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CONSTANT RESISTANCE ONE-PORTS
WHICH INCLUDE NONLINEAR TIME-VARYING ELEMENTS

by

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ABSTRACT

Methods for generating constant resistance one-ports with nonlinear and time-varying elements are proposed. The first step is a general characterization of constant resistance one-ports with linear time-invariant elements. This characterization is then used to generate classes of one-ports whose elements may be nonlinear and time-varying. Examples are given of constant resistance one-ports that include one-ports which neither need to be current-controlled nor voltage-controlled. In one case, the necessary and sufficient conditions for constant resistance are given.

This paper deals with the following question: is it possible to have a one-port with nonlinear and time-varying characteristics which still exhibits the property of constant resistance. Constant resistance one-ports with linear time-invariant elements have been known for a long time.¹ E. C. Cherry² has exhibited purely resistive one-ports which are constant resistance. We have shown³ that there were constant resistance one-ports which had nonlinear and time-varying reactive elements. Some preliminary results of the present research have been published (without proof) and reported orally.⁴ In the present paper we use the state-space approach to exhibit a general theorem which shows how given any constant resistance one-port it is possible to make the element characteristics time-varying and often nonlinear while still maintaining the property of constant resistance: all the constant resistance

one-ports that we have previously considered are now special cases of this general theorem. It is of interest to point out how deeply the state space approach, recently used successfully in nonlinear circuits,^{5,6,7} is indispensable in order to obtain the results: the concepts of reachability⁸ and unobservability are essential. In the last part, we give three cases in a far more general context: we deal then no longer with nonlinear elements but with nonlinear one-ports which need not be zero-state current-controlled nor zero-state voltage-controlled.

I. Preliminaries

Given a one-port \mathcal{N} it is usually possible to determine a state of the one-port such that if \mathcal{N} is in that state at time t_0 and if \mathcal{N} is either open or short-circuited for $t \geq t_0$, then its port voltage and current are identically zero for $t \geq t_0$. For example, if \mathcal{N} is made of linear elements such a state would correspond to having all its branch voltages and currents set to zero. It is also true that some linear and some nonlinear one-ports may have more than one state with the property above: if such is the case, we assume that one of these states is chosen and henceforth called the zero-state of the one-port.⁹ Throughout the paper we assume that each one-port is z-s (zero-state) determinate under both voltage source and current source drive; by this we mean that each of its branch voltage and currents are uniquely determined by the source waveform (the function e or i), given that the one-port is in its z-s.

when the source is applied.

Suppose a one-port \mathcal{N} has the following properties: (1) it is in the z-s at time t_0 , (2) if it is connected at time t_0 to an arbitrary one-port \mathcal{N}' then its port-voltage and port-current satisfy $v(t) = i(t)$ for $t \geq t_0$. The one-port \mathcal{N} will be said to be constant resistance iff these properties hold for all t_0 . The theory of constant resistance one-port will be greatly simplified by the following

Assertion. Let the one-port \mathcal{N} be z-s determinate and z-s equivalent to a one ohm resistor when it is driven by a voltage source. Under these conditions, if it is connected to an arbitrary one-port \mathcal{N}' then its terminal voltage and current are uniquely determined and the one-port \mathcal{N} is constant resistance.

Proof. Let at time t_0 \mathcal{N} be in the z-s and be connected to \mathcal{N}' as shown on Fig. 1a. We wish to show that $v_0 = i_0$ on $[t_0, \infty)$. Consider all the KVL and all the KCL equations of the circuit. Call \mathcal{V} (\mathcal{C} , resp.) the subset of the KVL (KCL, resp.) equations which include branch voltages (branch-currents, resp.) of \mathcal{N} . Change the equations of \mathcal{V} and \mathcal{C} as follows: (1) in each equation of \mathcal{V} replace the sum of branch voltages of \mathcal{N}' by v_0 (or $-v_0$, as required); (2) in each equation of \mathcal{C} replace the sum of the branch currents of \mathcal{N}' by i_0 (or $-i_0$, as required). This modified set of equations consists of all the KCL and KVL equations of the circuit of Fig. 1b. By assumption, this circuit has a unique solution

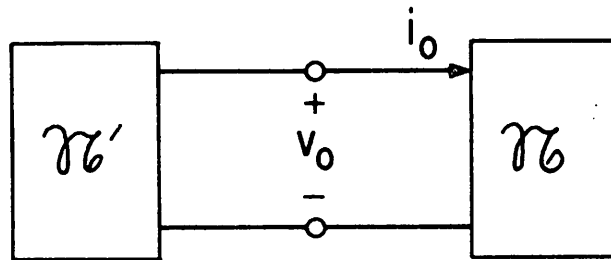


Fig. 1(a). The one-port \mathcal{R} is zero-state equivalent to a 1-ohm resistor when driven by a voltage source as shown in (b).

