PERFORMANCE OF SUBOPTIMUM FEEDBACK FUNCTIONS

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Abstract--The performances of the optimum and several suboptimum feedback functions for sequential binary communication systems with a feedback link are evaluated and compared. It is shown that the power advantage attained by the optimum system can be achieved without considerable loss by the suboptimum feedback functions.

INTRODUCTION

In recent papers [1], [2], Turin formulates the problem of the design of signals for both sequential and nonsequential binary communication systems with a delayless, infinite-bandwidth, forward channel disturbed by additive, white, Gaussian noise, and a delayless, noiseless, feedback link. The transmitter is subject to both peak- and average-power constraints. The receiver continuously informs the transmitter, through the feedback link, of the state of its uncertainty concerning which signal was sent. The transmitter, in turn, uses the output of the feedback link to modify its transmission so as to hasten the receiver's decision. The feedback link is used also, in the sequential case, to synchronize the transmitter when the receiver has reached a decision.

SEQUENTIAL CASE

The system considered is shown in Fig. 1.

At time t_0, the transmitter of the system starts transmitting either the signal s_+ or the signal s_-, which have *a priori* probabilities P_+ and P_-, respectively. A delayless channel adds to the transmitted signal, white, Gaussian noise whose single-sided power density is N_0 watts/cps. On the basis of its observation of the channel output z, the receiver computes, for every $t \geq t_o$, the logarithm of the *a posteriori* probability ratio,

$$y(t) \triangleq \ln \frac{\Pr[s_+/z_t]}{\Pr[s_-/z_t]}, \tag{1}$$

where z_t denotes the sample of signal plus noise observed over the interval $[t_0, t]$. The receiver continues to compute $y(t)$ as long as

$$Y_- < y(t) < Y_+.$$

It makes its decision in favor of s_+ (or s_-) when y first reaches Y_+ (or Y_-). Such a receiver is a so-called sequential detector,

$y(t)$ is a measure of the receiver's uncertainty at time t, and the receiver transmits $y(t)$ back to the transmitter over a delayless, noiseless, feedback link. The transmitter, in turn, uses $y(t)$ in order to control its transmission; that is, $s_{\pm} = s_{\pm}[y(t), t]$.

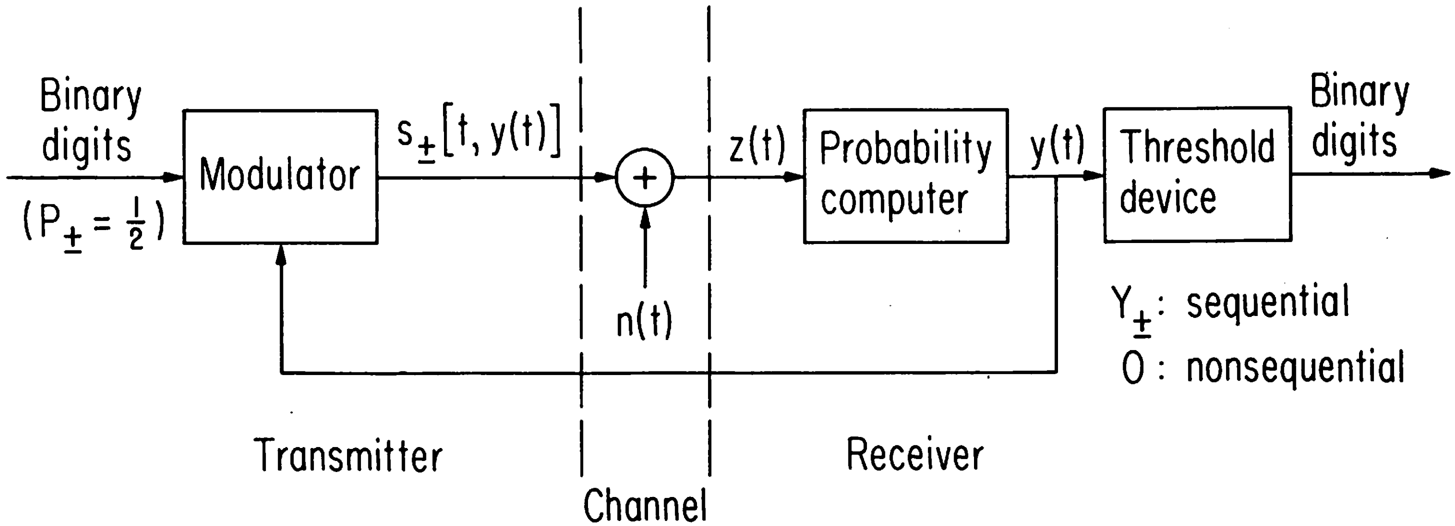

Fig. 1. Feedback communication system.

The following problem is posed: Given upper bounds P_{av} and P_{peak} on the average and peak powers available to the transmitter, and given a prescribed probability $P(e)$ of an error in the receiver's decision, find the signals $s_{\pm}(y, t)$ which minimize the average time $\overline{T}$ that it takes for the receiver to come to a decision.

