FREQUENCY CRITERIA FOR BOUNDED-INPUT - BOUNDED-OUTPUT STABILITY OF NONLINEAR SAMPLED-DATA SYSTEMS

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Frequency Criteria for Bounded-Input - Bounded-Output Stability of Nonlinear Sampled-Data Systems*

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ABSTRACT

Sufficient conditions for absolute stability in the bounded-input - bounded-output sense are obtained for a class of nonlinear sampled-data systems. The criteria are identical to those establishing absolute stability for the same class of autonomous nonlinear sampled-data systems.

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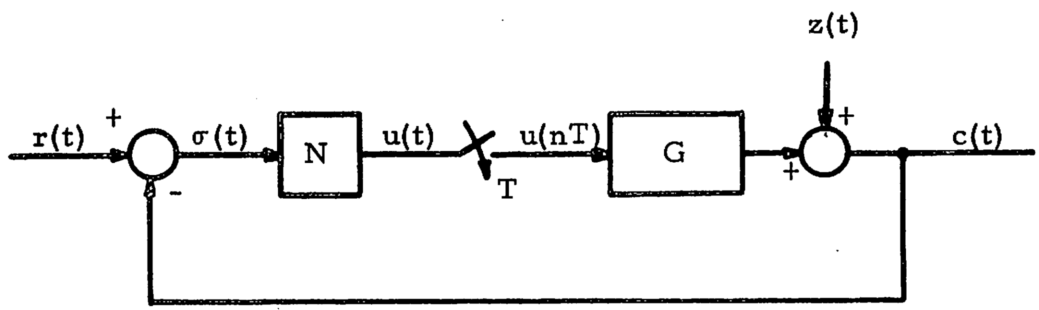

Fig. 1. System S.

I. Introduction

Recently Sandberg,[1] and Bergen, Iwens and Rault[2] obtained conditions for bounded-input - bounded-output (b. i. b. o.) stability of continuous nonlinear feedback systems. In this paper, the absolute stability in the b. i. b. o. sense of nonlinear sampled-data (NSD) feedback systems is investigated, and the previous results obtained by the authors[3] are extended. In Reference 3 the main emphasis is on NSD systems with a monotone nonlinear element, a restriction that is relaxed in this paper.

Conditions for absolute stability of certain classes of autonomous NSD systems are summarized by Jury and Lee.[4]

It is shown in this paper that the same conditions that Jury and Lee give also establish absolute stability in the b. i. b. o. sense.

Notation and Definitions.

$f(n) \triangleq f(nT)$, $n = 0, 1, 2, \cdots$, the value of $f(t)$ at the $\underline{n}$th sampling instant for a sampler with sampling period T;

$f(n) \triangleq 0, \ \forall n < 0$

$\nabla f(n) \triangleq f(n) - f(n-1)$, the backward difference;

$e^{j\bar{\omega}} \triangleq e^{j\omega T}$, for values of $z = e^{sT}$ on the unit circle.

II. Description of System

Consider the single input, single output sampled-data feedback system S shown in Fig. 1. The nonlinear gain element N is memoryless, the linear plant G is nonanticipative, time-invariant, and completely controllable and observable.

Assumption 1. The nonlinear element N is characterized by a piecewise continuous, integrable function $\varphi(\cdot)$ defined on $(-\infty, +\infty)$ satisfying

$$0 \leq \frac{\varphi(\sigma)}{\sigma} \leq k < \infty, \ \forall \sigma \neq 0 \tag{1}$$

$$\varphi(0) = 0 \tag{2}$$

and
$$-k_1 \leq \frac{d\varphi}{d\sigma} \leq k_2 \tag{3}$$

The output of N is given by $u(t) = \varphi(\sigma(t))$.

Assumption 2. The linear plant is characterized by its transfer function $G(s)$, which has no poles in the right-half s-plane. Hold circuits and any continuous or discrete compensation networks may be thought of as being included in $G(s)$. However, $G(s)$ must have a z-transform, $Z[G(s)] = G^*(z)$, which is a rational fraction in z whose numerator polynomial is, at most, of the same degree as the denominator. Furthermore $G^*(z)$ has poles only inside the unit circle in the z-plane (principal case), or has some poles on the unit circle (particular cases), but is analytic everywhere outside the unit circle. $z(t)$ is the zero input response of the linear plant.

Following the notation of Jury and Lee,[4] an NSD system S satisfying these assumptions for specific nonnegative k, k_1, k_2 is referred to as an NSD system S of $\Gamma(0, k; -k_1, k_2)$.

III. Main Results

Theorem 1. The NSD system S of $\Gamma(0, k; -\infty, k_2)$, for the principal case, and of $\Gamma(\epsilon, k; -\infty, k_2)$ for the particular cases ($\epsilon > 0$ arbitrarily small), is absolutely b.i.b.o. stable if the following inequality is satisfied for all $|z| = 1$ and some finite nonnegative number q.

$$\mathrm{Re}\,H_1^*(z) = \mathrm{Re}\{[1+q(z-1)]G^*(z)\} + \frac{1}{k} - \frac{k_2|q|}{2}\,|(z-1)G^*(z)|^2 \geq \delta > 0 \qquad (T_1)^\dagger$$

In addition, for particular cases, the conditions for stability-in-the-limit* must be satisfied. (δ may assume the value of any positive number).