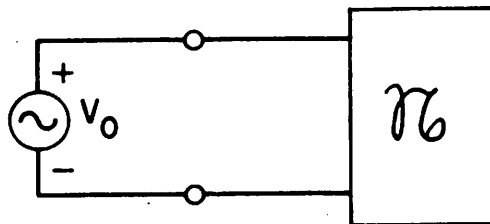


Fig. 1(b). \mathcal{R} is still zero-state equivalent to a 1-ohm resistor when it is connected to an arbitrary one-port \mathcal{R} as shown in (a).

and $v_0 = i_0$ on $[t_0, \infty)$. Hence the same must hold for the circuit of Fig. 1 b.

Thus, from now on, if we show that a one-port is z-s equivalent to a one ohm resistor under voltage source drive, then this one-port has the same property under all one-port terminations, i. e., it is constant resistance.

II. General Method

Our general method for obtaining constant resistance one-ports made of nonlinear time-varying elements is based on a general characterization of such one-ports made of linear time-invariant passive elements.

Characterization. Let \mathcal{N} be a one-port made of linear time-invariant passive RLC elements (inductive coupling is allowed). Let its input be a voltage source $e(\cdot)$ and the current through the source be $i(\cdot)$. Under these conditions, (a) if \mathcal{N} is a constant resistance one-port, then there is a representation of \mathcal{N} of the form

$$\dot{\underline{x}} = \underline{A} \underline{x} + \underline{b} e \tag{1}$$

$$i = \langle \underline{c}, \underline{x} \rangle + e \tag{2}$$

where \underline{A} , \underline{b} , \underline{c} are, respectively, constant $n \times n$, $n \times 1$, $n \times 1$ matrices subject to the conditions that

$$\langle \underline{c}, \underline{A}^k \underline{b} \rangle = 0 \quad \text{for } k = 0, 1, \dots, (n-1); \quad (3)$$

(b) conversely if \mathcal{R} has such a representation satisfying (3), then it is a constant resistance one-port.

Proof. A. We write the state equations following the idea of Bryant.¹⁰ We use the notations of Kuh and Rohrer.⁷ \mathcal{R} is driven by a voltage source e . We pick a tree so that it includes the voltage source, the maximum number of capacitors, then the maximum number of resistors and, finally, the minimum number of inductors. Since \mathcal{R} is constant resistance, there cannot be a capacitor-only path connecting the terminals of the voltage source, for otherwise the driving point impedance would go to zero as $s \rightarrow \infty$. Therefore, in the notation of the Kuh-Rohrer paper, $\underline{e}_s = 0$ and the state equations read (see their (56) and (57)).

$$\begin{pmatrix} \dot{\underline{q}}_C \\ \dot{\underline{\phi}}_L \end{pmatrix} = \begin{pmatrix} -\underline{Y} & \underline{K} \\ -\underline{K}' & -\underline{Z} \end{pmatrix} \begin{pmatrix} \underline{C}^{-1} & \underline{0} \\ \underline{0} & \underline{L}^{-1} \end{pmatrix} \begin{pmatrix} \underline{q}_C \\ \underline{\phi}_L \end{pmatrix} + \begin{pmatrix} \underline{b}_C \\ \underline{b}_L \end{pmatrix} e \quad (4)$$

where the second term in the right hand side of (4) is

$$\begin{pmatrix} \underline{F}'_{RC} & \underline{Q}^{-1} \underline{e}_R \\ -\underline{F}_{LG} & \underline{G}^{-1} \underline{F}'_{RG} \underline{G}_1 \underline{e}_R + \underline{e}_L \end{pmatrix} \quad (5)$$

The passivity of the elements is required in order to guarantee the

invertibility of \underline{Q} and \underline{G} . Let us now establish (2). Consider the fundamental cut-set defined by the voltage source: in view of the way the tree was selected, this cut-set includes only resistive and inductive links, and

$$\underline{i}_R = \underline{Q}^{-1} \underline{e}_R - \underline{Q}^{-1} \underline{F}_{RC} \underline{C}^{-1} \underline{q}_C - \underline{Q}^{-1} \underline{F}_{RG} \underline{R}_2 \underline{F}_{LG}' \underline{L}^{-1} \underline{\phi}_L \quad (6)$$

$$\underline{i}_L = \underline{L}^{-1} \underline{\phi}_L \quad (7)$$

Thus the current through the cut-set is a linear function of the state $(\underline{q}_C, \underline{\phi}_L)$ and of \underline{e}_R , which in our case is a vector whose components are e , $-e$ or 0 since there is only one voltage source in the circuit.

