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ON THE EXISTENCE OF SOLUTIONS TO A DIFFERENTIAL GAME

by

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ABSTRACT

In this paper we consider the problem of the existence of a "min-sup" strategy to a pursuit-evasion game. The dynamics of the players have been modeled by a general dynamical system rather than by a differential system. This has helped to achieve mathematical simplicity as well as clarification of the problems involved in a competitive situation. We have discussed the relation between the two models and the relevance of our results to time-optimal control problems.

I. Introduction. In this paper we study the problem of the existence of a solution to a pursuit-evasion game. The rules of the game can be framed as follows:

There are two players, one called the pursuer and the other called the evader. The states of these players at any time t , $0 \leq t < \infty$ are represented by n -dimensional vectors $p(t)$ and $e(t)$, respectively. The game starts at time $t = 0$. The dynamics of the two players are given by certain axioms. These axioms are a weaker version of those given in [1] but more restrictive than those given in [2]. The basic notion used in the formulation of these axioms is that of the attainability function $P(p_0, t_0, t)(E(e_0, t_0, t))$ which represents the set of states that can be reached at time t by the pursuer (evader) starting in state $p_0(e_0)$ at time t_0 . A motion for the pursuer (evader) is therefore a mapping $u(\cdot)$ ($v(\cdot)$) of an interval of $[0, \infty)$ into R^n such that

$$\begin{aligned} u(t) \in P(u(t_0), t_0, t), & \quad t \geq t_0, \\ \left(v(t) \in P(v(t_0), t_0, t), \right. & \quad \left. t \geq t_0 \right), \end{aligned}$$

The evader is informed of the dynamics of the pursuer and the initial state of the pursuer (as well as his own dynamics, of course). This is the extent of the evader's knowledge. A strategy for the evader therefore, consists in selecting, a priori, a motion which satisfies his constraints. The pursuer, on the other hand, together with being supplied with the dynamics of both players, is also told at each instant of time the motion of the evader up to that time. Based on this knowledge, the pursuer selects a course of action which takes him within a specified region (called the endzone) of the evader in the shortest possible time. The evader, of course, tries to escape from this predicament as long as possible. The game ends as soon as the pursuer has achieved his goal. For each strategy (= motion) v

of the evader and each strategy g of the pursuer, let $\tau(g, v)$ be the time (possibly $+\infty$) when the game ends. Let

$$T(g) = \sup\{\tau(g, v) \mid v \text{ is a strategy of the evader}\}$$

and let

$$T^* = \inf\{T(g) \mid g \text{ is a strategy of the pursuer}\},$$

i. e.,

$$T^* = \inf_g \sup_v \tau(g, v).$$

We say that the game has a solution if there is a pursuit strategy g^* such that $T^* = T(g^*)$.

The main result of this paper (see Sec. V) consists in showing that if $T^* < \infty$, then there exists a solution.

In Sec. VI we consider the appropriateness of our model and discuss the relation of our results with the known [5], [6], [7] existence results on time optimal control. Section II deals with the postulates of the dynamics of the two players. In Secs. III and IV we investigate the properties of the motion space and the strategy space, respectively.

II. Dynamics of the Players. Instead of giving the dynamics of the two players by means of differential equations, we adopt the axiomatics of Roxin [1], [2]. The basic notion of these postulates is the attainability function $P(p_0, t_0, t)$ for the pursuer, ($E(e_0, t_0, t)$ for the evader) which represents the set attainable by the pursuer (evader) at time t , which starting in state $p_0(e_0)$ at time t_0 . There are two reasons for using this model. First of all the mathematics is greatly simplified. More important is the belief that this simplified development enables us to distinguish the special problems arising in differential games, as opposed to optimal control.

The attainability functions have to satisfy the following axioms: (we only give the axioms for the pursuer since those for the evader can be obtained by replacing P by E and p by e).

A1. $P(p_0, t_0, t_1)$ is defined for all p_0 in R^n and for all t_0 and t_1 with $0 \leq t_0 \leq t_1 < \infty$. For each value of the argument,

$P(p_0, t_0, t)$ is a nonempty compact subset of R^n .

