

i.e.,

$$Q + \{y\} \subseteq (Q \cap x^{\perp}) + \{y\}.$$

Hence (2.3) is satisfied by this y so that the theorem is proved.

Q.E.D.

Since the n-dimensional diameter and the ε-determinability of a set is invariant under translation, we immediately have, from Lemmas 2.2 and 2.1, the following corollary.

Corollary 2.1. Let C be a convex set, symmetric about a point. If C is ε-determinable by an n-experiment, its n-dimensional diameter is less than ε, and for $\delta > 0$, C is $(\varepsilon+\delta)$-determinable by a linear n-experiment.

For arbitrary convex sets we have another corollary.

Corollary 2.2. Let C be a convex set which is ε-determinable by an n-experiment. Then $d_n(C) \leq 2\varepsilon$ and for $\delta > 0$, C is $(2\varepsilon+\delta)$-determinable by a linear n-experiment.

Proof. Let the n-experiment of the hypothesis be $(y_1, \cdots, y_n; f)$, and let N be the subspace generated by $\{y_1, \cdots, y_n\}$. Then for x_1 and x_2 in C with $P_N(x_1-x_2) = 0$ we must have $|x_1-x_2| \leq 2\varepsilon$. Without loss of generality we assume that the origin is in C, and we consider the convex symmetric closure $[C]$, of C. If the underlying field of the Hilbert space is real, then

$$[C] \triangleq [C]_R = \left\{ k_1x_1 - k_2x_2 \middle| x_i \in C,\ k_i \geq 0,\ k_1 + k_2 = 1 \right\}.$$

If the underlying field is the complexes, then

$$[C] \triangleq [C]_C = \left\{ \lambda x \middle| x \in [C]_R,\ \ |\lambda| \leq 1 \right\}.$$

In either case it is easy to see that for any $x \in [C]$ with $P_N x = 0$ we must have $|x| \leq 2\varepsilon$. From the proof of Lemma 2.2 we see then that $d_n([C]) \leq 2\varepsilon$ so that $d_n(C) \leq 2\varepsilon$. The second assertion follows from Lemma 2.1. Q.E.D.

Our final result of this section deals with determinable sets. As an alternative characterization of convex, determinable sets we have Theorem 2.2.

Theorem 2.2. A convex set C is determinable iff $C \subseteq N + K$ for some finite-dimensional subspace N and some compact set K.

Proof. From the proof of Corollary 2.2 it suffices to prove this statement for symmetric convex sets. Now C is determinable if and only if $d_n(C) \to 0$ as $n \to \infty$. Hence for each n there exists an n-dimensional subspace N_n such that

$$\lim_{n\to\infty} \sup_{x\in C} \left| x - P_{N_n} x \right| = 0.$$

We can assume that $N_n \subseteq N_{n+1}$ for each n. Let n_0 be such that

$$\sup_{x\in C} \left| x - P_{n_0} x \right| < \infty,$$

and let

$$K = \left\{ x - P_{N_{n_0}} x \middle| x \in C \right\}.$$

Then K is bounded and $C \subseteq N_{n_0} + K$. It is easy to see that K is also determinable and hence by a result of Prosser and Root [2], K is compact. The argument is trivially reversible so that the theorem is proved. Q.E.D.

Remark. It is conjectured that Theorem 2.2 is true without the convexity assumption. It is worth noting that we have also shown that if a set, convex or not, contains a sphere of radius bigger than ε, then the set is not ε-determinable. Furthermore, we have shown that a symmetric convex set is ε-determinable if and only if it is contained in the ε-neighborhood of an n-dimensional subspace, for some $n < \infty$, and in this case it is ε-determinable by a linear n-experiment.

III. DETERMINABLE CLASSES OF CHANNELS

Henceforth we take the signal space to be the real or complex Hilbert space H. As before, we denote the norm of a signal x in H by $|x|$ and the inner product of x and y in H by $\langle x, y\rangle$. By a channel we shall mean an element k of the Banach space B(H) of continuous endomorphisms on H. The norm of k in B(H) will be denoted by $\|k\|$ where $\|k\| = \sup\{|kx| \,|\, x \in H, |x| \leq 1\}$.

By an n-measurement we mean a fixed n-tuple of pairs of vectors $((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n))$ in H. The n-measurement is said to be practical if $x_1 = x_2 = \cdots = x_n$. An n-estimator function is a continuous mapping f from E^n into B(H). The estimator function is said to be linear if f is affine. An n-experiment is an n-measurement together with an n-estimator function; it is said to be linear or practical if the corresponding estimator is linear or practical.