Thus the form of (2) is established; the coefficient of e in (2) is unity because the driving point impedance of \mathcal{N} is one ohm. It remains to prove (3). Let r be the degree of the minimal polynomial of \underline{A} , then¹²

$$e^{\underline{A}t} = \sum_{k=0}^{r-1} \alpha_k(t) \underline{A}^k \quad (8)$$

The zero-state response is then easily computed from (1) and (2):

$$i(t) = \sum_{k=0}^{r-1} \langle \underline{c}, \underline{A}^k \underline{b} \rangle \int_0^t \alpha_k(t-t') e(t') dt' + e(t) \quad (9)$$

For any fixed $t > 0$ and for any $0 < t_1 < t$ there is an input $\hat{e}(\cdot)$

identically zero over $(t_1, t]$ such that the r integrals of (2) are equal to r arbitrarily prescribed numbers $\gamma_0, \gamma_1, \dots, \gamma_{r-1}$: this follows immediately from that the r function $\alpha_k(\cdot)$ are linearly independent over any open interval. Hence for such input $\hat{e}(\cdot)$

$$i(t) = \sum_{k=0}^{r-1} \langle \underline{c}, \underline{A}^k \underline{b} \rangle \gamma_k \quad (10)$$

The constant resistance condition requires that $i(t) = 0$, hence (3) must hold.

B. The converse follows immediately from (9).

Comment: the condition (3) states that the set of reachable states⁸ from the origin lies in the subspace orthogonal to \underline{c} . It is easy to see that if e is considered to be the input and i the response, then any state in that set is equivalent to the zero state.

We turn now to a theorem which will allow us to construct constant resistance one-ports made of nonlinear time-varying elements. Using the notations of the proof above, and if we call \underline{v}_C and \underline{i}_L the tree-branch capacitor voltages and the link inductor currents, then for the one-port of the above theorem we have

$$\begin{pmatrix} \underline{v}_C \\ \underline{i}_L \end{pmatrix} = \begin{pmatrix} \underline{C}^{-1} & 0 \\ 0 & \underline{\mathcal{L}}^{-1} \end{pmatrix} \begin{pmatrix} \underline{q}_C \\ \underline{\phi}_L \end{pmatrix} \quad (11)$$

Suppose we let the capacitors and inductors be nonlinear and time-varying such that

$$\begin{pmatrix} \underline{v}_C \\ \underline{i}_L \end{pmatrix} = \begin{pmatrix} \underline{C}^{-1} & \underline{0} \\ \underline{0} & \underline{L}^{-1} \end{pmatrix} \underline{f}(\underline{x}, t) \quad (12)$$

where $\underline{x} = (\underline{q}_C, \underline{\phi}_L)'$

$$\underline{f}(\underline{0}, t) = \underline{0} \quad \text{for all } t \quad (13)$$

\underline{f} is Lipschitz in \underline{x} and, for each fixed \underline{x} , is a regulated function of t . (14)

The resulting one-port has the representation

$$\dot{\underline{x}} = \underline{A} \underline{f}(\underline{x}, t) + \underline{b} e \quad (15)$$

$$i = \langle \underline{c}, \underline{f}(\underline{x}, t) \rangle + e \quad (16)$$

Theorem. Let \mathcal{R} be the subspace spanned by \underline{b} , $\underline{A} \underline{b}$, \dots , $\underline{A}^{n-1} \underline{b}$.

Let \mathcal{M} be a linear subspace of R^n such that (i) $\mathcal{M} \supset \mathcal{R}$, (ii) $\mathcal{M} \perp \underline{c}$, (iii) $\underline{A} \mathcal{M} \subset \mathcal{M}$. Under these conditions, if, for all t , $\underline{f}(\mathcal{M}, t) \subset \mathcal{M}$, then the one-port is constant resistance.

Proof. The differential equation (15) is equivalent to

$$\underline{x}(t) = \int_{t_0}^t \underline{A} \underline{f}(\underline{x}(t'), t') dt' + \underline{b} \int_{t_0}^t e(t') dt' \quad t \geq t_0 \quad (17)$$

where we assumed $\underline{x}(t_0) = \underline{0}$ since we consider only zero-state responses.

We study (17) by considering the following successive approximation

scheme: let $\underline{\xi}$ be the zero-state solution of (1) to the same input, then

we set

$$\underline{x}_1(t) = \int_{t_0}^t \underline{A} \underline{f}(\underline{\xi}(t'), t') dt' + \underline{b} \int_{t_0}^t e(t') dt' \quad (18)$$

and for $k = 1, 2, \dots$

$$\underline{x}_{k+1}(t) = \int_{t_0}^t \underline{A} \underline{f}(\underline{x}_k(t'), t') dt' + \underline{b} \int_{t_0}^t e(t') dt' \quad (19)$$

The curve $\underline{\xi}(t)$, $t_0 \leq t < \infty$, lies in \mathcal{R} , hence in \mathcal{M} , by (i). Since \underline{f} maps \mathcal{M} into \mathcal{M} , since $\underline{A} \mathcal{M} \subset \mathcal{M}$ and since $\underline{b} \in \mathcal{M}$, it follows that $\underline{x}_1(t)$, $t_0 \leq t < \infty$ lies in \mathcal{M} . The same considerations apply to (19) for each k : all curves \underline{x}_k lie in \mathcal{M} . Since over any finite interval the convergence is uniform and \mathcal{M} is closed, the solution \underline{x} of (15) is in \mathcal{M} . So is $\underline{f}(\underline{x}, (t), t)$, then by (ii) and (16) we get $i(t) = e(t)$ for all $t \geq t_0$.