<u>Corollary 1.</u> The NSD system S of $\Gamma(0, k; -k_2, k_2)$, for the principal case, and of $\Gamma(\epsilon, k; -k_2, k_2)$ for the particular cases, is absolutely b. i. b. o. stable if the conditions of Theorem 1 are satisfied for some finite number q, positive or negative.

<u>Theorem 2.</u> The NSD system S of $\Gamma(0, k; -k_1, \infty)$, for the principal case, and of $\Gamma(\epsilon, k; -k_1, \infty)$ for the particular cases, is absolutely b. i. b. o. stable if the following inequality is satisfied for all $|z| = 1$ and some nonnegative finite number q.

$$\mathrm{Re}\,H_2^*(z) = \mathrm{Re}\{[1+q\,\frac{z-1}{z}]G^*(z)\} + \frac{1}{k} - \frac{k_1|q|}{2}\,|(z-1)G^*(z)|^2 \geq \delta > 0 \qquad (T_2)$$

In addition, for particular cases, the conditions for stability-in-the-limit must be satisfied.

<u>Corollary 2.</u> The NSD system S of $\Gamma(0, k; -k_1, k_1)$, for the principal case, and of $\Gamma(\epsilon, k; -k_1, k_1)$ for the particular cases, is absolutely b. i. b. o.

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† The absolute bars are added to the second q in view of Corollary 1.

* These conditions require that the system under consideration be asymptotically stable for a linear gain $\varphi(\sigma) = \epsilon\sigma$, $\epsilon > 0$, arbitrarily small. This is a linear problem which has been extensively treated by Jury[5,6] and others. Root locus techniques, for instance, could be used to check whether the conditions for stability-in-the-limit are satisfied.

stable if the conditions of Theorem 2 are satisfied for some finite number q, positive or negative.

Corollary 1 and Corollary 2 predict stability for exactly the same class of systems, but their criteria are different. This is no contradiction since only sufficient conditions for stability are given. Corollary 1 gives stronger results for most systems.[4,7]

Because of the negative squared term $|(z-1)G^*(z)|^2$ in inequalities (T_1) and (T_2), it is obvious that for particular cases, (T_1) and (T_2) can only be satisfied for the simplest particular case, i.e., $G^*(z)$ has only a simple pole at $z = 1$.

Note that the stability theorems of Reference 3 can be obtained as special cases of Theorem 2 (except for the fact that the nonlinearity can also be time-varying when the stability inequality is satisfied for $q \equiv 0$).

Remarks. Without loss of generality, the theorems need only be proved for

(i) principal cases of $G^*(z)$,

(ii) the nonlinearity $\varphi(\sigma)$ in the reduced sector $[\epsilon, k-\epsilon]$, i.e.,

$$\epsilon \leq \frac{\varphi(\sigma)}{\sigma} \leq k-\epsilon, \quad \forall\, \sigma \neq 0, \text{ where } \epsilon > 0 \text{ is arbitrarily small.}$$

These remarks are justified in Appendix I in a similar manner as shown previously.[3] The method used is the technique of system transformations of Aizerman and Gantmacher,[8] adapted to sampled-data systems.

Auxiliary Lemma. The following lemma plays an important part in the proof of Theorems 1 and 2 and is considered to be the main contribution of this paper.

Main Lemma. If the NSD system S of $\Gamma(0, k; -\infty, k_2)$ $[\Gamma(0, k; -k_1, \infty)]$, which is a principal case, satisfies all the conditions of Theorem 1 [Theorem2], then the following inequality holds for sufficiently small $\alpha > 0$

$$\left(\sum_{\ell=0}^{n} e^{2\alpha\ell} u^2(\ell)\right)^{1/2} \le \frac{1}{\delta}\left(c_0 e^{2\alpha n} + c_1 \sum_{\ell=0}^{n} e^{2\alpha\ell} x^2(\ell) + c_2 \sum_{\ell=0}^{n} e^{2\alpha\ell}[r(\ell) - z(\ell)]^2\right)^{1/2}, \quad \forall n \ge 0 \tag{L}$$

where

$$x(n) = \sum_{\ell=0}^{n} g(n-\ell)[r(\ell) - z(\ell)]$$

and $g(n)$ is the inverse z-transform of $G^*(z)$ of the linear plant. c_0, c_1 and c_2 are finite positive constants independent of n.

Proof of Theorems 1 and 2. Referring to the remarks we need only to prove the theorems for principal cases of $G^*(z)$. Denote $g(n)$ as the inverse z-transform of $G^*(z)$. At the nth sampling instant the system S is described by the equation

$$\sigma(n) = r(n) - z(n) - \sum_{\ell=0}^{n} g(n-\ell)u(\ell) \tag{4}$$

or equivalently

$$\sigma(n) = r(n) - z(n) - \sum_{\ell=0}^{n} e^{\alpha(n-\ell)} g(n-\ell) e^{-\alpha(n-\ell)} u(\ell) \tag{5}$$

Using the triangle inequality and the Schwarz inequality, we obtain

$$|\sigma(n)| \leq |r(n) - z(n)| + \left(\sum_{\ell=0}^{\infty} e^{2\alpha\ell} g^2(\ell)\right)^{1/2} e^{-\alpha n}\left(\sum_{\ell=0}^{n} e^{2\alpha\ell} u^2(\ell)\right)^{1/2} \tag{6}$$

