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NETWORK ANALYSIS AND SYNTHESIS VIA STATE VARIABLES

by

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INTRODUCTION

The state space approach to network analysis has generated considerable amount of interest recently. For linear time-invariant RLC networks the analysis problem is completely solved.[1] Methods exist for writing state equations in explicit form and for determining the order of complexity of an arbitrary given network. Extensions to general linear passive networks which contain multi-winding transformers and gyrators are, however, far from trivial, and are given in this paper.

While the analysis problem for linear time-invariant passive networks has essentially been solved, the synthesis problem is only at a beginning stage. We can divide the synthesis problem into two sub-problems depending on whether the given information is in terms of a state characterization or an input-output characterization.

Consider a lumped, linear, time-invariant and passive network, the familiar state characterization is

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u}$$

$$\underline{y} = \underline{C}\,\underline{x} + \underline{D}\,\underline{u}$$

where $\underline{x}$ is the state vector and usually has as its components capacitor voltages and inductor currents, $\underline{u}$ is the input vector, and $\underline{y}$ the output vector. $\underline{A}$ is characterized by network topology and element values, $\underline{B}$ specifies a relation between the input and the state, $\underline{C}$ gives a relation between the output and the state, while $\underline{D}$ describes the direct input-output relation which is independent of the state. From the analysis point-of-view, the main problem is to express $\underline{A}$ in terms of given network topology and elements. The determination of $\underline{B}$, $\underline{C}$, and $\underline{D}$ introduces no additional difficulties. From the synthesis point-of-view we propose two problems. The first one is, from a given $\underline{A}$, find a network which has a set of state variables specified by $\dot{\underline{x}} = \underline{A}\,\underline{x}$. Thus we are interested in only the zero-input response of the state $\underline{x}$. The analysis and synthesis problems of $\underline{A}$ will be treated in Part I.

In Part II we deal with the second problem of synthesis. From the state equations, we obtain the input-output characterization:

$$\underline{y} = [\underline{D} + \underline{C}(p\underline{1} - \underline{A})^{-1}\underline{B}]\,\underline{u}$$

Let $\underline{u}$ be a current source vector and $\underline{y}$ be a voltage response vector, then the impedance matrix is

$$\underline{Z}(p) = \underline{D} + \underline{C}(p\underline{1} - \underline{A})^{-1}\underline{B}$$

The problem is, from a specified positive real square matrix $\underline{Z}(p)$, to determine the matrices $\underline{A}$, $\underline{B}$, $\underline{C}$ and $\underline{D}$ which in turn lead to all equivalent passive network realizations.

PART I

Analysis and Synthesis of A-Matrix

1. Passive and reciprocal networks

1.1 Network elements, characterization of general multi-winding transformers:

Since we are interested in the zero-input responses, we can assume that there are no sources in the network. For passive and reciprocal networks the allowed elements are R, L, C and T (transformers). Let us give a brief but general treatment of the characterization of multi-winding transformers.[2] Consider a typical five-winding transformer as shown in Fig. 1a, we wish to determine the electric properties of the transformer in terms of 5-dimensional voltage and current vectors, $\underline{v}_T$ and $\underline{i}_T$. The magnetic property of the transformer can be characterized by a magnetic graph shown in Fig. 1.b and its branch flux vector $\underline{\psi}$ and m.m.f. vector $\underline{\xi}$. Let $\underline{K}$ be a diagonal matrix which specifies the turns of windings. Then Kirchhoff laws for the magnetic circuit states

$$\underline{Q}_M \underline{\psi} = \underline{0} \tag{1a}$$

$$\underline{B}_M \underline{\xi} = \underline{0} \tag{1b}$$

where $\underline{Q}_M$ and $\underline{B}_M$ are the fundamental cut-set and loop matrices for a chosen tree of the magnetic graph, M. The relation between $\underline{v}_T$ and $\underline{\psi}$ is

$$\underline{v}_T = \underline{K} \frac{d\underline{\psi}}{dt} \quad \text{or} \quad \frac{d\underline{\psi}}{dt} = \underline{K}^{-1} \underline{v}_T \tag{2a}$$

and the relation between $\underline{i}_T$ and $\underline{\xi}$ is

$$\underline{\xi} = \underline{K}\, \underline{i}_T \tag{2b}$$

Combining (1) and (2), we obtain the two basic equations which characterize an arbitrary multi-winding transformer:

$$\underline{Q}_W \underline{v}_T = \underline{0}, \quad \underline{Q}_W \triangleq \underline{Q}_M \underline{K}^{-1} \tag{3a}$$

$$\underline{B}_W \underline{i}_T = \underline{0}, \quad \underline{B}_W \triangleq \underline{B}_M \underline{K} \tag{3b}$$