A subset K of B(H) is said to be ε-determinable if there exists an n-experiment $((x_1, y_1), \cdots, (x_n, y_n); f)$ such that

$$\|k - f(\langle kx_1, y_1\rangle, \cdots, \langle kx_n, y_n\rangle)\| \leq \varepsilon, \quad \text{for } k \in K. \tag{3.1}$$

K is ε-determinable in practice if there is a practical n-experiment which satisfies (3.1). K is said to be determinable (in practice) if for each $\varepsilon > 0$ it is ε-determinable (in practice). For a motivation of these definitions and further elaborations the reader is referred to references [1] and [2].

We first obtain a generalization of two results of Prosser and Root [2].

Theorem 3.1. Let $K \subseteq B(H)$ be a bounded set of channels. Then K is determinable if and only if the closure of K, $\overline{K}$, is compact.

Proof. The proof for necessity is the same as that of Prosser and Root [2]. By definition of determinability for each $\varepsilon > 0$, there is a linear map $g_\varepsilon : B(H) \to E^n$ and a continuous map $f_\varepsilon : E^n \to B(H)$ such that

$$\|k - f_\varepsilon(g_\varepsilon(k))\| \leq \varepsilon, \quad \text{for all } k \text{ in } K. \tag{3.2}$$

Since K is bounded and g_ε is linear, $g_\varepsilon(K)$ is bounded in E^n and hence totally bounded. Since f_ε is continuous, $f_\varepsilon(g_\varepsilon(K)) = K_\varepsilon$ is totally bounded and also from (3.2), $K \subseteq K_\varepsilon + S_\varepsilon$, where S_ε is the sphere in $B(H)$ of radius ε. Therefore, K is totally bounded, i.e., $\overline{K}$ is compact.

We prove sufficiency through the following lemma.

Lemma 3.1. Let K be a totally bounded subset of $B(H)$. Then for each $\varepsilon > 0$, there exists an n-measurement $((x_1, y_1), \cdots, (x_n, y_n))$ with $|x_i| \leq 1$, $|y_i| \leq 1$ such that for every pair (k, k') in K

$$\text{if } |\,(k-k')x_i, y_i\,| \leq \varepsilon, \quad \text{for } i=1, \cdots, n$$

$$\text{then } \|k-k'\| \leq 12\varepsilon. \tag{3.3}$$

Proof. Let $\varepsilon > 0$ be fixed. Let $K_\varepsilon = \{k_1, \cdots, k_m\}$ be a finite set such that $K \subseteq K_\varepsilon + S_\varepsilon$. For each pair k_i, k_j in K_ε let x_{ij} be an element of H such that $|x_{ij}| \leq 1$ and $|(k_i-k_j)x_{ij}| \geq \|k_i-k_j\| - \varepsilon$. Now let k, k' be in K and suppose that $\|k-k'\| > 6\varepsilon$ By definition of K_ε there exist i and j such that $\|k-k_i\| \leq \varepsilon$ and $\|k'-k_j\| \leq \varepsilon$. It follows that $\|k_i-k_j\| > 4\varepsilon$. Moreover,

$$\begin{aligned}
|(k-k')x_{ij}| &= |(k-k_i)x_{ij} - (k'-k_j)x_{ij} + (k_i-k_j)x_{ij}| \\
&\geq |(k_i-k_j)x_{ij}| - |(k-k_i)x_{ij}| - |(k'-k_j)x_{ij}| \\
&\geq \|k_i-k_j\| - \varepsilon - \|k-k_i\|\,|x_{ij}| - \|k'-k_j\|\,|x_{ij}| \\
&> 4\varepsilon - \varepsilon - \varepsilon - \varepsilon = \varepsilon.
\end{aligned} \tag{3.4}$$

Thus we have shown that if $\|k-k'\| > 6\varepsilon$ then $|(k-k')x_{ij}| > \varepsilon$ for some x_{ij}. Now, since K is totally bounded, the set $Q_{ij} = \{kx_{ij} | k \in K\}$

is totally bounded for each i, j, and hence forms a determinable subset of H. Hence, there exists a finite set Y_{ij} in H such that $|kx_{ij} - k'x_{ij}| > \varepsilon$ implies that $|\langle (k-k')x_{ij}, y\rangle| > \varepsilon/2$ for some y in Y_{ij}. Combining this fact with (3.4) yields (3.3). Q.E.D.

We return to the proof of Theorem 3.1. The n-measurement of Lemma 3.1 yields a linear mapping $g: B(H) \to E^n$ with the ith coordinate g_i of g given by $g_i(k) = \langle kx_i, y_i\rangle$. We will construct a function $f: E^n \to B(H)$ with the following properties: (1) f is continuous and (2) $|g_i(k) - g_i(f \circ g(k))| \leq \varepsilon$ for $i=1, \cdots, n$ and for all k in K. By Lemma 3.1, therefore, $\|k-f(\langle kx_1, y_1\rangle, \cdots, \langle kx_n, y_n\rangle)\| \leq 12\varepsilon$ for k in K and the theorem would be proved.