Using inequality (L) of the Main Lemma, we obtain

$$|\sigma(n)| \leq |r(n) - z(n)| + \left(\sum_{\ell=0}^{\infty} e^{2\alpha\ell} g^2(\ell)\right)^{1/2}$$

$$\cdot \frac{1}{\delta}\left(c_0 + c_1 \sum_{\ell=0}^{n} e^{-2\alpha(n-\ell)} x^2(\ell) + c_2 \sum_{\ell=0}^{n} e^{-2\alpha(n-\ell)} [r(\ell) - z(\ell)]^2\right)^{1/2} \tag{7}$$

Since $G^*(z)$ is a principal case, there exist positive constants K_0, K_1 such that $|g(n)| \leq K_1 e^{-K_0 n}$, $\forall n \geq 0$. Therefore, there exists an α, $0 < \alpha < K_0$, such that $\Sigma_{\ell=0}^{\infty} e^{2\alpha\ell} g^2(\ell) \leq A < \infty$. The second and third sums are bounded for all $n \geq 0$ since each of them is the discrete convolution of a strictly stable, linear, sampled-data system with a bounded input. (Note that $x(n)$ and $z(n)$ are bounded for principal cases). Thus the r.h.s. of inequality (7) is bounded for all $n \geq 0$. Therefore,

$$|\sigma(n)| \leq B < \infty, \quad \forall\, n \geq 0 \tag{8}$$

which implies that the output $c(n)$ of the system is bounded. This completes the proof of Theorems 1 and 2.

Proof of Corollary 1. Suppose the NSD system S of $\Gamma(0, k; -k_2, k_2)$ satisfies inequality (T_1) with a negative q. The transformation

$$\hat{\varphi}(\sigma) = k\sigma - \varphi(\sigma) \tag{9}$$

changes this system into an equivalent system $\hat{S}$ of $\Gamma(0, k; -k_2 + k, k_2 + k)$ with $\hat{G}^*(z) = \dfrac{-G^*(z)}{1+kG^*(z)}$. To see this, substitute (9) into (4) and take z-transforms. Theorem 1 is now applied to test for the stability of the transformed system with a positive $\hat{q} = -q$. Forming $\mathrm{Re}\,\hat{H}_1^*(z)$ and substituting for $\hat{G}^*(z)$, we obtain

$$\mathrm{Re}\,\hat{H}_1^*(z) = \mathrm{Re}\{[1-q(z-1)]\hat{G}^*(z)\} + \frac{1}{k} - \frac{(k_2+k)|q|}{2}\left|(z-1)\hat{G}^*(z)\right|^2$$

$$= \frac{\mathrm{Re}\,H_1^*(z)}{|1+kG^*(z)|^2} - kq(1-\cos\bar{\omega})\left|\frac{G^*}{1+kG^*}\right|^2 - k|q|(1-\cos\bar{\omega})\left|\frac{G^*}{1+kG^*}\right|^2$$

Since $q < 0$,

$$\mathrm{Re}\,\hat{H}_1^*(z) = \frac{\mathrm{Re}\,H_1^*(z)}{|1+kG^*(z)|^2} \tag{10}$$

It is clear from (10) that the satisfaction of (T_1) for a negative q implies that there exists a $\hat{\delta} > 0$ such that

$$\mathrm{Re}\,\hat{H}_1^*(z) \geq \hat{\delta} > 0, \quad \forall\ |z| = 1 \quad \text{with} \quad \hat{q} = -q > 0 \tag{11}$$

For particular cases, $\frac{1}{k}$ in $\hat{H}_1^*(z)$ must be replaced by $\frac{1}{k-\epsilon}$ in order that (11) hold. Inequality (11) implies absolute b.i.b.o. stability of the transformed system, by Theorem 1. The original system is then also stable. This completes the proof of Corollary 1.

Proof of Corollary 2. The proof of Corollary 2 follows the same procedure as the proof of Corollary 1. By applying transformation (9) to the original system S of $\Gamma(0, k; -k_1, k_1)$, we obtain an equivalent system $\hat{S}$ of $\Gamma(0, k; -k_1 + k, k_1 + k)$, to which Theorem 2 may now be applied with a positive $\hat{q} = -q$.

Proof of Main Lemma. Let

$$f(n) = r(n) - z(n) \tag{12}$$

Denote for any positive integer N

$$f_N(n) = \begin{cases} f(n), & 0 \leq n \leq N \\ 0, & \text{otherwise} \end{cases} \qquad u_N(n) = \begin{cases} u(n), & 0 \leq n \leq N \\ 0, & \text{otherwise} \end{cases}$$

Then define $\sigma_N(n)$ and $\nabla\sigma_N(n)$ by the following equations.

$$\sigma_N(n) = f_N(n) - \sum_{\ell=0}^{n} g(n-\ell)\, u_N(\ell) \tag{13}$$

$$\nabla\sigma_N(n) = \nabla f_N(n) - \sum_{\ell=0}^{n} \nabla g(n-\ell)\, u_N(\ell) \tag{14}$$

Clearly, $\sigma_N(n) = \sigma(n)$ for $0 \leq n \leq N$ and $\nabla\sigma_N(n) = \nabla\sigma(n)$ for $0 \leq n \leq N$. Note that $\sigma_N(n)$ and $\nabla\sigma_N(n)$ are not identically zero for $n \geq N + 1$ but satisfy the following inequalities.