1.2 Network equations and the order of complexity:

Let us partition the network branches into two classes depending on whether they are R, L and C, or transformers. We use subscript Σ to denote the former and subscript T to denote the latter: Thus KCL states

$$\underline{Q}\, \underline{i} = \underline{0}$$

or

$$[\underline{Q}_{\Sigma}, \underline{Q}_T] \begin{bmatrix} \underline{i}_{\Sigma} \\ \underline{i}_T \end{bmatrix} = \underline{0} \tag{4a}$$

and KVL states

$$\underline{B}\,\underline{v} = \underline{0}$$

or

$$[\underline{B}_{\Sigma}, \underline{B}_T] \begin{bmatrix} \underline{v}_{\Sigma} \\ \underline{v}_T \end{bmatrix} = \underline{0} \tag{4b}$$

where $\underline{i}$ and $\underline{v}$ are branch current and voltage vectors, $\underline{Q}$ is the fundamental cut-set matrix and $\underline{B}$ is the fundamental loop matrix for a chosen tree of the network graph N. The branch relations for R, L and C elements can be simply stated as follows:

$$\underline{Z}_{LR}\underline{i}_{\Sigma} - \underline{Y}_C\,\underline{v}_{\Sigma} = \underline{0} \tag{5a}$$

where

$$\underline{Z}_{LR} \triangleq \begin{bmatrix} \underline{1}_C & & \\ & \underline{R} & \\ & & p\underline{L} \end{bmatrix} \tag{5b}$$

and

$$\underline{Y}_C = \begin{bmatrix} p\underline{C} & & \\ & \underline{1}_R & \\ & & \underline{1}_L \end{bmatrix} \qquad (5c)$$

$\underline{R}$, $\underline{C}$, and $\underline{L}$ are branch parameter matrices for the resistors, capacitors, and inductors, respectively. Combining Eqs. (3), (4) and (5), we obtain the complete characterization of the network in terms of the equation

$$\left[\begin{array}{cc|cc} \underline{Q}_\Sigma & \underline{Q}_T & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{B}_\Sigma & \underline{B}_T \\ \hline \underline{Z}_{LR} & \underline{0} & -\underline{Y}_C & \underline{0} \\ \underline{0} & \underline{B}_W & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{Q}_W \end{array}\right] \left[\begin{array}{c} \underline{i}_\Sigma \\ \underline{i}_T \\ \hline \underline{v}_\Sigma \\ \underline{v}_T \\ \\ \end{array}\right] = \underline{0} \qquad (6)$$

Definition 1.1. The natural frequencies of a linear time-invariant network are defined as the roots of the polynomial, $\det[\underline{\theta}(p)]$ where

$$\underline{\theta}(p) = \begin{bmatrix} \underline{Q}_{\xi} & \underline{Q}_T & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{B}_{\Sigma} & \underline{B}_T \\ \underline{Z}_{LR} & \underline{0} & -\underline{Y}_C & \underline{0} \\ \underline{0} & \underline{B}_W & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{Q}_W \end{bmatrix} \tag{7}$$

The total number of natural frequencies is called the order of complexity of the network.

1.3 Fundamental results:

Definition 1.2. A proper tree of an RLCT network is any set of branches which forms a tree and such that its transformer branches form a tree in its magnetic graph M.

Definition 1.3. A maximal proper tree of an RLCT network is defined as a proper tree having $(n_C - n_L)$ a maximum, where n_C is the number of tree capacitors and n_L is the number of tree inductors.

Theorem 1.1. A necessary condition for an RLCT network N to have a unique solution is that N contains at least one proper tree.

Theorem 1.2. The (maximum) order of complexity of an RLCT network is equal to the sum of the number of capacitors in any maximal proper tree and the number of inductors in the corresponding co-tree.

The proof of the two theorems is straightforward. It depends on the expansion of $\det[\underline{\theta}(p)]$ by the rule of Laplace expansion and is omitted here.[3]

Theorem 1.3. If a maximal proper tree is chosen, let us denote the tree-branch capacitor voltages by $\underline{v}_C$ and link inductor currents by $\underline{i}_L$, then the explicit form of the zero-input state equation of an RLCT network is

$$\begin{bmatrix} \dot{\underline{v}}_C \\ \dot{\underline{i}}_L \end{bmatrix} = - \begin{bmatrix} \mathcal{C}^{-1} & \underline{0} \\ \underline{0} & \mathcal{L}^{-1} \end{bmatrix} \begin{bmatrix} \mathcal{Y} & -\mathcal{H} \\ \mathcal{H}^t & \mathcal{Z} \end{bmatrix} \begin{bmatrix} \underline{v}_C \\ \underline{i}_L \end{bmatrix} \tag{8}$$

where the matrices $\mathcal{C}$, $\mathcal{L}$, $\mathcal{Y}$, $\mathcal{H}$, $\mathcal{Z}$ are given explicitly in terms of the element values of the capacitors, inductors, resistors, transformer turn matrix, and the submatrices of $\underline{Q}$ and $\underline{Q}_M$. Note that (8) is of the same form as the state equations for RLC network without transformers. We refer to Reference 3 for the explicit forms and the proof.