This theorem calls for the following remarks:

1. \mathcal{R} is the subspace of all states reachable from the origin (in the linear one-port described by (1) and (2)).¹¹
2. Conditions (ii) and (iii) of the theorem imply that \mathcal{M} is included

in the set of all unobservable states (call it \mathcal{U}); indeed \mathcal{U} is the orthogonal complement of the subspace spanned by \underline{c} , $\underline{A}^t \underline{c}$, \dots , $(\underline{A}^t)^{n-1} \underline{c}$.

3. If $\mathcal{R} = \mathcal{U}$, then in the theorem \mathcal{M} must be \mathcal{R} itself.

A Necessary Condition. Let $\tilde{\mathcal{N}}$ be a one-port described by Eqs. (15) and (16) subject to the additional condition that $\underline{f}(\underline{x}, t)$ be continuous in t for each \underline{x} . Then $\tilde{\mathcal{N}}$ is constant resistance if and only if $\langle \underline{c}, \underline{f}(\underline{x}(t), t) \rangle = 0$ for all t and for all z -s responses $\underline{x}(\cdot)$. Let t_0 be arbitrary, $\underline{x}(t_0^-) = \underline{0}$, and $e(t) = \lambda \delta(t - t_0)$, where λ is an arbitrary constant. By (15), $\underline{x}(t_0^+) = \lambda \underline{b}$. For $t > t_0$, put $\underline{x}(t) = \lambda \underline{b} + \underline{\xi}(t)$, then $\underline{\xi}(t_0^+) = \underline{0}$ and

$$\begin{aligned} \dot{\underline{\xi}}(t) &= \underline{A} \underline{f}(\underline{x}(t), t) \\ &= \underline{A} [\underline{f}(\lambda \underline{b}, t) + \underline{f}'(\lambda \underline{b}, t) \underline{\xi}(t) + \dots] \end{aligned}$$

where \underline{f}' denotes the derivative of the mapping $\underline{f}(\cdot, t)$. Solving by iteration, the solution may be written as a series whose k th term is of the order of $(t - t_0)^k$:

$$\begin{aligned} \underline{\xi}(t) &= \int_{t_0}^t \underline{A} \underline{f}(\lambda \underline{b}, t') dt' + \int_{t_0}^t \underline{A} \underline{f}'(\lambda \underline{b}, t') \left[\int_{t_0}^t \underline{A} \underline{f}(\lambda \underline{b}, t'') dt'' \right] dt' \\ &\quad + \dots \end{aligned}$$

Using this result in the constant resistance condition, we get

$$\begin{aligned}
\langle \underline{c}, \underline{f}(\lambda \underline{b} + \underline{\xi}(t), t) \rangle &= \langle \underline{c}, \underline{f}(\lambda \underline{b}, t) \rangle + \langle \underline{c}, \underline{f}'(\lambda \underline{b}, t) \underline{\xi}(t) \rangle + \dots \\
&= \langle \underline{c}, \underline{f}(\lambda \underline{b}, t) \rangle + \int_{t_0}^t \langle \underline{c}, \underline{f}'(\lambda \underline{b}, t) \underline{A} \underline{f}(\lambda \underline{b}, t') \rangle dt' + \dots
\end{aligned}$$

Since the leading terms of this series dominate only if $(t - t_0)$ is small and since \underline{f} is continuous in t , we conclude with the

Corollary. If for some λ and for some t , either $\langle \underline{c}, \underline{f}(\lambda \underline{b}, t) \rangle \neq 0$ or $\langle \underline{c}, \underline{f}'(\lambda \underline{b}, t) \underline{A} \underline{f}(\lambda \underline{b}, t) \rangle \neq 0$, then the one-port $\tilde{\mathcal{N}}$ is not constant resistance.

We turn now to examples of the theorem and of the corollary.

Example 1.⁴ Start with \mathcal{N} , a constant resistance one-port, made of lumped, linear, time-invariant passive RLC elements. Construct $\tilde{\mathcal{N}}$ from \mathcal{N} as follows: let ψ be a real-valued positive continuous function which is bounded away from zero; replace each capacitor C_k of \mathcal{N} by a capacitor $C_k/\psi(t)$ and each set of coupled inductors with inductance matrix \underline{L}_k of \mathcal{N} by a set with $\underline{L}_k/\psi(t)$ as inductance matrix. Clearly referring to (11) and (12) we have

$$\underline{f}(\underline{x}, t) = \underline{x} \psi(t) \quad (20)$$

All the conditions of the theorem are satisfied, hence the one port $\tilde{\mathcal{N}}$

made of linear time-invariant resistors and linear time-varying reactive elements is constant resistance.