Under the assumptions[1*]

(a) $s_{\pm}(y, t) = \pm \sigma(t) U_{\pm}(y)$,

(b) $U_{+}(y) + U_{-}(y) = 1$,

(c) $P_{+} = P_{-} = \frac{1}{2}$,

(d) $Y_{+} = -Y_{-} = Y$,

Turin[2] shows that the problem can be simplified as follows: Find $U(y)$ in $\mathcal{U}$ which minimizes

$$T_0(U) \triangleq \max_{\mathcal{U}}[T_1(U), T_2(U)], \tag{2}$$

where

$$\mathcal{U} \triangleq \left\{U(y): U(-y) = 1 - U(y) \text{ for } |y| \leq Y\right\}, \tag{3}$$

$$T_1(U) = \frac{N_0}{2 P_{av}} \int_{-Y}^{Y} \left\{U^2(y) e^{\frac{1}{2}y} + [1 - U(y)]^2 e^{-\frac{1}{2}y}\right\} \times \frac{\sinh \frac{1}{2}(Y - |y|)}{\cosh \frac{Y}{2}} dy, \tag{4}$$

$$T_2(U) = \frac{N_0}{P_{av}} \frac{Y \tanh \frac{Y}{2}}{\alpha} U^2_{max}, \quad \alpha \geq 1, \tag{5}$$

and

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* Footnotes are listed on p. 25.

$$U^2_{max} \triangleq \max_y U^2(y). \tag{6}$$

The symbol $U(y)$ is used for $U_+(y)$. The quantities $T_1(U)$ and $T_2(U)$ are the lower bounds on $\overline{T}$ set by the average-power and the peak-power constraints, respectively.

We note that the assumptions (c) and (d) imply[3]

$$Y = \ln \frac{1 - P(e)}{P(e)}. \tag{7}$$

The optimum feedback functions are:

Case $\alpha = 1$ [4]

$$U(y) = \frac{1}{2} \quad \text{for all } |y| \leq Y, \tag{8}$$

Case $1 < \alpha < \alpha'$ [5]

$$U(y) = \begin{cases} \dfrac{1}{1 + e^{y_m}} & y_m < y \leq Y, \\ \dfrac{1}{1 + e^{y}} & |y| \leq y_m, \\ \dfrac{1}{1 + e^{-y_m}} & -Y \leq y < -y_m, \end{cases} \tag{9}$$

where[6]

$$\alpha \triangleq \frac{Y[e^Y/(e^Y+1)]^2 (e^Y-1)/(e^Y+1)}{(Ye^Y)/(e^Y+1) + \ln[2/(e^Y+1)]}, \tag{10}$$

and

$$y_m \triangleq \ln \frac{\max_y U(y)}{1 - \max_y U(y)}. \tag{11}$$

The quantities $V \triangleq \max_y U(y)$, Y and α are related by[7]

$$\alpha = \frac{V^2 Y(e^Y - 1)}{f(V, Y)}, \tag{12}$$

where

$$f(V, Y) = V(1-V)(1+e^Y) + \left(\ln \frac{V}{1-V}\right)[V^2 - e^Y(1-V)^2]$$
$$- (Y+1)V^2 + (Y-1)(1-V)^2 e^Y + \ln 2(1-V)$$
$$+ e^Y \ln 2V. \tag{13}$$

Case $\alpha > \alpha'$ [8]

$$U(y) = \frac{1}{1 + e^y} \quad \text{for } |y| \leq Y. \tag{14}$$

A typical optimum feedback function is shown in Fig. 2 (with $P(e) = 10^{-6}$ and $\alpha = 10$).

NONSEQUENTIAL CASE

The system considered is similar to that of the sequential case except for the receiver, which now acts as a nonsequential detector. The receiver makes its decision at time $t_0 + T$ according to whether $y(t_0+T) - y(t_0)$ is positive (decision s_+) or negative (decision s_-), where T is a constant.

As in the sequential case, the following problem is posed: Given upper bounds P_{av} and P_{peak} on the average and peak powers available to the transmitter, and given a prescribed probability $P(e)$ of an error in the receiver's decision, find the signals $s_{\pm}(y, t)$ which minimize the