$$|\sigma_N(n)| \leq K_2 e^{-K_0 n}, \quad \forall n \geq N+1$$

$$|\nabla\sigma_N(n)| \leq K_3 e^{-K_0 n}, \quad \forall n \geq N+1$$

where K_2, K_3 are positive constants (depending on N) and K_0 was defined in $|g(n)| \leq K_1 e^{-K_0 n}$, $\forall n \geq 0$. Define the following auxiliary functions:

$$\lambda_N(n) = \sigma_N(n) - \left(\frac{1}{k} - \gamma\right) u_N(n) \tag{15}$$

$$\psi_N(n) = \frac{k_2 q}{2} \nabla\sigma_N(n) + q u_N(n-1) \tag{16}$$

$$v_N(n) = \nabla\sigma_N(n) \tag{17}$$

where, in (15), γ is such that $0 < \gamma < \delta$, but otherwise arbitrary. Because of the truncation of u(n) and f(n) at N, the z-transforms of these auxiliary functions are analytic on and outside the unit circle in the z-plane. Using (13) and (14) we see that they are given by

$$\Lambda_N^*(z) = F_N^*(z) - [G^*(z) + \frac{1}{k} - \gamma] U_N^*(z) \tag{18}$$

$$\Psi_N^*(z) = \frac{k_2 q}{2}\left[\frac{z-1}{z} F_N^*(z) - \frac{z-1}{z} G^*(z) U_N^*(z)\right] + q z^{-1} U_N^*(z) \tag{19}$$

$$V_N^*(z) = \frac{z-1}{z} F_N^*(z) - \frac{z-1}{z} G^*(z) U_N^*(z) \tag{20}$$

Next, define a Popov function[4,7] $\rho(N)$.

$$\rho(N) = \sum_{n=0}^{\infty} \left[\left(e^{\alpha n} u_N(n)\right)\left(e^{\alpha n} \lambda_N(n)\right) + \left(e^{\alpha n} \psi_N(n)\right)\left(e^{\alpha n} v_N(n)\right)\right] \tag{21}$$

where $0 < \alpha < K_0$, so that the z-tranform of each term under the summation sign is still analytic on and outside the unit circle. Note that in general[6] for any $y(n)$ with z-tranform $Y^*(z)$,

$$\mathcal{Z}[e^{\alpha n} y(n)] = Y^*(e^{-\alpha T} z) \tag{22}$$

If (22) is used it follows immediately from (A 2.1) in Appendix II that

$$\rho(N) \leq \left[\frac{(1+2q)^2}{2(\delta - \gamma)} + 2k_2 q\right] \sum_{n=0}^{N} e^{2\alpha n} f^2(n)$$

$$+ \frac{8(k_2 q)^2}{\delta - \gamma} \sum_{n=0}^{\infty} \left[\sum_{\ell=0}^{n} e^{\alpha(n-\ell)} g(n-\ell) e^{\alpha \ell} f_N(\ell)\right]^2 \tag{23}$$

if

$$\mathrm{Re}\, H_1^*(e^{-\alpha T} z) \geq \delta > 0, \quad \forall |z| = 1 \tag{T_1'}$$

It is shown in Appendix III that for sufficiently small $\alpha > 0$, satisfaction of (T_1) implies (T_1').

Since by hypothesis $f(n)$ is bounded for $n \geq 0$ and $|g(n)| \leq K_1 e^{-K_0 n}$, it is true that

$$\sum_{\ell=0}^{n} g(n-\ell) f_N(\ell) = \sum_{\ell=0}^{n} g(n-\ell) f(\ell), \quad 0 \leq n \leq N$$

and

$$\left|\sum_{\ell=0}^{n} g(n-\ell)\, f_N(\ell)\right| \leq K_4\, e^{-K_0(n-N-1)}, \quad n \geq N+1$$

where K_4 is a positive constant, independent of N, given by

$$K_4 = \sup_{0 \leq n \leq \infty} \left|f(n)\right| K_1 (e^{K_0} - 1)^{-1}$$

Then

$$\rho(N) \leq \left[\frac{(1+2q)^2}{2(\delta-\gamma)} + 2k_2 q\right] \sum_{n=0}^{N} e^{2\alpha n} f^2(n)$$

$$+ \frac{8(k_2 q)^2}{\delta - \gamma} \sum_{n=0}^{N} e^{2\alpha n} \left[\sum_{\ell=0}^{n} g(n-\ell) f(\ell)\right]^2 + \frac{c_0 e^{2\alpha N}}{4(\delta-\gamma)} \tag{24}$$

where c_0 is a positive constant, independent of N, given by

$$c_0 = 32(k_2 q)^2 e^{2\alpha} K_4^2 \sum_{n=0}^{\infty} e^{-2(K_0 - \alpha)n}$$