A2. For all $p_0, t_0, P(p_0, t_0, t_0) = \{p_0\}$.

A3. For all p_0, t_0, t_1, t_2 with $t_0 \leq t_1 \leq t_2$,

$$P(p_0, t_0, t_2) = \bigcup_{p_1 \in P(p_0, t_0, t_1)} P(p_1, t_1, t_2).$$

A4. For fixed $p_0, t_0, P(p_0, t_0, t_1)$ is continuous in t_1 , i.e., for each p_0, t_0, t_1 and $\varepsilon > 0$, there is a $\delta > 0$ such that

$$P(p_0, t_0, t_1) \subseteq P(p_0, t_0, t') + S_\varepsilon$$

and

$$P(p_0, t_0, t') \subseteq P(p_0, t_0, t_1) + S_\varepsilon$$

when

$$|t' - t_1| \leq \delta \text{ and } t' \geq t_0.$$

A5. $P(p_0, t_0, t_1)$ is upper semicontinuous in the triple (p_0, t_0, t_1) , i.e., for each p_0, t_0, t_1 and $\varepsilon > 0$ there is a $\delta > 0$ such that

$$P(p'_0, t'_0, t'_1) \subseteq P(p_0, t_0, t_1) + S_\varepsilon$$

whenever

$$|t'_1 - t_1| \leq \delta, \quad |t'_0 - t_0| \leq \delta, \quad |p'_0 - p_0| \leq \delta^2 \text{ and } t'_0 \leq t'_1.$$

III. Motions of the Players. In this section we define the motions of the two players subject to the dynamical constraints of the previous section. The most important result (Theorem 3.1) is that the set of motions of each player over a fixed time interval $[0, T]$, $T < \infty$ is a compact set under a suitable topology. We assume throughout that each player starts from a fixed initial state.

Definition 3.1.

a. A motion of the pursuer (evader) is a mapping $u(\cdot)$ ($v(\cdot)$) of a subinterval I of $[0, \infty)$ into R^n such that

(1) $0 \in I$ and $u(0) = p_0$ ($v(0) = e_0$) where p_0 (e_0) is the fixed initial state of the pursuer (evader), and

(2) for t_0, t_1 in I with $t_0 \leq t_1$ we have,

$$u(t_1) \in P(u(t_0), t_0, t_1) \left(v(t_1) \in E(v(t_0), t_0, t_1) \right).$$

We will say that the motion is defined on I .

b. Let $u(v)$ and $u_1(v_1)$ be motions defined on I and I_1 , respectively. We say that $u_1(v_1)$ is a prolongation of $u(v)$ if

(1) $I \subseteq I_1$, and

(2) $u(t) = u_1(t)$ ($v(t) = v_1(t)$) for t in I .

c. An entire motion is a motion defined on the entire interval $[0, \infty)$ of interest.

d. A pursuer (evader) motion defined on $[0, T]$, $T < \infty$ will be denoted by $u_T(v_T)$. The set of all such motions will be denoted by $U_T(V_T)$. An entire pursuer (evader) motion will be denoted by $\hat{u}(\hat{v})$ whereas the space of all such motions will be called $\hat{U}(\hat{V})$.

Remark. In the main, we will be only interested in motions on a finite interval, i.e., in the spaces U_T and V_T .

For a proof of the following fact the reader is referred to Lemma 6.1 of reference [2].

Lemma 3.1. A motion defined on an interval I is necessarily a continuous mapping of I into R^n .

Definition 3.2. For $T < \infty$ let C_T denote the Banach space (see [3], pp. 261-281) of all continuous mappings of the interval $[0, T]$ into R^n where for $\xi \in C_T$, the norm of ξ is given by

$$\|\xi\| = \sup_{0 \leq t \leq T} |\xi(t)|.$$

Because of Lemma 3.1, U_T and V_T can be considered to be subsets of C_T . We consider U_T and V_T as subspaces of C_T .

The next result follows directly the axioms of Section II.

Lemma 3.2. U_T and V_T are bounded subsets of C_T .