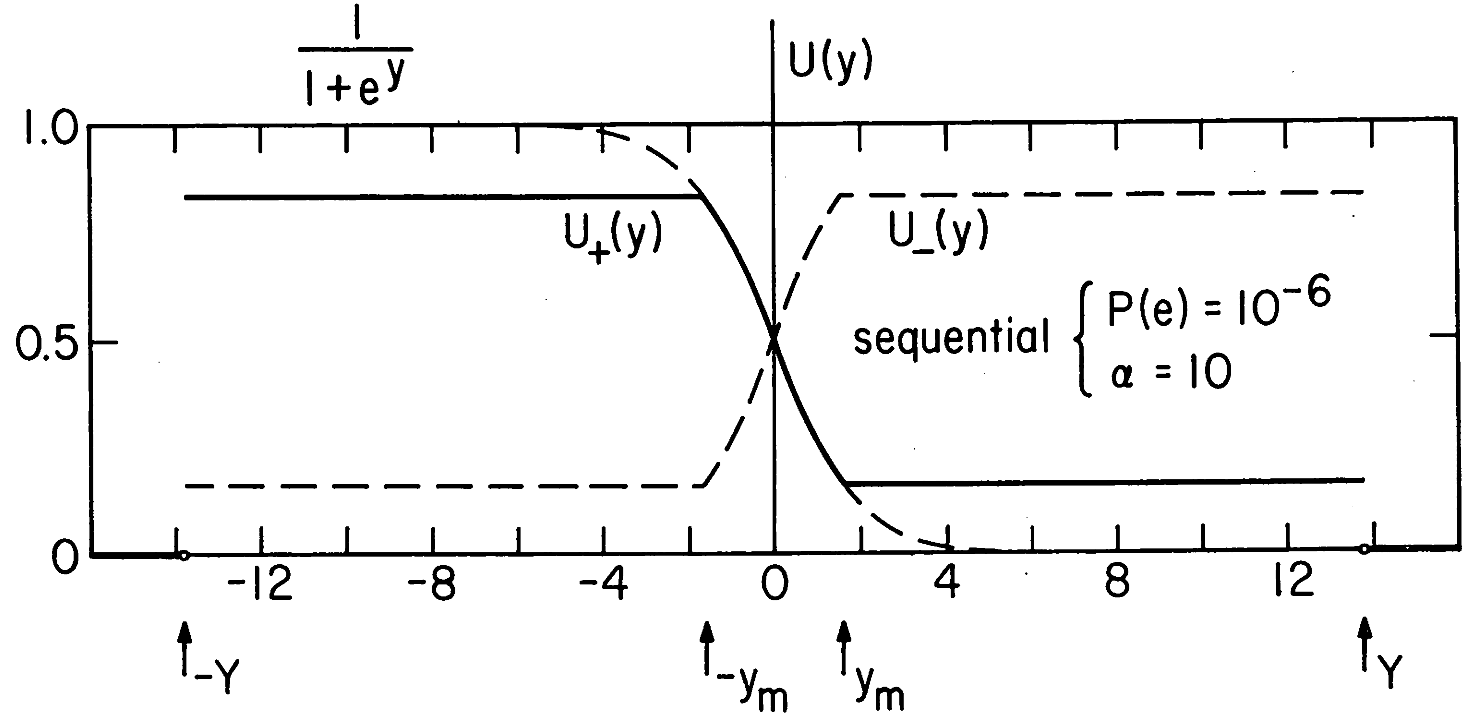

Fig. 2. A typical feedback function.

time T, the duration of a binary transmission.

Assumptions (a), (b), and (c), which were for the sequential case, are considered to hold.

The problem can be stated as follows:[9] Find U(y) in $\hat{\mathcal{U}}$ which minimizes

$$\hat{T}_0(U) \triangleq \max_{\hat{\mathcal{U}}} [\hat{T}_1(U), \hat{T}_2(U)], \tag{15}$$

where

$$\hat{\mathcal{U}} = \{U(y): U(-y) = 1 - U(y) \text{ for all } y\}, \tag{16}$$

$$\hat{T}_1(U) = \frac{N_0}{2 P_{av}} \int_{-\infty}^{\infty} \{U^2(y)\hat{Q}_+(y) + [1-U(y)]^2 \hat{Q}_-(y)\} \, dy, \tag{17}$$

$$\hat{T}_2(U) = \frac{N_0}{P_{av}} \frac{\tilde{R}}{\alpha} U^2_{max}, \tag{18}$$

$$\hat{Q}_{\pm}(y) = \frac{e^{\pm y/2}}{\sqrt{\pi}} \int_0^{\sqrt{\tilde{R}}} \exp\left[-\frac{1}{4}(\tau^2 + y^2/\tau^2)\right] d\tau \tag{19}$$

and

$$\tilde{R} = \frac{1}{N_0} \int_{t_0}^{t_0+T} \sigma^2(\tau) d\tau.$$

The quantities $\tilde{R}$ and P(e) are related by

$$P(e) = \frac{1}{2}\left[1 - \operatorname{erf}(\sqrt{\tilde{R}}/2)\right], \tag{20}$$

and U^2_{max} is given by (6).[10] $\hat{T}_1(U)$ and $\hat{T}_2(U)$ are the lower bounds on T set by the average-power and peak-power constraints, respectively.

The optimum feedback functions are:

<u>Case</u> $\alpha = 1$ [11]

$$U(y) = \frac{1}{2} \text{ for all } y. \tag{21}$$

<u>Case</u> $1 < \alpha < \hat{\alpha}'$ [12]

$$U(y) = \begin{cases} \dfrac{1}{1+e^{\hat{y}_m}} & y \geq \hat{y}_m, \\ \dfrac{1}{1+e^{y}} & |y| < \hat{y}_m, \\ \dfrac{1}{1+e^{-\hat{y}_m}} & y \leq -\hat{y}_m, \end{cases} \tag{22}$$

where[13]

$$\hat{\alpha}' = \frac{[e^y/(1+e^y)]^2 \tilde{R}}{\frac{1}{2}\int_{-\infty}^{\infty}\left\{\left(\frac{1}{1+e^y}\right)^2 \hat{Q}_+(y) + \left(\frac{e^y}{1+e^y}\right)^2 \hat{Q}_-(y)\right\} dy}, \tag{23}$$

and

$$\hat{y}_m \triangleq \ln \frac{\sup_y U(y)}{1 - \sup_y U(y)}. \tag{24}$$

<u>Case</u> $\alpha \geq \hat{\alpha}'$ [14]

$$U(y) = \frac{1}{1+e^y} \text{ for all } y. \tag{25}$$

PERFORMANCE OF THE OPTIMUM FEEDBACK FUNCTIONS

We evaluate the performance of the optimum feedback functions by considering relationship of the error probability to the normalized transmission rate,

$$\eta \triangleq \frac{R_{max}}{C}, \tag{26}$$

where C is the channel capacity of the system and R_{max} is the maximum transmission rate.

Some of these evaluations are similar to Horstein's but since we present them in different forms and for the sake of completeness, we include them here.

The channel capacity is given by[15]

$$C = \frac{P_{av}}{N_0 \ln 2} \qquad \text{(bits/sec)}, \tag{27}$$

and the maximum transmission rate R_{max} is given, for the sequential case, by

$$R_{max} = \frac{1}{\min_{\mathcal{U}} T_0(U)},$$

or, for the nonsequential case, by

$$R_{max} = \frac{1}{\min_{\hat{\mathcal{U}}} \hat{T}_0(U)}.$$

Therefore,

$$\eta = \frac{N_0 \ln 2}{P_{av} \min_{\mathcal{U}} T_0(U)}, \tag{28a}$$

for the sequential case, and

$$\eta = \frac{N_0 \ln 2}{P_{av} \min_{\hat{\mathcal{U}}} \hat{T}_0(U)}, \tag{28b}$$

for the nonsequential case.