Denote the r.h.s. of inequality (24) by C(N). Substituting (15) - (17) into (21), inequality (24) becomes

$$\sum_{n=0}^{N} e^{2\alpha n}\left(\sigma(n) - \frac{u(n)}{k}\right) u(n) + \gamma \sum_{n=0}^{N} e^{2\alpha n} u^2(n)$$

$$+ \sum_{n=0}^{N} e^{2\alpha n}\left\{q\, u(n-1)\, \nabla\sigma(n) + \frac{k_2 q}{2}[\nabla\sigma(n)]^2\right\} + \frac{k_2 q}{2} \sum_{n=N+1}^{\infty} e^{2\alpha n}[\nabla\sigma_N(n)]^2 \leq C(N) \tag{25}$$

which is rewritten as

$$S_1 + S_2 + S_3 + S_4 \leq C(N) \tag{26}$$

by identifying the corresponding terms on the l. h. sides of (25) and (26) with each other. Because of the constraint $-\infty \leq \frac{d\varphi}{d\sigma} \leq k_2$, the following area inequality, due to Jury and Lee,[4] applies.

$$q\,u(n-1)\,\nabla\sigma(n) + \frac{k_2 q}{2}\,[\nabla\sigma(n)]^2 \geq q \int_{\sigma(n-1)}^{\sigma(n)} \varphi(\sigma)d\sigma, \quad \forall n \geq 0 \tag{27}$$

Using this inequality, we show in Appendix IV that

$$S_3 \geq -\frac{1}{2}\,q\,k(e^{2\alpha}-1) \sum_{n=0}^{N} e^{2\alpha n}\,\sigma^2(n) \tag{28}$$

For any finite integer N, S_4 satisfies

$$0 \leq S_4 < \infty$$

and may be deleted from the l.h.s. of (25). Remember that $u(n) = \varphi(\sigma(n))$ and note that

$$\left(\sigma - \frac{\varphi(\sigma)}{k}\right)\varphi(\sigma) \geq \frac{\epsilon^2}{k}\,\sigma^2 \tag{29}$$

since it may be assumed that $\varphi(\sigma)$ lies in the reduced sector $[\epsilon, k-\epsilon]$, $\epsilon > 0$ arbitrarily small. Substituting (28) and (29) into (25) yields

$$\sum_{n=0}^{N} e^{2\alpha n}\left[\frac{\epsilon^2}{k} - \frac{1}{2}\,q\,k(e^{2\alpha}-1)\right]\sigma^2(n) + \gamma \sum_{n=0}^{N} e^{2\alpha n}\,u^2(n) \leq C(N) \tag{30}$$

The first sum in (30) is nonnegative if $\frac{\epsilon^2}{k} - \frac{1}{2} qk(e^{2\alpha} - 1) \geq 0$. For any $\epsilon > 0$, $0 \leq q < \infty$, $0 \leq k < \infty$ one can always find an $\alpha > 0$, sufficiently small, such that

$$0 < (e^{2\alpha} - 1) \leq \frac{2\epsilon^2}{qk^2} \tag{31}$$

Delete then the first sum in (30) and set $\gamma = \frac{\delta}{2}$, since γ is arbitrary as long as $0 < \gamma < \delta$. Using (12) and (24), inequality (30) becomes

$$\left(\sum_{n=0}^{N} e^{2\alpha n} u^2(n)\right)^{1/2} \leq \frac{1}{\delta}\left(c_0 e^{2\alpha N} + c_1 \sum_{n=0}^{N} e^{2\alpha n} x^2(n) + c_2 \sum_{n=0}^{N} e^{2\alpha n} [r(n) - z(n)]^2\right)^{1/2}, \quad \forall\, N \geq 0 \tag{32}$$

where

$$x(n) = \sum_{\ell=0}^{n} g(n-\ell)[r(\ell) - z(\ell)] \tag{33}$$

which is bounded for all $n \geq 0$, since $|g(n)| \leq K_1 e^{-K_0 n}, \forall n \geq 0$. The constants c_1 and c_2 are given by

$$c_1 = 32(k_2 q)^2 \tag{34}$$

and

$$c_2 = 2(1 + 2q)^2 + 4\delta k_2 q \tag{35}$$

This completes the proof of the Main Lemma as far as its application to Theorem 1 is concerned.

To prove the Main Lemma for application to Theorem 2, redefine the auxiliary function $\psi_N(n)$, first defined in (16), by

$$\psi_N(n) = \frac{1}{2} k_1 q \nabla\sigma_N(n) + q u_N(n) \tag{36}$$

and follow the same procedure as above. Need will then arise for another area inequality (see Jury and Lee)[4] given by

$$q u(n) \nabla\sigma(n) + \frac{1}{2} k_1 q [\nabla\sigma(n)]^2 \geq q \int_{\sigma(n-1)}^{\sigma(n)} \varphi(\sigma) d\sigma, \quad \forall\, n \geq 0 \tag{37}$$

Following the steps of the derivation given above, one obtains inequality (32) under the condition that

$$\mathrm{Re}\, H_2^*(e^{-\alpha T} z) \geq \delta > 0, \quad \forall\, |z| = 1 \tag{T_2'}$$

Using the same arguments as in Appendix III we can show that for sufficiently small $\alpha > 0$, satisfaction of (T_2) implies (T_2').

This completes the proof of the Main Lemma.