Lemma 3.3. U_T and V_T are closed subsets of C_T .

Proof. It is enough to prove the assertion for U_T since the proof for V_T is identical. Thus, let $\{u_{T,n}\}_{n=1}^{\infty}$ be a sequence in U_T which converges to an element ξ in C_T , i.e.,

$$(3.1) \quad \lim_n \sup_{0 \leq t \leq T} |u_{T,n}(t) - \xi(t)| = 0.$$

We have to show that $\xi \in U_T$. First of all, since $u_{T,n}(0) = p_0$ for each n , $\xi(0) = p_0$, so that by Def. 3.1a, it remains to show that for all t_0, t_1 with $0 \leq t_0 \leq t_1 \leq T$,

$$\xi(t_1) \in P(\xi(t_0), t_0, t_1)$$

or, since $P(\xi(t_0), t_0, t_1)$ is closed, we have to show that for each $\varepsilon > 0$,

$$(3.2) \quad \xi(t_1) \in P(\xi(t_0), t_0, t_1) + S_\varepsilon.$$

Let $\varepsilon > 0$. Because of A5 and (3.1), for sufficiently large n , say $n > n_1$,

$$u_{T,n}(t_1) \in P(u_{T,n}(t_0), t_0, t_1) \subseteq P(\xi(t_0), t_0, t_1) + S_{\varepsilon/2}.$$

But again for large n , say $n > n_2$,

$$|\xi(t_1) - u_{T,n}(t_1)| \leq \frac{\varepsilon}{2}.$$

Therefore, for $n > n_1 + n_2$, (3.2) is satisfied. Q.E.D.

Lemma 3.4. U_T and V_T are equicontinuous subsets of C_T .

Proof. Again we prove the assertion for U_T only. We have to show that for each $\varepsilon > 0$ and for each $t \in [0, T]$, there is a $\delta > 0$, depending on ε, t such that

$$(3.2) \quad |u_T(t) - u_T(t')| \leq \varepsilon \text{ for all } u_T \text{ in } U_T \text{ and all } t' \text{ in } [0, T] \text{ with } |t' - t| \leq \delta.$$

Suppose the assertion is false. Then there is an $\varepsilon > 0$, $t \in [0, T]$, and sequences $\{u_{T,n}\}_{n=1}^{\infty} \subseteq U_T$, $\{t_n\}_{n=1}^{\infty} \subseteq [0, T]$ such that

$$|t - t_n| \leq \frac{1}{n}, \quad |u_{T,n}(t_n) - u_{T,n}(t)| \geq \varepsilon.$$

Taking subsequences, if necessary, we can assume that there are $x \in R^n$, $y \in R^n$ such that

$$(3.3) \quad u_{T,n}(t) \rightarrow x, \quad u_{T,n}(t_n) \rightarrow y \text{ as } n \rightarrow \infty$$

so that

$$(3.4) \quad |x - y| \geq \varepsilon.$$

Again, taking subsequences if necessary, we may assume that either

(i) $t_n \leq t$ for all n or (ii) $t_n \geq t$ for all n .

Case (i). $t_n \leq t$ for all n .

By axiom A5 of Section II, using (3.3) we see that for large n ,

$$P(u_{T,n}(t_n), t_n, t) \subseteq P(y, t, t) + S_{\varepsilon/2}.$$

But $t_n \leq t$ implies that $u_{T,n}(t) \in P(u_{T,n}(t_n), t_n, t)$ so that, for large n

$$|u_{T,n}(t) - y| \leq \frac{\varepsilon}{2}$$

and hence $|x - y| \leq \varepsilon/2$ which contradicts (3.4). Interchanging t_n and t in the above argument yields a contradiction of (3.4) for Case (ii) also. Hence, the assertion must be true. Q.E.D.

Combining Lemmas 3.2, 3.3 and 3.4, the Ascoli-Arzelà theorem [3], p. 266, yields the following theorem.

Theorem 3.1. U_T and V_T are compact subsets of C_T .

Remark. For an alternative proof of Theorem 3.1 see Roxin [2], Theorem 6.2.