Construction of f. Let $K_\varepsilon = \{k_1, \cdots, k_m\}$ be defined as in Lemma 3.1, and let $[K_\varepsilon]$ be the convex hull of K_ε. Let $Q_\varepsilon = g(K_\varepsilon)$. Then Q_ε is a finite set in E^n and $[Q_\varepsilon] = g([K_\varepsilon])$ since g is linear. Let $Q = g(K)$. For each q in Q let $\overline{q}$ be the unique point in $[Q_\varepsilon]$ closest to q. The mapping $d: q \to \overline{q}$ of Q onto $[Q_\varepsilon]$ is continuous, and furthermore $|q_i-\overline{q}| \leq \varepsilon$ for $i=1, \cdots, n$. If we treat $[Q_\varepsilon]$ as a simplicial complex we can easily construct a continuous map $h: [Q_\varepsilon] \to [K_\varepsilon]$ such that $g(h \circ g(k)) = g(k)$ for k in $[K_\varepsilon]$. Putting $f = g \circ d$ we see that f has the required properties. Q.E.D.

The next result characterizes bounded sets of channels which are determinable in practice.

Theorem 3.3. Let $K \subseteq B(H)$ be a bounded set. Then K is determinable in practice if and only if (1) K is determinable, i.e., K is totally bounded, and (2) for each $\varepsilon > 0$, there is an x in H such that for each pair (k, k') in $\overline{K}$, the closure of K,

$$\|k-k'\| \geq \varepsilon \Longrightarrow |kx - k'x| > 0. \tag{3.5}$$

Proof. The necessity of the two conditions follows from the definition. It remains to prove sufficiency. Let $\varepsilon > 0$ be fixed and let $x \in H$ satisfy (3.5). We can assume that $|x| \leq 1$. Since $\overline{K}$ is compact in $B(H)$, $\overline{K} \times \overline{K}$ is compact in $B(H) \times B(H)$. The set $P \subseteq \overline{K} \times \overline{K}$ given by $P = \{(k, k') | k, k' \text{ in } K \text{ and } \|k-k'\| \geq \varepsilon\}$ is also compact. Let $\pi: P \rightarrow H$ be the map given by $\pi(k, k') = (k-k')x$. Because P is compact and from (3.5) we see that there is a number $\eta > 0$ such that $|\pi(k, k')| \geq \eta$ for $(k, k') \in P$ so that we have for each pair (k, k') in $\overline{K}$ that

$$\|k-k'\| \geq \varepsilon \Longrightarrow |kx - k'x| \geq \eta. \qquad (3.6)$$

Now the set $C = \{kx | k \in \overline{K}\}$ is compact since $\overline{K}$ is compact, hence there exists a finite set $\{y_1, \cdots, y_n\} \subseteq H$, $|y_i| \leq 1$ such that

$$|kx - k'x| \geq \eta \Longrightarrow |\langle kx - k'x,\ y_i \rangle| \geq \frac{\eta}{2} \text{ for some } y_i. \qquad (3.7)$$

Combining (3.6) and (3.7) we get for each k, k' in $\overline{K}$

$$|\ (k-k')x,\ y_i\ | < \frac{\eta}{2} \quad i=1, \cdots, n \Longrightarrow \|k-k'\| < \varepsilon, \qquad (3.8)$$

i.e., the practical n-measurement $((x, y_1), \cdots, (x, y_n))$ defines a linear mapping $g: B(H) \rightarrow E^n$ with the ith coordinate g_i given by $g_i(k) = \langle kx,\ y_i \rangle$ such that for all k, k' in K

$$|g_i(k-k')| < \frac{\eta}{2} \quad i=1, \cdots, n \Longrightarrow \|k-k'\| < \varepsilon.$$

Let $K_{\eta/2} = \{k_1, \cdots, k_m\}$ be a finite set such that $\overline{K} \subseteq K_{\eta/2} + S_{\eta/2}$. Using $K_{\eta/2}$ we can construct a continuous map $f: E^n \rightarrow B(H)$ (as in the proof of Theorem 3.1), such that for k in K, $\|k - f \circ g(k)\| < \varepsilon$. Hence, the practical n-experiment $((x, y_1), \cdots, (x, y_n); f)$ constitutes an ε-determination of K. **Q.E.D.**

The next sequence of results deals with special classes of unbounded sets of channels.