IV. Additional Results

It is worthwhile to note that the proofs given in the previous section do not only establish absolute stability in the b. i. b. o. sense, but also prove absolute stability of the null solution of the autonomous NSD system S. To see this, set $r(t) \equiv 0$ and rederive the term $c_0 e^{2\alpha N}$ appearing in (32). Note also that for the principal case $z(t) \to 0$ exponentially at the same rate as $g(n) \to 0$. From Equation (23) it is clear that the term of interest is

$$S_{N+1} = \sum_{n=N+1}^{\infty} e^{2\alpha n} \left[\sum_{\ell=0}^{N} g(n-\ell) z(\ell) \right]^2 \tag{38}$$

It is easy to show that

$$S_{N+1} \leq K_1^2 \left[\sup_n |z(n)| \right]^2 \left[e^{2(K_0 - \alpha)} - 1 \right]^{-1} N^2 e^{-2(K_0 - \alpha)N} \tag{39}$$

Therefore the term $c_0 e^{2\alpha N}$ in (32) can be replaced by $c_0' N^2 e^{-2(K_0 - \alpha)N}$, where

$$c_0' = 32(k_2 q)^2 K_1^2 \left[\sup_n |z(n)| \right]^2 \left[e^{2(K_0 - \alpha)} - 1 \right]^{-1} \tag{40}$$

With these modifications in mind, for the case $r(t) \equiv 0$ inequality (7) becomes

$$|\sigma(n)| \leq |z(n)| + \left(\sum_{\ell=0}^{\infty} e^{2\alpha\ell} g^2(\ell) \right)^{1/2}$$

$$\cdot \frac{1}{\delta} \left(c_0' n^2 e^{-2K_0 n} + c_1 \sum_{\ell=0}^{n} e^{-2\alpha(n-\ell)} x^2(\ell) + c_2 \sum_{\ell=0}^{n} e^{-2\alpha(n-\ell)} z^2(\ell) \right)^{1/2} \tag{41}$$

where now

$$x(n) = -\sum_{\ell=0}^{n} g(n-\ell) z(\ell)$$

Clearly, $\sigma(n)$ is stable in the sense of Liapunov and $\sigma(n) \rightarrow 0$

exponentially as $n \to \infty$. Because the linear plant is assumed to be completely controllable and observable, absolute asymptotic stability-in-the-large follows.

CONCLUSIONS

Popov-type frequency criteria for absolute stability in the b.i.b.o. sense of nonlinear sampled-data systems have been obtained. The class of systems considered are feedback systems containing a single memoryless nonlinear element described by a function $\varphi(\sigma)$. The bounds on the slope of the nonlinear function $\varphi(\sigma)$ play an important role in the stability criteria. The stability criteria obtained in this paper are identical* to those obtained by other researchers[4,7,9,10,11] for the absolute stability of the null solution of autonomous nonlinear sampled-data systems of the same class. For this reason no examples demonstrating the use of the developed stability criteria are given, since several examples already exist in the literature.[4,7,9,10,11]

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* Some authors differ by exchanging the position of the signs, " $\geq$ " and " > ", regarding the sector restriction of the nonlinear element and the stability inequality. The results obtained, however, are the same.

APPENDIX I

Justification of Remarks

To justify (i), assume that $G^*(z)$ is a particular case satisfying all the conditions of Theorem 1. Make the change of variable

$$\varphi(\sigma) = \tilde{\varphi}(\sigma) + \epsilon\sigma \tag{A1.1}$$

which transforms the system S of $\Gamma(\epsilon, k; -\infty, k_2)$ into an equivalent system $\tilde{S}$ of $\Gamma(0, k-\epsilon; -\infty, k_2-\epsilon)$ with the z-transform of the linear plant given by $\tilde{G}^*(z) = \dfrac{G^*(z)}{1+\epsilon G^*(z)}$, which is a principal case. Consider $\mathrm{Re}\,\tilde{H}_1^*(z)$ for all $|z| = 1$, i.e., $z = e^{j\bar{\omega}}$, $-\pi \leq \bar{\omega} \leq \pi$, and substitute for $\tilde{G}^*(z)$. Then one obtains

$$\mathrm{Re}\,\tilde{H}_1^*(z) = \frac{\mathrm{Re}\,H_1^*(z)}{|1+\epsilon G^*|^2} + \frac{k\epsilon}{k-\epsilon}\left|\frac{G^* + \frac{1}{k}}{1+\epsilon G^*}\right|^2 \tag{A1.2}$$

It is clear from (A1.2) that the satisfaction of (T_1) in Theorem 1 implies that there exists a $\tilde{\delta} > 0$ such that*

$$\mathrm{Re}\,\tilde{H}_1^*(z) \geq \tilde{\delta} > 0, \quad \forall\, |z| = 1 \tag{A1. 3}$$

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* If $G^*(z)$ has a pole at $z = e^{j\bar{\omega}_0}$, then the r.h.s. of (A1.2) at $\bar{\omega} = \bar{\omega}_0$ becomes $k/[\epsilon(k-\epsilon)] > 0$, $-\pi \leq \bar{\omega}_0 \leq \pi$.

If Theorem 1 has been proved for principal cases then (A1.3) establishes stability of the transformed system $\tilde{S}$. The original system S of $\Gamma(\epsilon, k; -\infty, k_2)$ is then also stable, which was to be shown.

The same arguments can be repeated for Theorem 2. Applying again the same transformation, one obtains the equivalent system $\tilde{S}$ of $\Gamma(0, k-\epsilon; -k_1 - \epsilon, \infty)$ with $\tilde{G}^*(z) = \dfrac{G^*(z)}{1+\epsilon G^*(z)}$. Formulation of $\mathrm{Re}\,\tilde{H}_2^*(z)$ and substitution for $\tilde{G}^*(z)$ yields

$$\mathrm{Re}\,\tilde{H}_2^*(z) = \frac{\mathrm{Re}\,H_2^*(z)}{|1+\epsilon G^*|^2} + \frac{k\epsilon}{k-\epsilon}\left|\frac{G^* + \frac{1}{k}}{1+\epsilon G^*}\right|^2, \quad \forall\, |z| = 1 \tag{A1.4}$$

from which the desired conclusion follows as before.