Example 2.³ The bridge network \mathcal{N}_2 shown on Fig. 2 is constant resistance since $S_1 = S_2 = \Gamma_3 = \Gamma_4 = 1$. Its equations are

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{\phi}_3 \\ \dot{\phi}_4 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} e \quad (21)$$

$$i = \langle \underline{c}, \underline{x} \rangle + e \quad (22)$$

with

$$\underline{x} = (q_1, q_2, \phi_3, \phi_4)' \quad (23)$$

$$\underline{c} = (-1, -1, 1, 1)' \quad (24)$$

Let f be a real-valued function satisfying the conditions of the theorem.

If the reactive elements have the characteristics

$$v_1 = f(q_1, t) \quad i_3 = f(\phi_3, t) \quad (25)$$

$$v_2 = f(q_2, t) \quad i_4 = f(\phi_4, t) \quad (26)$$

we see that, in the notation of (12) we have

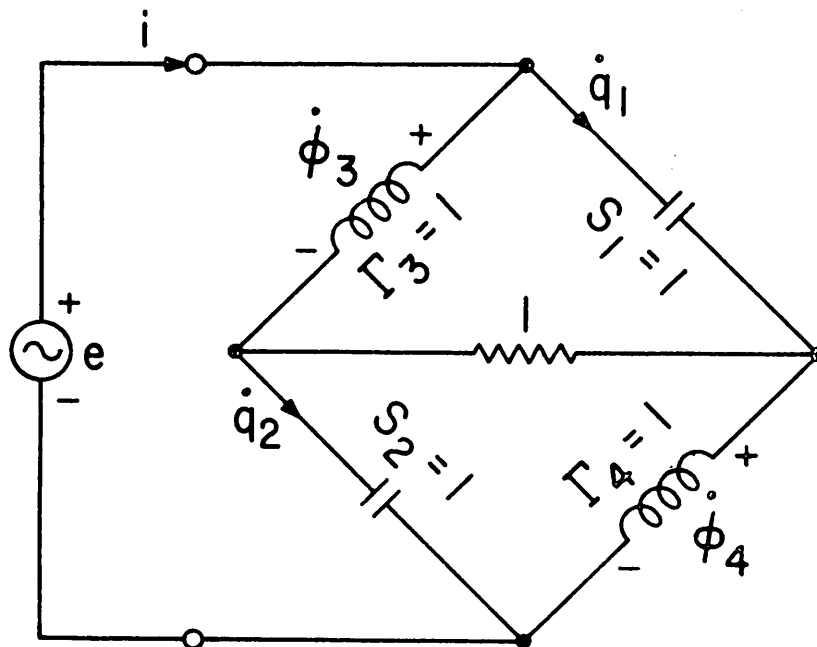


Fig. 2. The one-port is made of linear time-invariant elements and it is constant resistance; yet if all the reactive elements have the same non-linear time-varying characteristic it is still constant resistance.

$$\underline{f}(\underline{x}, t) = \left(f(q_1, t), f(q_2, t), f(\phi_3, t), f(\phi_4, t) \right)' \quad (27)$$

From (21), \mathcal{R} is spanned by $(1, 1, 1, 1)'$. By (27) and with $\mathcal{M} = \mathcal{R}$, we see $\underline{f}(\underline{x}, t)$ defined by (27) satisfies the conditions (i), (ii) and (iii) of the theorem. Hence the one-port $\tilde{\mathcal{N}}_2$, whose topology is that of \mathcal{N}_2 shown on Fig. 2 and whose element characteristics are given by (25), (26), is a constant resistance one-port.

Example 3. Let \mathcal{N}_3 be the ladder shown on Fig. 3. Its equations are

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{\phi}_3 \\ \dot{\phi}_4 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} e \quad (28)$$

and

$$\underline{c} = (-1, 0, 1, 0)' \quad (29)$$

It is easy to check that \mathcal{R} is spanned by $(1, 0, 1, 0)'$ and $(0, 1, 0, 1)'$.

Clearly by (29), the conditions (3) hold and hence \mathcal{N}_3 is constant resistance. Let now $f_1(\cdot, t)$, $f_2(\cdot, t)$ be two real valued functions satisfying the conditions of the theorem and let

$$\underline{f}(\underline{x}, t) = \left(f_1(q_1, t), f_2(q_2, t), f_1(\phi_3, t), f_2(\phi_4, t) \right)'$$