<u>Theorem 3.3.</u> Let N be an n-dimensional subspace in B(H) generated by the linearly independent channels $\{k_1, \cdots, k_n\}$. Then N is determinable by a linear n-experiment. Furthermore N is determinable in practice if and only if there is a vector x in H such that the vectors $\{k_1x, \cdots, k_nx\}$ are linearly independent.

<u>Proof.</u> Since $\{k_1, \cdots, k_n\}$ is a linearly independent set there exist n pairs of vectors $((x_1, y_1), \cdots, (x_n, y_n))$ in H such that the $n \times n$ matrix $A = \{a_{ij}\}$ with $a_{ij} = \langle k_jx_i, y_i\rangle$ is nonsingular. The n-measurement $((x_1, y_1), \cdots, (x_n, y_n))$ defines a one-one mapping g, from N into E^n as follows:

$$g: k = \sum_{i=1}^{n} \alpha_i k_i \to (\langle kx_1, y_1\rangle, \cdots, \langle kx_n, y_n\rangle) = A\alpha$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$. Clearly, the mapping $f: E^n \to N$ given by

$$f(\lambda_1, \cdots, \lambda_n) = \sum_{i=1}^{n} \alpha_i k_i$$

where $\alpha = A^{-1}\lambda$ is continuous and the composite mapping $g \circ f$ is the identity operator on N. This proves the first assertion. In the second assertion the necessity is clear. Thus, suppose $x \in H$ is such that the vectors $k_1x, k_2x, \cdots, k_nx$ are linearly independent. Choose vectors y_i in H such that $\langle k_jx, y_i\rangle = 0$ for $i \neq j$ and $\langle k_ix, y_i\rangle = 1$. Then the matrix A is the identity matrix and the rest of the proof follows as in the previous case. **Q.E.D.**

The final result is given without proof. It can be proved by a combination of the techniques used in the proof of the two previous theorems.

<u>Theorem 3.4.</u> Let K be a set of channels contained in the vector sum of a finite-dimensional subspace and a compact set. Then K is determinable. K is determinable in practice if and only if for each $\varepsilon > 0$ there is a vector x in H such that for k and k' in K

$$\|k-k'\| \geq \varepsilon \Longrightarrow |kx - k'x| > 0.$$

APPENDIX

Proof of Theorem 2.1 for $n = 1$

Let C be a symmetric, closed convex set in H. Let $x \in H$. We have to show that there exists a vector y in H, depending on x such that

$$P_{y^\perp}(C) \subseteq P_{y^\perp}(C \cap x^\perp). \tag{1}$$

Proof. Equation 1 is equivalent to showing that there is a vector y such that

$$C + \{y\} \subseteq (C \cap x^\perp) + \{y\}. \tag{2}$$

Let M be any subset of H. We define the polar of ~~M~~ to be the set

$$M^0 = \left\{ x \in H \,\middle|\, \sup_{m \in M} |\langle m, x \rangle| \leq 1 \right\}.$$

Since the sets in Eq. 2 are closed, convex and symmetric, using the Bipolar Theorem it is enough to show that

$$(C + \{y\})^0 \supseteq \Big((C \cap x^\perp) + \{y\}\Big)^0. \tag{3}$$

From the definition of the polar we see that Eq. 3 is equivalent to

$$C^0 \cap y^\perp \supseteq (C \cap x^\perp)^0 \cap y^\perp \tag{4}$$

which in turn is equivalent to

$$C^0 \cap y^\perp \supseteq (C^0 + \{x\}) \cap y^\perp. \tag{5}$$

Let $Q = C^0$. Q is a convex symmetric set and we have to show that there exists a vector y such that

$$Q \cap y^\perp \supseteq (Q + \{x\}) \cap y^\perp. \tag{6}$$

If $Q \supseteq (Q + \{x\})$ the assertion is trivial. Therefore, suppose that $\alpha x \notin Q$ for some α. Now define

$$P_+ = \{q + \alpha x \mid \alpha > 0,\ q \in Q,\ q + \alpha x \notin Q\}$$

and

$$P_- = \{q + \alpha x \mid \alpha < 0,\ q \in Q,\ q + \alpha x \notin Q\}.$$

Then $P_- = -P_+$, and $0 \notin P_+$. It can also be verified that 0 does not belong to the convex hull $[P_+]$ of P_+. Therefore, $0 \notin [P_-]$. 0 can therefore be separated from $[P_+]$, i.e., there exists a vector y in H such that

$$0 < \langle y, p\rangle, \quad \text{for } p \in [P_+].$$

It can be checked that y satisfies Eq. 6. **Q.E.D.**

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