To justify (ii), assume $G^*(z)$ is a principal case and make the change of variable

$$\varphi(\sigma) = \varphi_\epsilon(\sigma) - \epsilon\sigma \tag{A1.5}$$

which transforms the system S of $\Gamma(0, k; -\infty, k_2)$ into an equivalent system S_ϵ of $\Gamma(\epsilon, k+\epsilon; -\infty, k_2+\epsilon)$ with $G_\epsilon^*(z) = \dfrac{G^*(z)}{1-\epsilon G^*(z)}$. For $\epsilon > 0$ sufficiently small, $G_\epsilon^*(z)$ will be a principal case. It can be shown, by using (A1.2) with a negative ϵ, that satisfaction of (T_1) in Theorem 1 implies that there exists a δ_ϵ, $0 < \delta_\epsilon < \delta$, such that the inequality

$$\mathrm{Re}\{[1+q(z-1)]G_\epsilon^*(z)\} + \frac{1}{k+2\epsilon} - \frac{(k_2+\epsilon)|q|}{2}\left|(z-1)G_\epsilon^*(z)\right|^2 \geq \delta_\epsilon > 0, \ \forall\, |z| = 1 \tag{A1.6}$$

is satisfied for a sufficiently small $\epsilon > 0$. Hence, if Theorem 1 has been proved for principal cases of $G^*(z)$ with the nonlinearity in a reduced sector $[\epsilon, k-\epsilon]$, then (A1.6) establishes the stability of S_ϵ in the sector $[\epsilon, k+\epsilon]$. The original system S of $\Gamma(0, k; -\infty, k_2)$ is then stable in the sector $[0, k]$, which was to be shown.

The same arguments, with slight modification, can be repeated for Theorem 2.

APPENDIX II

Proposition. Let

$$\rho_0(N) = \sum_{n=0}^{\infty} [u_N(n)\lambda_N(n) + \psi_N(n)v_N(n)]$$

If inequality (T_1) of Theorem 1, i. e., $\operatorname{Re} H_1^*(z) \geq \delta > 0$, is satisfied for all $|z| = 1$, then

$$\rho_0(N) \leq \left[\frac{(1+2q)^2}{2(\delta-\gamma)} + 2k_2 q\right] \sum_{n=0}^{N} f^2(n) + \frac{8(k_2 q)^2}{\delta - \gamma} \sum_{n=0}^{\infty} \left[\sum_{\ell=0}^{n} g(n-\ell) f_N(\ell)\right]^2 \qquad \text{(A 2.1)}$$

Proof. By the Liapunov - Parseval Theorem[4, 7]

$$\rho_0(N) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \overline{U_N^*(e^{j\bar{\omega}})}\, \Lambda_N^*(e^{j\bar{\omega}}) + \overline{\Psi_N^*(e^{j\bar{\omega}})}\, V_N^*(e^{j\bar{\omega}}) \right\} d\bar{\omega}$$

Substituting (18) - (20) and using (T_1)

$$\rho_0(N) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\{ - |U_N^*|^2 [H_1^* - \gamma] + [1+q(e^{j\bar{\omega}} - 1)]\, F_N^* \bar{U}_N^*$$

$$- \frac{k_2 q}{2} |e^{j\bar{\omega}} - 1|^2 [F_N^* \bar{G}^* \bar{U}_N^* + \bar{F}_N^* G^* U_N^*] + \frac{k_2 q}{2} \left|(e^{j\bar{\omega}} - 1) F_N^*\right|^2 \Big\} d\bar{\omega}$$

Noting that the r.h.s. must be real and completing two squares

$$\rho_0(N) = \frac{-1}{2\pi} \int_{-\pi}^{\pi} \left| \sqrt{\tfrac{1}{2} \mathrm{Re}[H_1^* - \gamma]}\; U_N^* - \frac{[1+q(e^{j\bar{\omega}} - 1)] F_N^*}{2\sqrt{\tfrac{1}{2} \mathrm{Re}[H_1^* - \gamma]}} \right|^2 d\bar{\omega}$$

$$+ \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{|[1+q(e^{j\bar{\omega}} - 1)] F_N^*|^2}{\mathrm{Re}[H_1^* - \gamma]}\, d\bar{\omega}$$

$$- \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sqrt{\tfrac{1}{2} \mathrm{Re}[H_1^* - \gamma]}\;\; U_N^* + \frac{k_2 q |e^{j\bar{\omega}} - 1|^2 F_N^* \bar{G}^*}{2\sqrt{\tfrac{1}{2} \mathrm{Re}[H_1^* - \gamma]}} \right|^2 d\bar{\omega}$$

$$+ \frac{(k_2 q)^2}{4\pi} \int_{-\pi}^{\pi} \frac{|e^{j\bar{\omega}} - 1|^4 |F_N^* G^*|^2}{\mathrm{Re}[H_1^* - \gamma]}\, d\bar{\omega}$$

$$+ \frac{k_2 q}{4\pi} \int_{-\pi}^{\pi} |(e^{j\bar{\omega}} - 1) F_N^*|^2 d\bar{\omega} \qquad \text{(A 2.2)}$$