IV. Pursuit Strategies and Feasible Pursuit Strategies.

Definition 4.1.

a. A pursuit strategy is a mapping $g_T: V_T \rightarrow U_T$ such that if v_T and v_T' are in V_T and

$$v_T(\tau) = v_T'(\tau) \quad \text{for } 0 \leq \tau \leq t$$

then

$$g_T v_T(\tau) = g_T v_T'(\tau) \quad \text{for } 0 \leq \tau \leq t.$$

We say that g_T is defined on $[0, T]$.

b. Let G_T denote the set of all strategies defined on $[0, T]$.

Definition 4.2.

a. Let $F(V_T, U_T)$ be the space of all mappings η from V_T into U_T . We give $F(V_T, U_T)$ the topology of pointwise convergence (see [4], p. 217). Thus a net $\{\eta_\alpha\} \subseteq F(V_T, U_T)$ converges to an element η of $F(V_T, U_T)$ if and only if

$$\eta_\alpha(v_T) \text{ converges to } \eta(v_T) \text{ in } U_T$$

for each v_T in V_T .

b. We can consider G_T as a subset of $F(V_T, U_T)$ and give G_T the relative topology.

Definition 4.3. Let Θ be a fixed closed subset of $[0, \infty) \times \mathbb{R}^n$. Θ is called the endzone.

Definition 4.4.

a. A pursuit strategy $g_T \in G_T$ is said to be feasible if for each v_T in V_T ,

$$(\tau, g_T v_T(\tau) - v_T(\tau)) \in \Theta,$$

for some τ , $0 \leq \tau \leq T$.

b. We will denote a feasible pursuit strategy defined on $[0, T]$ by f_T , and the set of all feasible strategies defined on a fixed interval $[0, T]$ by F_T . We consider F_T as a subspace of $F(V_T, U_T)$.

Thus a feasible pursuit strategy f_T is a strategy which guarantees the pursuer that the game will end in time at most T independent of the strategy used by the evader.

Lemma 4.1. F_T is a closed subset of $F(V_T, U_T)$.

Proof. Let $\{f_{T, \alpha}\}$ be a net in F_T which converges to an η of $F(V_T, U_T)$. We must show that $\eta \in F_T$.

By the definition of convergence in $F(V_T, U_T)$ we have that for each v_T in V_T ,

$$(4.1) \quad f_{T, \alpha}(v_T) \rightarrow \eta(v_T) \text{ in } U_T.$$

First of all, let v_T and v_T^i be in V_T such that $v_T(\tau) = v_T^i(\tau)$ for $0 \leq \tau \leq t$. Since $f_{T, \alpha}$ is a pursuit strategy for each α , we must have $f_{T, \alpha} v_T(\tau) = f_{T, \alpha} v_T^i(\tau)$ for $0 \leq \tau \leq t$ so that (4.1) implies that $\eta v_T(\tau) = \eta v_T^i(\tau)$ for $0 \leq \tau \leq t$. Therefore η is certainly a pursuit strategy.

Now let v_T be a fixed element of V_T . For each α there is a τ_α , $0 \leq \tau_\alpha \leq T$ such that (see Definition 4.4a),

$$\left(\tau_\alpha, f_{T, \alpha} v_T(\tau_\alpha) - v_T(\tau_\alpha) \right) \in \Theta$$

Taking subnets if necessary, we can assume that τ_α converges to τ^* for some τ^* with $0 \leq \tau^* \leq T$. We now show that,

$$(4.2) \quad f_{T, \alpha} v_T(\tau_\alpha) - v_T(\tau_\alpha) \rightarrow \eta v_T(\tau^*) - v_T(\tau^*)$$

$$(4.3) \quad \left| \eta v_T(\tau^*) - v_T(\tau^*) - f_{T, \alpha} v_T(\tau_\alpha) - v_T(\tau_\alpha) \right| \\ \leq \left| \eta v_T(\tau^*) - \eta v_T(\tau_\alpha) \right| + \left| \eta v_T(\tau_\alpha) - f_{T, \alpha} v_T(\tau_\alpha) \right| \\ + \left| v_T(\tau^*) - v_T(\tau_\alpha) \right|.$$