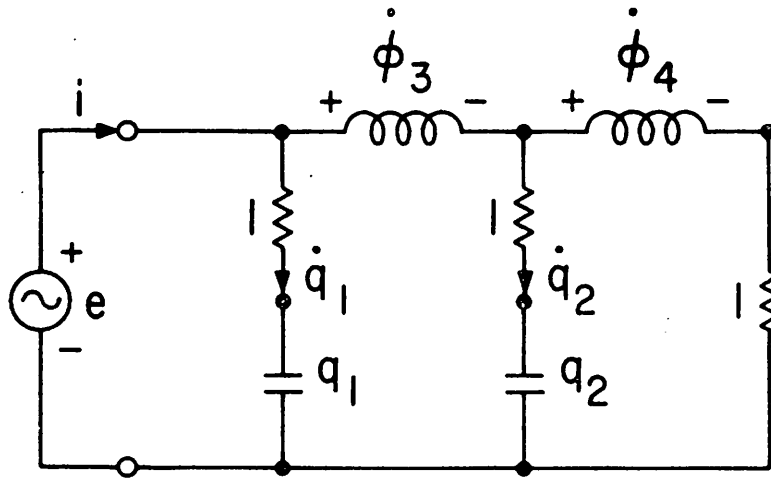


Fig. 3. The one-port shown is made of linear time-invariant elements and it is constant resistance. It is still constant resistance when the reactive elements are nonlinear and time-varying, the first pair of reactive elements having one characteristic and the second pair having another one.

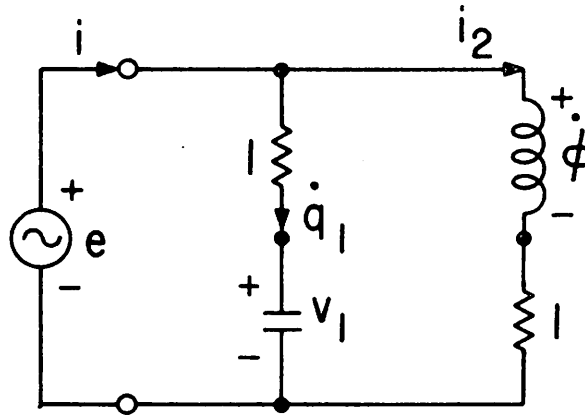


Fig. 4. The one-port shown is constant resistance if and only if the reactive elements have the same characteristic.

With $\mathcal{M} = \mathcal{R}$, the conditions of the theorem are satisfied. Therefore the one-port $\tilde{\mathcal{N}}_3$, whose topology is that of \mathcal{N}_3 shown on Fig. 3 and whose elements have characteristics given by

$$\begin{aligned} v_1 &= f_1(q_1, t) & i_3 &= f_1(\phi_3, t) \\ v_2 &= f_2(q_2, t) & i_4 &= f_2(\phi_4, t), \end{aligned}$$

is a constant resistance one-port.

Example 4.³ The linear time-invariant one-port \mathcal{N}_4 is shown on Fig. 4.

In this case

$$\underline{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \underline{c} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (30)$$

Let f be a scalar valued function satisfying the conditions of the theorem then with $\underline{f}(\underline{x}, t) = \begin{pmatrix} f(q_1, t), f(\phi_2, t) \end{pmatrix}'$, it follows from the theorem that the corresponding one-port $\tilde{\mathcal{N}}_4$ is constant resistance.

The converse is also true. The corollary requires that for all real λ and t , $\langle \underline{c}, \underline{f}(\lambda \underline{b}, t) \rangle = 0$. Since the subspace orthogonal to \underline{c} is spanned by \underline{b} , this condition is equivalent to

$$f_1(\lambda, t) = f_2(\lambda, t) \quad \text{for all } \lambda \text{ and all } t \geq 0.$$

Thus we conclude: the one-port $\tilde{\mathcal{N}}_4$ whose topology is that shown on Fig. 4

is constant resistance if and only if the reactive elements have characteristics

$$v_1 = f(q_1, t) \qquad i_2 = f(\phi_2, t)$$

where f is a scalar valued function satisfying the conditions of the theorem.

III. Special Classes

All the previous examples were concerned with circuits which allowed a detailed description in terms of nonlinear ordinary differential equations. It is of great interest to show that one can also exhibit the property of constant resistance for some one-ports which themselves include very general one-ports whose description is in terms of a relation \mathcal{K} which gives the zero-state response v (or i) in terms of the prescribed input i (or v): thus we write $i = \mathcal{K}(v)$. Here i and v are real valued functions of time and to each v there may be several possible i 's. It is for this reason we called \mathcal{K} a relation rather than a function. Of course if the one-port \mathcal{K} is embedded in a network and the resulting network is zero-state determinate, there will be for each v , only one branch current i through \mathcal{K} . The class of one-ports described by a relation such as \mathcal{K} is very large, in particular, it includes the lumped determinate one-ports with nonlinear and time-varying elements.

Example 1. The one-port \mathcal{N}_1 is made by interconnecting a one-ohm resistor, two identical one-ports N and two identical one-ports N^* as shown by Fig. 5. N^* is the z-s dual of N . \mathcal{N}_1 is assumed to be z-s determinate under voltage source drive. We assert that \mathcal{N}_1 is constant resistance.