First, we consider the sequential case.

<u>Case</u> $\alpha = 1$

Since, when $\alpha = 1$, $T_1 \leq T_2$, for all U, we have $T_0 = T_2$. Therefore, from (5), (6), and (8),

$$\min_{\mathcal{U}} T_0(U) = \frac{N_0}{P_{av}} \frac{Y \tanh Y/2}{4}. \tag{29}$$

Substituting (29) into (28a), and manipulating the result, we get

$$\eta = \frac{4}{G(P(e)) - H(P(e))} \tag{30}$$

where

$$G(x) \triangleq -x \log_2(1-x) - (1-x)\log_2 x \quad \text{for} \quad 0 < x < 1, \tag{31}$$

and

$$H(x) \triangleq -x \log_2 x - (1-x)\log_2(1-x) \quad \text{for} \quad 0 < x < 1. \tag{32}$$

<u>Case</u> $1 < \alpha < \alpha'$

It can be shown[16] that, when $1 < \alpha < \alpha'$, the minimum of T_0 occurs at U such that $T_1(U) = T_2(U)$, and that (9) satisfies the equation. Therefore, from (5) and (9),

$$\min_{\mathcal{U}} T_0(U) = \frac{N_0}{P_{av}} \frac{Y \tanh Y/2}{\alpha} U_{max}^2. \tag{33}$$

Substituting (33) into (28a), and manipulating the result, we get

$$\eta = \frac{1}{G(P(e)) - H(P(e))} \frac{1}{U_{max}^2}, \tag{34}$$

where P(e) and U_{max} are related by (12) with α as a parameter.

Case $\alpha \geq \alpha'$

Since, when $\alpha \geq \alpha'$, $T_1 \geq T_2$[17] for all U, we have $T_0 = T_1$. Therefore, min $T_0(U)$ does not depend upon $T_2(U)$, i.e., α and the performance of the system is the same for any $\alpha \geq \alpha'$. From (4) and (14),

$$\min T_0(U) = \frac{N_0}{P_{av}} \left[\frac{Ye^Y}{e^Y + 1} + \ln \frac{2}{1 + e^Y} \right]. \tag{35}$$

Substituting (35) into (28a), and manipulating the result, we get

$$\eta = \frac{1}{1 - H(P(e))}. \tag{36}$$

Next, we consider the nonsequential case.

Case $\alpha = 1$

Since, when $\alpha = 1$, $\hat{T}_1 \leq \hat{T}_2$ for all U, we have $T_0 = T_2$. Therefore, from (10) and (21),

$$\min_{\hat{\mathcal{U}}} \hat{T}_0(U) = \frac{N_0}{P_{av}} \frac{\tilde{R}}{4}. \tag{37}$$

Substituting (37) into (28b), we have

$$\eta = \frac{4 \ln 2}{\tilde{R}} \tag{38}$$

Case $1 < \alpha < \hat{\alpha}'$

It can be shown[18] that, when $1 < \alpha < \hat{\alpha}'$, the minimum of $\hat{T}_0$ occurs at U such that $\hat{T}_1(U) = \hat{T}_2(U)$, and that (22) satisfies the equation. Therefore, from (17),

$$\min_{\hat{\mathcal{U}}} \hat{T}_0(U) = \frac{\bar{S}_{min}(U_{max})}{P_{av}}, \tag{39}$$

where

$$\bar{S}_{min}(U_{max}) = \frac{N_0}{2}\int_{-\infty}^{\infty}\{U^2(y)\hat{Q}_+(y) + [1 - U(y)]^2\hat{Q}_-(y)\}\,dy, \tag{40}$$

U(y) being given by (22). Substituting (39) into (28b), we have

$$\eta = \frac{2 \ln 2}{2\,\bar{S}_{min}(U_{max})/N_0}. \tag{41}$$

The values of the denominator are given in Fig. 5 of Horstein [3].

Case $\alpha \geq \hat{\alpha}'$

Since, when $\alpha \geq \hat{\alpha}'$, $\hat{T}_1(U) \geq \hat{T}_2(U)$ for all U, $\min T_0(U)$ does not depend upon $\hat{T}_2(U)$, i.e., α. Therefore, the performance of the system is the same for any $\alpha \geq \hat{\alpha}'$. The quantity η is obtained by putting $U_{max} = 1$ in (41). Notice that, when $\alpha > \hat{\alpha}'$, U(y) given by (25) minimizes $\hat{T}_1(U)$, i.e., $\hat{T}_0(U)$.

Fig. 3 shows the P(e) vs η characteristic with α as a parameter when the optimum feedback functions are used. These curves can be considered as lower bounds on the error probability that can be