Now v_T , ηv_T , $f_{T, \alpha} v_T$ are continuous functions on $[0, T]$; furthermore, convergence in U_T means uniform convergence over $[0, T]$ so that from (4.1) we see that each of the three terms in (4.3) converges to zero. Hence (4.2) is true. Therefore,

$$(4.4) \quad \left(\tau_\alpha, f_{T, \alpha} v_T(\tau_\alpha) - v_T(\tau_\alpha) \right) \rightarrow \left(\tau^*, \eta v_T(\tau^*) - v_T(\tau^*) \right)$$

in $[0, \infty) \times R^n$.

But each term in the left-hand side of (4.4) belongs to Θ and Θ is a closed subset so that the right-hand term also belongs to Θ . This proves that η is a feasible pursuit strategy (see Definition 4.4a), i.e., $\eta \in F_T$, so that F_T is closed. Q.E.D.

Theorem 4.1. F_T is a compact subset of $F(V_T, U_T)$.

Proof. The topology on $F(V_T, U_T)$ is the product topology on the product space of $|V_T|^3$ copies of the space U_T . But by Theorem 3.1, U_T is compact. By the Tychonoff product theorem [4], p. 143, $F(V_T, U_T)$ is compact. By Lemma 4.1, F_T is a closed subset of $F(V_T, U_T)$ so that F_T is compact. Q.E.D.

V. Existence of Solutions. Suppose that there is a T , $0 < T < \infty$ such that F_T is nonempty, i. e., suppose that there exists a feasible pursuit strategy.

Definition 5.1. For each v_T in V_T and each f_T in F_T let $\tau(f_T, v_T)$ be the smallest number τ such that $(\tau, f_T v_T(\tau) - v_T(\tau)) \in \Theta$

Remark. Since Θ is a closed subset of $[0, \infty) \times \mathbb{R}^n$ and $f_T v_T(\tau) - v_T(\tau)$ is a continuous function on $[0, T]$, $\tau(f_T, v_T)$ is well defined.

Definition 5.2

a. For each f_T in F_T let

$$T(f_T) = \sup_{v_T \in V_T} \tau(f_T, v_T).$$

b. Let

$$T^* = \inf_{f_T \in F_T} T(f_T).$$

c. We say that the game has a well solution if there is an f_T^* in F_T such that

$$T(f_T^*) = T^*.$$

We will call f_T^* a well solution of the game.

Lemma 5.1. For fixed v_T , $\tau(f_T, v_T)$ is a lower semicontinuous function ([4], pp. 101-102) on F_T , i. e., for each a the set $\{f_T \mid \tau(f_T, v_T) \leq a\}$ is a closed subset of F_T .

Proof. Let a be fixed. Let $\{f_{T, \alpha}\} \subseteq F_T$ be a net such that

- (i) $\tau(f_{T, \alpha}, v_T) \leq a$ for each α , and
- (ii) $f_{T, \alpha}$ converges to an element f_T of F_T .

We have to show that $\tau(f_T, v_T) \leq a$, i. e., we must show that there is a τ , with $0 \leq \tau \leq a$ such that

$$(5.1) \quad \left(\tau, f_{T, \alpha} v_T(\tau) - v_T(\tau) \right) \in \mathbb{H}.$$

Because of (i), for each α , there is a τ_α with $0 \leq \tau_\alpha \leq a$ such that

$$\left(\tau_\alpha, f_{T, \alpha} v_T(\tau_\alpha) - v_T(\tau_\alpha) \right) \in \mathbb{H}.$$

Taking subnets, if necessary, we can assume that $\tau_\alpha \rightarrow \tau^*$ and $0 \leq \tau^* \leq a$. But then the same argument as in Lemma 4.1 shows that (5.1) is satisfied with $\tau = \tau^*$. Q.E.D.

Lemma 5.2. (See Definition 5.2a.) $T(f_T)$ is a lower semicontinuous function on F_T .