Analysis. Call v_k, i_k the z-s branch voltages and the z-s branch currents due to the source e_s . The branch relations are

$$v_1 = \mathcal{K}(i_1) \quad (1a)$$

$$v_3 = \mathcal{K}(i_3) \quad (1b)$$

$$i_2 = \mathcal{K}(v_2) \quad (1c)$$

$$i_4 = \mathcal{K}(v_4) \quad (1d)$$

$$v_5 = i_5 \quad (1e)$$

It should be stressed that in the equations above, the v_k 's and i_k 's denote functions and not the values of the corresponding functions at some time. In (1a) and (1b), \mathcal{K} is not necessarily a function: it states that v_1 is the voltage across N when i_1 flows through N ; it does not assert that N is z-s current-controlled; however, given the network of Fig. 2, given e_s and given that \mathcal{N}_1 is in the zero-state when e_s is applied, then by assumption all the i_k 's and all the v_k 's are uniquely

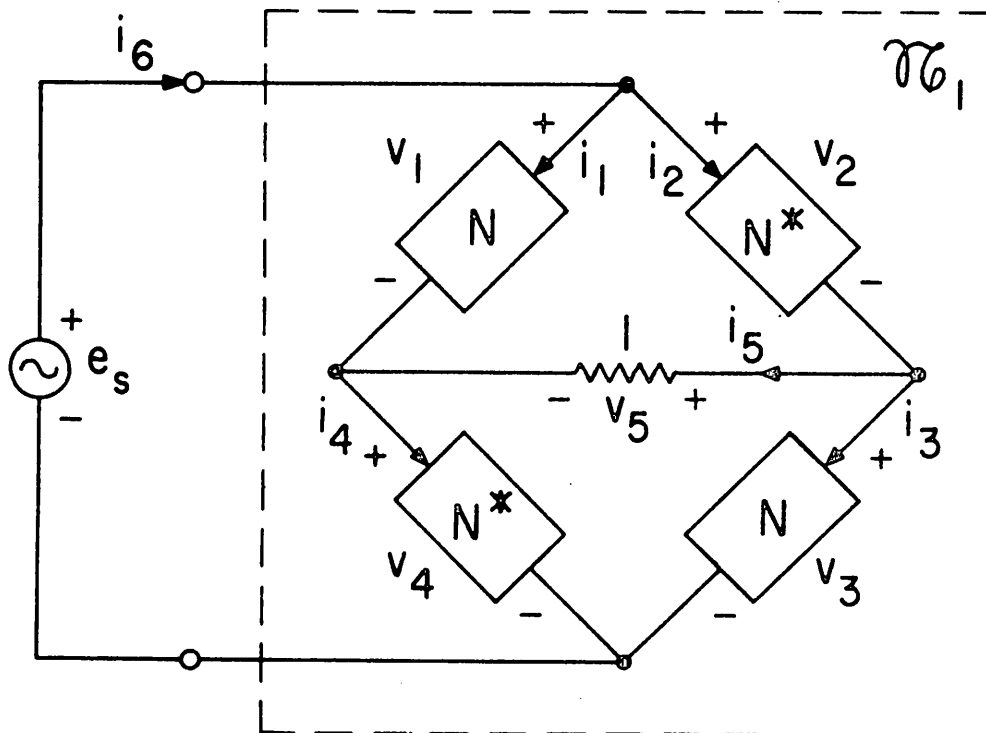


Fig. 5. With N and N^* being dual one-ports, the one-port \mathcal{N}_1 is constant resistance.

specified. (1-c) and (1-d) follow from (1-a) and (1-b) by duality. Pick branches 6, 1 and 3 as a tree for the graph of \mathcal{G}_1 with e_s connected to it. The fundamental loop and cut-set equations are

$$e_s = v_2 + \mathcal{K}(i_3) \quad (2-a)$$

$$e_s = v_4 + \mathcal{K}(i_1) \quad (2-b)$$

$$e_s = \mathcal{K}(i_1) + \mathcal{K}(i_3) - v_5 \quad (2-c)$$

$$i_6 = \mathcal{K}(v_2) + \mathcal{K}(v_4) - v_5 \quad (2-d)$$

$$v_5 = \mathcal{K}(v_4) - i_1 \quad (2-e)$$

$$v_5 = \mathcal{K}(v_2) - i_3 \quad (2-f)$$

Let us repeat (2-a) and (2-b), eliminate v_5 among (2-c), (2-e) and (2-f), and drop (2-d); we thus obtain the system

$$e_s = v_2 + \mathcal{K}(i_3) \quad (3-a)$$

$$e_s = v_4 + \mathcal{K}(i_1) \quad (3-b)$$

$$e_s = i_1 + \mathcal{K}(i_1) + \mathcal{K}(i_3) - \mathcal{K}(v_4) \quad (3-c)$$

$$e_s = i_3 + \mathcal{K}(i_1) + \mathcal{K}(i_3) - \mathcal{K}(v_2) \quad (3-d)$$