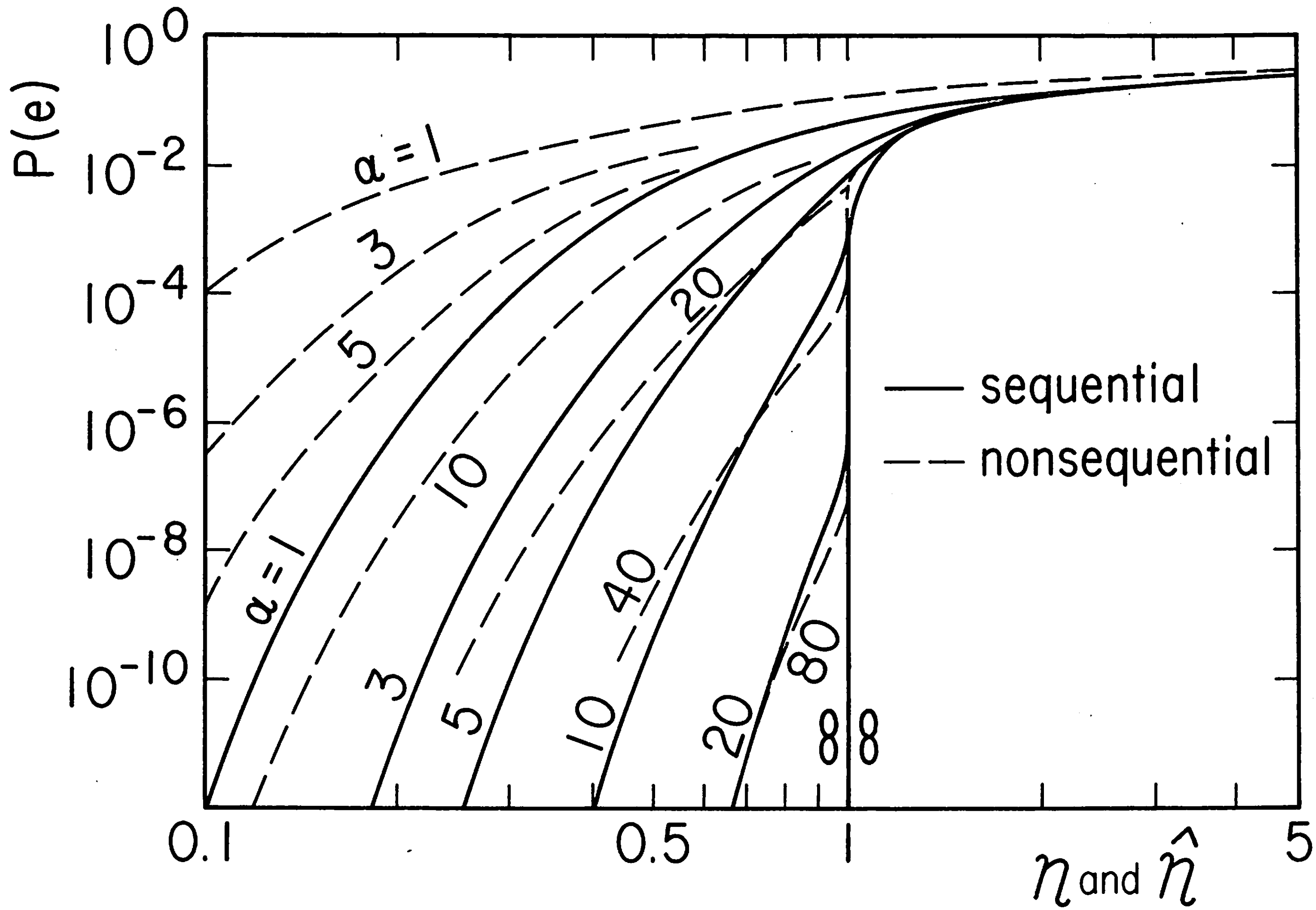

Fig. 3. P(e) vs. η (Optimum feedback functions in $\mathcal{U}$ and $\hat{\mathcal{U}}$).

achieved with feedback systems of the type considered. The lower bounds are achieved with the optimum feedback functions.

PERFORMANCE OF SUBOPTIMUM FEEDBACK FUNCTIONS

We have seen that the optimum feedback functions have, in general, the form $1/(1+e^y)$. Since it is very difficult to realize the optimum functions in practical systems, it is useful to evaluate the performance of some feedback functions which are more easily implemented. The evaluation may give us some idea about the degredation of the performance of the systems caused by using nonoptimum feedback functions.

Since the expression for $\hat{T}_1(U)$ is rather involved, we restrict ourselves to the sequential case.

First, we consider the following class of feedback functions:

$$\mathcal{U}_K \triangleq \left[U(y): U(y) \triangleq \begin{cases} \frac{1}{2} - K, & 0 < y \leq Y \\ \frac{1}{2}, & y = 0 \\ \frac{1}{2} + K, & -Y \leq y < 0 \\ 0 & \text{elsewhere} \end{cases} \quad \text{where } 0 \leq K \leq \frac{1}{2}. \right]$$

We compute the value K for which $T_0(U)$ is minimized for the given α and $P(e)$, i.e., Y. Notice that since $\mathcal{U}_K$ includes $U(Y) = 1/2$, the expression for η for $\alpha = 1$ is given by (30).

Substitution of $U(y)$ of (42) into (4) and (5) yields

$$T_1(U) = \frac{N_0}{P_{av}} \frac{1}{e^Y+1}\left\{\left(\frac{1}{2}+K\right)^2 (e^Y-Y-1) + \left(\frac{1}{2}-K\right)^2\left[(Y-1)e^Y+1\right]\right\}, \qquad (43)$$

and

$$T_2(U) = \frac{N_0}{P_{av}} \frac{Y(e^Y-1)}{\alpha(e^Y+1)}\left(\frac{1}{2}+K\right)^2, \qquad (44)$$

respectively.