Proof. Let a be a fixed real number and let $\{f_{T, \alpha}\} \subseteq F_T$ be a set such that

(i) $T(f_{T, \alpha}) \leq a$ for each α , and

(ii) $f_{T, \alpha}$ converges to an element f_T of F_T .

We must show that $T(f_T) \leq a$. Let v_T be an arbitrary element of V_T . Because of (i) above,

$$\tau(f_{T, \alpha}, v_T) \leq a \text{ for each } \alpha.$$

But then by Lemma 5.1, $\tau(f_T, v_T) \leq a$, so that

$$T(f_T) = \sup_{v_T \in V_T} \tau(f_T, v_T) \leq a.$$

Q.E.D.

Theorem 5.1. If there exists a feasible pursuit strategy, then there exists a well solution to the game.

Proof. By the hypothesis there is a T , $0 \leq T < \infty$ such that F_T is nonempty. By Theorem 4.1, F_T is a nonempty compact set. By Lemma 5.2, $T(f_T)$ is a lower semicontinuous function on F_T and

hence it has a minimum at some point f_T^* of F_T . Clearly, f_T^* is a well solution to the game.. Q. E. D.

VI. Discussion of the Model and Relation with Time-Optimal Control Problems. Suppose that the dynamics of the players are given by differential equations instead of via the axioms of Section II. For example, let the pursuer dynamics be given by

$$(S) \quad \dot{p}(t) = f(p(t), \sigma(t), t) \quad (p(0) = p_0,$$

where $\sigma(t) \in \Sigma \subseteq R^m$ is the control vector. Suppose that Σ is bounded, and f satisfies enough conditions to insure uniqueness and boundedness of the solution for each measurable control function $\sigma(t)$ with range in Σ . We can define the attainability function $P(p_0, t_0, t)$ for this differential system (S) to be the set of all states p which can be reached at time t , starting in p_0 at time t_0 , by using an admissible control. Then under mild conditions on f , $P(p_0, t_0, t)$ is bounded for each value of its argument and the attainability function satisfies axioms A2-A5. However, in general $P(p_0, t_0, t)$ is not a closed subset of R^n (see Ref. [8]). Let us assume that conditions are imposed on f (see [5]-[8]) such that $P(p_0, t_0, t)$ is closed. Now the system (S) has a well-defined notion of trajectory which is any solution of (S) arising from a measurable control. The attainability function P , on the other hand, gives rise to the concept of motion as in Section III. A little reflection shows that every trajectory is a motion. However, the converse is not true in general. The precise relations between the set of trajectories of (S) and the set of "derived motions" will be investigated in another paper. This relationship is very similar to the one between the "original curves" and the "relaxed curves" discussed by J. Marga [9]. In References [5]-[8], varying sets of sufficient conditions are imposed on f (see (S)) which insure that (a) the attainable set is closed, and (b) every motion is a trajectory. If any

of these sets of conditions are satisfied by (S), Theorem 5.1 tells us that if there exists a feasible solution to the time optimal problem, there exists an optimal solution.

VII. Summary and Conclusion. In this paper we have considered the problem of the existence of a "min-sup strategy" to a pursuit-evasion game. The dynamics of the players have been modeled by a general dynamical system (as developed by Zubov, Roxin, and others) instead of a differential system, the purpose being to clarify the problems arising in the competitive situation of a game as distinct from an optimal control problem. Usually in game theory, interest centers around the existence of a more symmetric solution, i. e., on both "minimax" and "max-min" strategies. The existence of such solutions, seems to the author, to be extremely unlikely in a differential game, except under very restrictive conditions. In any case, it is hoped that the results of this paper will evoke interest among both game theoreticians and people working in the theory of optimal control.

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FOOTNOTES

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¹ Throughout S_ϵ represents the closed sphere in R^n of radius ϵ and center 0. Also, if A, B are subsets of R^n , then $A + B = \{a + b \mid a \in A, b \in B\}$.

² Throughout for $x \in R^n$, $|x|$ represents the Euclidean norm of x .

³ $|V_T|$ denotes the cardinality of the set V_T .