Clearly any solution of the set (2-a) to (2-f) is a solution of the set (3-a) to (3-d), conversely any solution of the set (3-a) to (3-d) is a

solution of (2-a) to (2-f) where i_6 is calculated by (2-d). The z-s determinateness of \mathcal{V}_1 implies that the system (3-a) to (3-d) has a unique solution. Let t_0 be the time at which e_s is applied, then we assert that the solution of (2-a) to (2-f) is such that

$$i_1 = v_2 = i_3 = v_4 = \xi \quad (4)$$

where $\xi(\cdot)$ is the unique solution of

$$e_s = \xi + \mathcal{K}(\xi) \quad \text{with} \quad \xi(t_0) = 0 \quad (5)$$

Let us proceed with the verification: $v_2 = v_4$ (from (4)) implies, by (3-a) and (3-b), that

$$\mathcal{K}(i_1) = \mathcal{K}(i_3). \quad (6)$$

This together with $i_1 = i_3$ (from (4)) implies, by (3-c) and (3-d), that

$$\mathcal{K}(v_2) = \mathcal{K}(v_4). \quad (7)$$

From (4), i_1 , v_2 , v_4 and i_3 have the common value ξ but we do not yet know that $\mathcal{K}(i_1) = \mathcal{K}(v_2)$. To show this use (3-a) with (6) and (3-d) with (6)

$$e_s = \xi + \mathcal{K}(i_1)$$

$$e_s = \xi + 2\mathcal{K}(i_1) - \mathcal{K}(v_2)$$

hence

$$\mathcal{K}(i_1) = \mathcal{K}(v_2) \quad (8)$$

Thus (6), (7) and (8) show that (3-a) to (3-d) reduce to

$$\mathcal{K}(i_1) = \mathcal{K}(v_2) = \mathcal{K}(\xi) = e_s - \xi \quad (9)$$

and we have shown that (4) and (5) determine the solution of (2-a) to (2-f) where

$$i_6 = 2\mathcal{K}(\xi) - (\mathcal{K}(\xi) - \xi) = e_s \quad (10)$$

Hence \mathcal{R}_1 is constant resistance.

Example 2. The one-port \mathcal{R}_2 is made by interconnecting two one-ohm resistors and two one-ports N and N^* as shown on Fig. 3. N^* is the z.s. dual one-port of N . \mathcal{R}_2 is assumed to be z-s. determinate under voltage source drive. We assert that \mathcal{R}_2 is constant resistance. We do not give a detailed proof of this fact since it is obtained by a method similar to that of the Example 1.

Example 3. If in any of the one-ports described above we replace one or more of the one-ohm resistors by another such one-port and if we repeat this process any number of times we obtain a one-port that is still constant resistance.

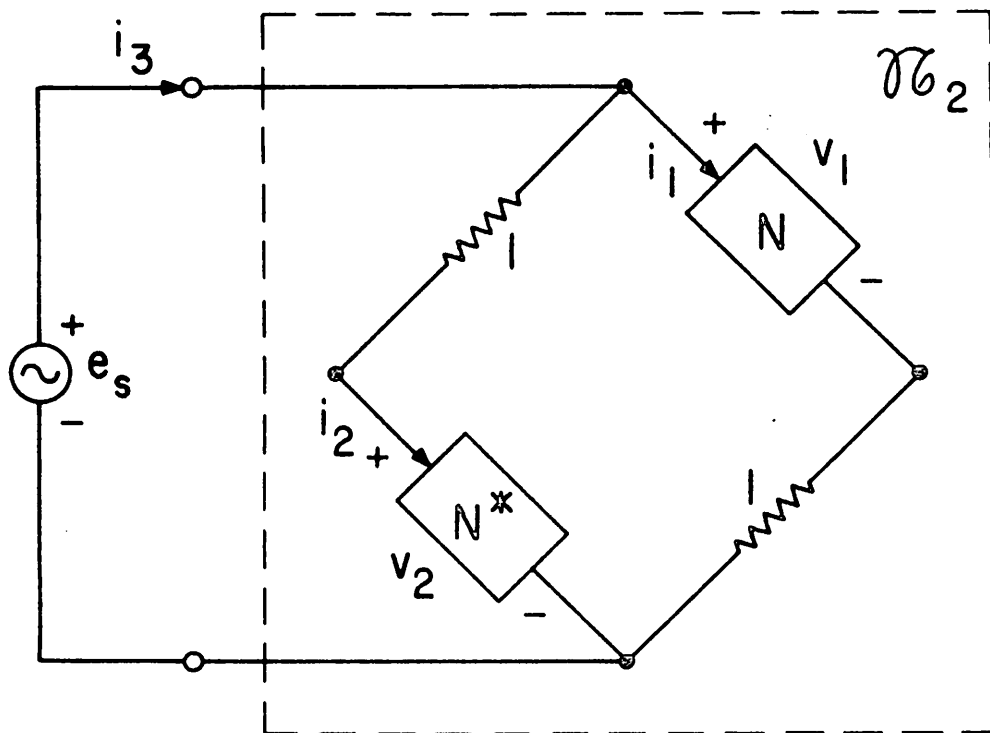


Fig. 6. N and N^* are dual one-ports and \mathcal{R}_2 is constant resistance.

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