If we restrict ourselves to the feedback function

$$U_s(y) \triangleq \frac{1}{2}[1 - \operatorname{sgn} y], \qquad (45)$$

we get, by putting $K = 1/2$ in (43) and (44),

$$T_1(U) = \frac{N_0}{P_{av}} \frac{e^Y - Y - 1}{e^Y + 1}, \qquad (46)$$

and

$$T_2(U) = \frac{N_0}{P_{av}} \frac{Y(e^Y - 1)}{e^Y + 1}. \qquad (47)$$

By computation, we have

$$T_1(U) \gtreqless T_2(U) \qquad \text{for } \alpha \gtreqless \alpha_s', \qquad (48)$$

where

$$\alpha_s' \triangleq \frac{Y(e^Y-1)}{e^Y - Y - 1}.$$

Therefore, we have, from (28a) and (46),

$$\eta = (\ln 2)\,\frac{e^Y + 1}{e^Y - Y - 1}$$

$$= \frac{\ln 2}{1 - 2P(e) - P(e)\ln\frac{1 - P(e)}{P(e)}} \quad \text{for } \alpha \geq \alpha_s' \qquad (49)$$

and from (28a) and (47)

$$\eta = (\alpha \ln 2)\,\frac{e^Y + 1}{Y(e^Y - 1)} = \frac{\alpha \ln 2}{[1 - 2P(e)]\ln\frac{1 - P(e)}{P(e)}} \quad \text{for } \alpha < \alpha_s'. \qquad (50)$$

Now we again consider $\mathcal{U}_K$. By computation, we see that $T_1(U)$ takes on a minimum value for $K = K_0$,

$$K_0 = \frac{1}{2} - \frac{e^Y - Y - 1}{Y(e^Y - 1)}. \qquad (51)$$

Notice that $0 < K_0 < 1/2$ for $Y > 0$. Since both $T_1(U)$ and $T_2(U)$ are quadratic and convex with respect to K and $T_2(U)$ is monotonically increasing for $K \geq 0$, if

$$T_1(U)\Big|_{K=K_0} > T_2(U)\Big|_{K=K_0}, \qquad (52)$$

then

$$\min_{\mathcal{U}_K} T_0(U) = T_1(U)\Big|_{K=K_0}. \qquad (53)$$

If (52) does not hold, then the value of K which minimizes $T_0(U)$ must satisfy $T_1(U) = T_2(U)$. It is clear that the smaller value of K, say K_1, among the two which satisfy $T_1(U) = T_2(U)$, minimizes $T_0(U)$.

Then

$$\min_{\mathcal{U}_K} T_0(U) = T_1(U)\Big|_{K=K_1} = T_2(U)\Big|_{K=K_1}. \tag{54}$$

From (43) and (51), we get

$$T_1(U)\Big|_{K=K_0} = \frac{N_0}{P_{av}} \frac{(Y-1)e^{2Y} - (Y^2-2)e^Y - (Y+1)}{Y(e^Y+1)(e^Y-1)}. \tag{55}$$

Inequality (52) together with (44), (51), and (55) imply that K_0 is optimum when

$$\alpha \geq \frac{[(Y-1)^2 e^Y + 1]^2}{(Y-1)e^{2Y} - (Y^2-2)e^Y - (Y+1)} \triangleq \alpha_k'. \tag{56}$$

K_1 is obtained from (43) and (44) as

$$K_1 = \frac{B - \sqrt{B^2 - A^2}}{2A}, \tag{57}$$

where

$$A \triangleq \left(1 - \frac{1}{\alpha}\right) Y(e^Y-1),$$

and

$$B \triangleq Y(e^Y+1) + \left(2 - \frac{Y}{\alpha}\right)(e^Y-1),$$

and

$$T_2(U)\Big|_{K=K_1} = \frac{N_0}{P_{av}} \frac{Y(e^Y-1)}{\alpha(e^Y+1)} \left[\frac{B + A - \sqrt{B^2 - A^2}}{2A}\right]^2. \tag{58}$$

The discussion above may be summarized by the expression

$$\eta = \begin{cases} \dfrac{4}{G(P(e)) - H(P(e))} & \text{for } \alpha = 1, \\ \dfrac{N_0 \ln 2}{P_{av} T_2(U)\big|_{K=K_1}} & \text{for } 1 < \alpha < \alpha_k', \\ \dfrac{N_0 \ln 2}{P_{av} T_1(U)\big|_{K=K_0}} & \text{for } \alpha \geq \alpha_k'. \end{cases} \tag{59}$$

Next, we consider the following class of functions:

$$\mathcal{U}_c(a) \triangleq \left[U(y): U(y) \triangleq \begin{cases} ca + \frac{1}{2}, & a \leq y \leq Y \\ cy + \frac{1}{2}, & -a < y < a \\ -ca + \frac{1}{2}, & -Y \leq y \leq -a \\ 0 & \text{elsewhere} \end{cases} \quad \text{where } -\frac{1}{2a} \leq c < 0 \text{ and } a \text{ is a fixed positive value.} \right] \tag{60}$$

We denote by $\mathcal{U}_c$ the collection of $\mathcal{U}_c(a)$ for all positive values of a.

Since $\mathcal{U}_c$ contains $\mathcal{U}_K$, it is clear that the performance of $\mathcal{U}_c$ is not worse than that of $\mathcal{U}_K$, and since $\mathcal{U}_c$ is contained in $\mathcal{U}$, the performance of $\mathcal{U}_c$ is not better than that of $\mathcal{U}$.

Substituting U(y) of (60) into (4) and (5), we get

$$T_1(U) = \frac{N_0}{P_{av} \cosh(Y/2)} (Ac^2 + Bc + C), \tag{61}$$

where

$$A = \frac{a^2}{3}(3Y - 2a)\sinh\frac{Y}{2} - (a^2 - 2)\cosh\frac{Y}{2}$$

$$- 2a \sinh\left(\frac{Y}{2} - a\right) - 2\cosh\left(\frac{Y}{2} - a\right),$$

$$B = -a \sinh\frac{Y}{2} - \left(\frac{a^2}{2} - aY + 1\right)\cosh\frac{Y}{2} + \cosh\left(\frac{Y}{2} - a\right),$$

and

$$C = \frac{Y}{4}\sinh\frac{Y}{2},$$

and

$$T_2(U) = \frac{N_0}{P_{av}} \frac{Y \tanh(Y/2)}{\alpha} \left(\frac{1}{2} - ca\right)^2, \tag{62}$$

respectively.