Since by hypothesis $\mathrm{Re}\, H_1^*(e^{j\bar{\omega}}) \geq \delta > 0$, $-\pi \leq \bar{\omega} \leq \pi$, it follows that $\mathrm{Re}\, H_1^*(e^{j\bar{\omega}}) - \gamma \geq \delta - \gamma > 0$, if $0 < \gamma < \delta$. Therefore,

$$\rho_0(N) \leq \frac{1}{4\pi(\delta - \gamma)} \int_{-\pi}^{\pi} \left| [1 + q(e^{j\bar{\omega}} - 1)] F_N^* \right|^2 d\bar{\omega}$$

$$+ \frac{(k_2 q)^2}{4\pi(\delta - \gamma)} \int_{-\pi}^{\pi} \left| e^{j\bar{\omega}} - 1 \right|^4 \left| F_N^* G^* \right|^2 d\bar{\omega}$$

$$+ \frac{k_2 q}{4\pi} \int_{-\pi}^{\pi} \left| (e^{j\bar{\omega}} - 1) F_N^* \right|^2 d\bar{\omega}$$

or strengthening the inequality further,*

$$\rho_0(N) \leq \frac{(1 + 2q)^2}{4\pi(\delta - \gamma)} \int_{-\pi}^{\pi} \left| F_N^* \right|^2 d\bar{\omega}$$

$$+ \frac{2^4 (k_2 q)^2}{4\pi(\delta - \gamma)} \int_{-\pi}^{\pi} \left| F_N^* G^* \right|^2 d\bar{\omega}$$

$$+ \frac{4 k_2 q}{4\pi} \int_{-\pi}^{\pi} \left| F_N^* \right|^2 d\bar{\omega}$$

Applying again the Liapunov - Parseval Theorem, one obtains (A 2.1).

Q. E. D.

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* This step remains true if $e^{j\bar{\omega}}$ is replaced by $e^{-\alpha T + j\bar{\omega}}$, $\alpha > 0$.

APPENDIX III

Proof That Satisfaction of Inequality (T_1) Implies (T_1'). In the expression of (T_1') replace δ by δ_α. It will be shown that this has no consequence and that if there exists a $\delta > 0$ satisfying (T_1), then there also exists a δ_α, $0 < \delta_\alpha < \delta$, satisfying (T_1') and $|\delta - \delta_\alpha| \to 0$ as α becomes arbitrarily small. Rewrite (T_1') as

$$\mathrm{Re}\{[1+q(e^{-\alpha T} z - 1)] G^*(e^{-\alpha T} z)\} + \frac{1}{k}$$

$$- \frac{k_2 |q|}{2} \left| (e^{-\alpha T} z - 1) G^*(e^{-\alpha T} z) \right|^2 \geq \delta_\alpha > 0, \quad \forall |z| = 1 \qquad (T_1')$$

Given any principal case $G^*(z)$, there exists a sufficiently small $\alpha > 0$ such that $G^*(z)$ is analytic in the domain $|z| \geq e^{-\alpha T}$. It follows that

$$|(e^{-\alpha T} z - 1) G^*(e^{-\alpha T} z) - (z-1) G^*(z)| \text{ and } |G^*(e^{-\alpha T} z) - G^*(z)|$$

approach zero uniformly for all $|z| = 1$ as $\alpha > 0$ becomes arbitrarily small. Then there exists δ_α satisfying (T_1') such that $0 < \delta_\alpha < \delta$ and $|\delta - \delta_\alpha| \to 0$.

APPENDIX IV

It is shown that the following inequality holds.

$$S_3 = \sum_{n=0}^{N} e^{2\alpha n} \left\{ q\, u(n-1)\, \nabla\sigma(n) + \frac{k_2 q}{2} [\nabla\sigma(n)]^2 \right\}$$

$$\geq -\frac{1}{2}\, q\, k(e^{2\alpha} - 1) \sum_{n=0}^{N} e^{2\alpha n} \sigma^2(n) \tag{A 4.1}$$

Proof. Using (27), one obtains

$$S_3 \geq q \sum_{n=0}^{N} e^{2\alpha n} \int_{\sigma(n-1)}^{\sigma(n)} \varphi(\sigma)\, d\sigma \tag{A 4.2}$$

Denote

$$\nabla w(n) = \int_{\sigma(n-1)}^{\sigma(n)} \varphi(\sigma)\, d\sigma$$

and sum (A 4.2) by parts.

$$S_3 \geq q\, e^{2\alpha N} \int_0^{\sigma(N)} \varphi(\sigma)\, d\sigma - q(1 - e^{-2\alpha}) \sum_{n=1}^{N} e^{2\alpha n} \int_0^{\sigma(n-1)} \varphi(\sigma)\, d\sigma$$

which yields because of (1) and (2)

$$S_3 \geq - \frac{1}{2} q k(1 - e^{-2\alpha}) \sum_{n=1}^{N} e^{2\alpha n} \sigma^2(n-1)$$

$$S_3 \geq - \frac{1}{2} q k(e^{2\alpha} - 1) \sum_{n=0}^{N} e^{2\alpha n} \sigma^2(n) \qquad \text{(A 4.3)}$$

which was to be shown.

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