<u>Case</u> $\alpha = 1$

Since the expressions for η for $\mathcal{U}$ and for $\mathcal{U}_K$ coincide, the same expression must hold for $\mathcal{U}_c$. In fact, letting $a \to 0$ in (60) gives (8).

<u>Case</u> $\alpha = \infty$

Since, when $\alpha = \infty$, $T_2(U) = 0$, the minimum value of $T_0(U)$ is equal to that of $T_1(U)$. It can be shown that $\mathcal{U}_c(a)$ is a convex function space and $T_1(U)$ is convex[19] on $\mathcal{U}_c(a)$. Therefore, for any given a and Y, A must be positive. Then, it is easily seen that

$$\min_{\mathcal{U}_c(a)} T_1(U) = k \frac{N_0}{P_{av}} \frac{Y(e^Y - 1)}{4(e^Y + 1)}, \tag{63}$$

where

$$k = \begin{cases} 1 - \dfrac{B^2}{4AC} & \text{if } -\dfrac{B}{2A} < 0, \\[2ex] 1 & \text{if } -\dfrac{B}{2A} \geq 0. \end{cases} \tag{64}$$

Since

$$\min_{\mathcal{U}_c} T_0(U) = \min_a \left[\min_{\mathcal{U}_c(a)} T_1(U) \right]$$

$$= \min_a k \frac{N_0}{P_{av}} \frac{Y(e^Y - 1)}{4(e^Y + 1)}, \tag{65}$$

we get from (28a),

$$\eta = \frac{(4 \ln 2)(e^Y + 1)}{\left(\min_a k\right) Y(e^Y - 1)}, \tag{66}$$

where k is given by (64).

<u>Case</u> $1 < \alpha < \infty$

It is easy to see from (62) that for any $a > 0$, $Y > 0$, and $\alpha \geq 1$, $T_2(U)$ is quadratic and convex with respect to c, and that $c = 1/2a > 0$ gives the minimum value of $T_2(U)$. Therefore, in the range c which we are interested in, i.e., $c < 0$, $T_2(U)$ decreases monotonically as c increases.

Moreover, for any finite a,

$$T_1(U)\Big|_{c=0} \triangleq \lim_{c \to 0} T_1(U) = \frac{N_0}{P_{av}} \frac{Y \tanh(Y/2)}{4}$$

and

$$T_2(U)\Big|_{c=0} \triangleq \lim_{c\to 0} T_2(U) = \frac{N_0}{P_{av}} \frac{Y \tanh(Y/2)}{4\alpha}.$$

Therefore,

$$T_1(U)\Big|_{c=0} > T_2(U)\Big|_{c=0}.$$

Then, if $T_1(U) = T_2(U)$ has valid solutions for c, at least one of them gives $\min_{\mathcal{U}_c(a)} T_0(U)$, and $\min_{\mathcal{U}_c} T_0(U)$ is given by

$$\min_{a} \min_{\mathcal{U}_c(a)} T_0(U).$$

If $T_1(U) = T_2(U)$ does not have any valid solution, $T_1(U) > T_2(U)$ for any value of c. Then $\min_{\mathcal{U}_c} T_0(U)$ is given by (65).

For both cases, η is obtained by (28a).

The relationship between the error probability, $P(e)$, and the normalized transmission rate, η, for the best functions within $\mathcal{U}$, $\mathcal{U}_K$ and $\mathcal{U}_c$ (for each α and $P(e)$), and also for the function U_s, is illustrated in Fig. 4.

Since the curves for $\mathcal{U}_K$ and $\mathcal{U}$ coincide well for $\alpha < 20$ and since the performance of the best function in $\mathcal{U}_c$ lies between the performances of the best functions in $\mathcal{U}_K$ and $\mathcal{U}$, curves for $\mathcal{U}_c$ for $\alpha = 3$, 5 and 10 are disregarded in the illustration.

DISCUSSION

It can be seen from Fig. 4 that the best feedback function in $\mathcal{U}_c$ gives an excellent performance for all the possible combinations

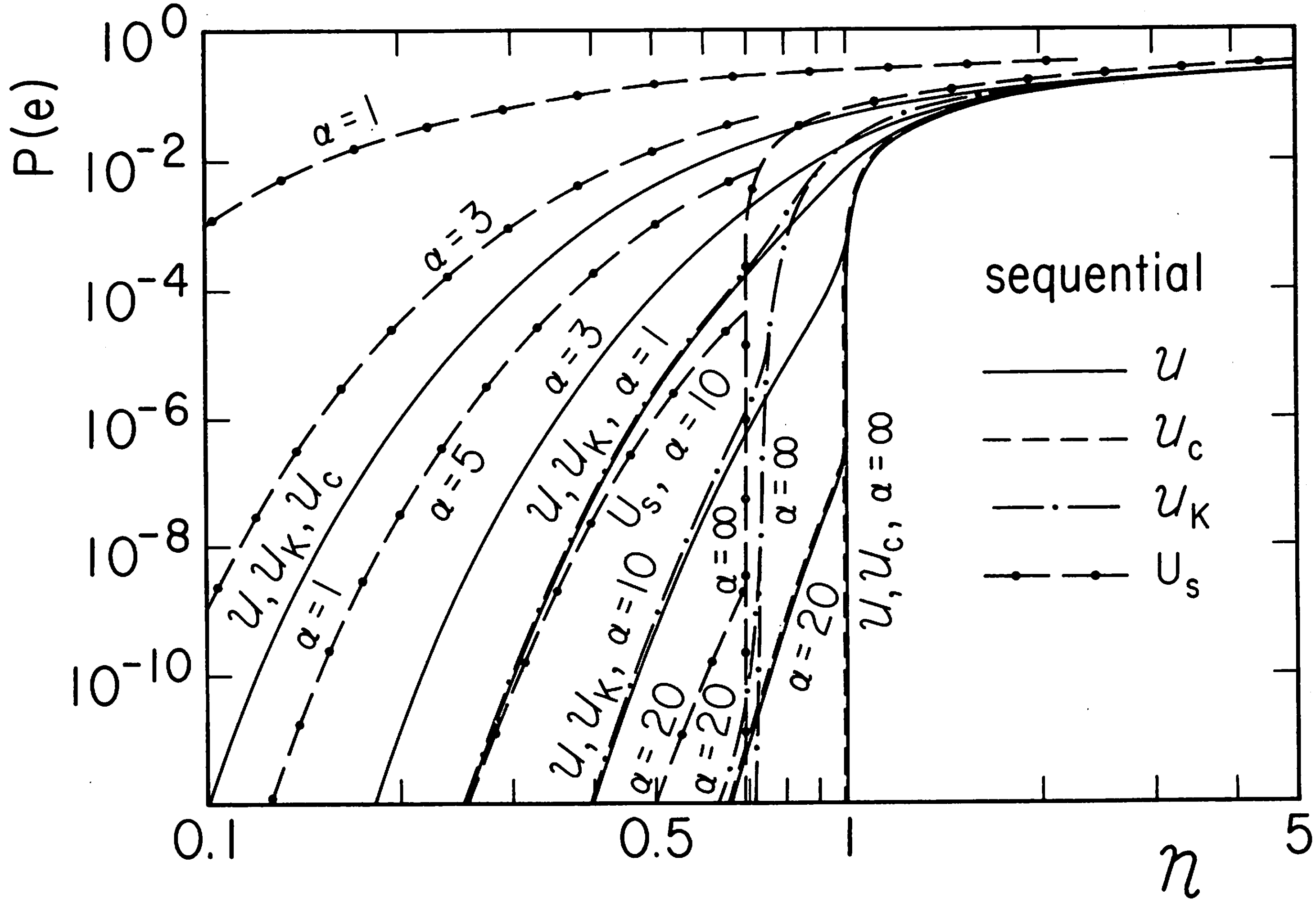

Fig. 4. P(e) vs. η (Best feedback functions in $\mathcal{U}$, $\mathcal{U}_c$, $\mathcal{U}_K$ and U_s).

of P(e) and α, and that the best feedback function in $\mathcal{U}_K$ also gives a good performance for the values of P(e) and α of practical interest, say $P(e) \leq 10^{-5}$ and $\alpha \leq 10$.

Note that when $\alpha < \infty$, it is impossible to achieve $P(e) = 0$ with any feedback function. However, if $\alpha = \infty$, we have $P(e) = 0$ with (see (36), (49), (59), (66))

$$\eta = 1 \text{ for the best function of } \mathcal{U}, \quad U(y) = \frac{1}{1+e^y},$$

$$\eta = \ln 2 \text{ for } U_s, \quad U(y) = \frac{1}{2}[1 - \operatorname{sgn} y],$$

$$\eta = \ln 2 \text{ for } \mathcal{U}_K, \quad U(y) = \frac{1}{2}[1 - \operatorname{sgn} y],$$

and

$$\eta = \eta_0 \text{ for the best function of } \mathcal{U}_c,$$

$$U(y) = \begin{cases} 0, & 2.55 \leq y \\ -0.392y + \frac{1}{2}, & |y| < 2.55 \\ 1, & y \leq -2.55, \end{cases}$$

where η_0 lies between 0.977 and 1.

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FOOTNOTES

1 Turin has formulated the problem in more general terms.

2 (40) and (41), [1].

3 (6), [1].

4 See p. 405, [1].

5 (27), [3]. Notice that (27) is valid for any $\alpha \geq 1$. It is mainly for convenience of discussion that we consider the cases separately.

6 (33), [3]. Also see [4]. Equation 10 is obtained by substituting (14) into (4) and (5), and equating the results. For further discussions, see pp. 13-14, [5].

7 Equation 12 is obtained by substituting (9) into (4) and (5), and equating the results.

8 (48a), [1]. Turin has shown that (14) is optimum if $\alpha \geq \alpha' \cong -\ln P(e)/\ln 2$, the approximation holding for $\alpha' >> 1$.

9 (15) and (17), [2].

10 In (6), $\max_y U^2(y)$ is to be interpreted as $\sup_y U^2(y)$ whenever it is necessary to do so.

11 See p. 9, [2].

12 (44), [3]. Notice that (44) is valid for any $\alpha \geq 1$. It is mainly for convenience of discussion that we consider the cases separately.

13 See pp. 13-14, [5], for a similar discussion.

14 (22), [2]. Turin has shown that (25) is optimum if $\alpha \geq \hat{\alpha}'$ $\cong -4 \ln P(e)/\ln 2$, the approximation holding for $\hat{\alpha}' >> 1$.

15 Shannon, C. E., "The zero-error capacity of a noisy channel," IRE Trans. on Information Theory, Vol. IT-2, pp. S8-S19, September 1956; also see footnote 4, [2].

16 See pp. 13-20, [5].

17 See p. 405, [1].

18 See pp. 13-16 and pp. 34-36, [5].

19 See p. 11, [5].

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