1.4 Synthesis of $\underline{A}$:

Theorem 1.4. A real matrix $\underline{A}$ in the state equation, $\dot{\underline{x}} = \underline{A}\,\underline{x}$, is the $\underline{A}$ matrix of an RLCT network if it admits a decomposition of the form

$$\underline{A} = - \begin{bmatrix} \mathcal{C}^{-1} & \underline{0} \\ \underline{0} & \mathcal{L}^{-1} \end{bmatrix} \begin{bmatrix} \mathcal{Y} & -\mathcal{H} \\ \mathcal{H}^t & \mathcal{Z} \end{bmatrix} \tag{9}$$

where $\mathcal{C}$ and $\mathcal{L}$ are symmetric and positive definite, $\mathcal{Y}$ and $\mathcal{Z}$ are symmetric and positive semi-definite; the state variables, that is, the elements of $\underline{x}$ are then certain voltages and currents in the network.

Proof: Since $\mathcal{C}$ and $\mathcal{L}$ are positive definite, by congruent transformations, we have

$$\underline{T}_C^t \mathcal{C} \underline{T}_C = \underline{\Lambda}_C, \quad \underline{T}_L^t \mathcal{L} \underline{T}_L = \underline{\Lambda}_L \tag{10}$$

where $\underline{\Lambda}_C$ and $\underline{\Lambda}_L$ are diagonal matrices with positive elements. Let $\underline{i}_C'$, $\underline{v}_C'$ and $\underline{i}_L'$, $\underline{v}_L'$ be the currents and voltages for the capacitors and inductors in the network as shown in Fig. 2. Then

$$\underline{i}_C' = \underline{\Lambda}_C \dot{\underline{v}}_C', \quad \underline{v}_L' = \underline{\Lambda}_L \dot{\underline{i}}_L' \tag{11}$$

The congruent transformations can be interpreted as defining equations for the transformers in the network. The equations are

$$\underline{i}_C' = \underline{T}_C^t \underline{i}_C, \quad \underline{v}_C = \underline{T}_C \underline{v}_C' \tag{12 a}$$

$$\underline{v}_L' = \underline{T}_L^t \underline{v}_L, \quad \underline{i}_L = \underline{T}_L \underline{i}_L' \tag{12 b}$$

Substituting (10), (11) and (12) in (8), we obtain

$$-\begin{bmatrix} \underline{i}_C \\ \underline{v}_L \end{bmatrix} = \begin{bmatrix} \mathcal{Y} & -\mathcal{H} \\ \mathcal{H}^t & \mathcal{Z} \end{bmatrix} \begin{bmatrix} \underline{v}_C \\ \underline{i}_L \end{bmatrix} \tag{13}$$

which is the hybrid characterization of an RT network as shown in Fig. 2.

2. Passive and nonreciprocal networks

2.1 Characterization of general passive network elements:

In the treatment of general passive networks we allow R, L, C, ideal transformers (T) and gyrators (G) as network elements. However, we will demonstrate that it is only necessary to consider RCG networks. First it is well-known that an inductor is equivalent to a gyrator which is terminated at a capacitor. Next we will show that any multi-winding transformer has an equivalent circuit with gyrators only. Let us consider the 5-winding transformer in Fig. 1a, we assert that it has an equivalent representation shown in Fig. 3. The proof is straightforward. Consider the gyrator (') branches which constitute a graph M, which is by construction identical to the magnetic graph of the transformer. Thus using the same tree, we have

$$\underline{Q}_M \underline{i}_G' = \underline{0}, \quad \underline{B}_M \underline{v}_G' = \underline{0} \tag{14}$$

Let $\underline{K}$ be the gyration constant matrix of the gyrators, then

$$-\underline{i}_G' = \underline{K}^{-1} \underline{v}_G, \quad \underline{v}_G' = \underline{K} \underline{i}_G \tag{15}$$

Combining (14) and (15), we obtain immediately

$$\underline{Q}_M \underline{K}^{-1} \underline{v}_G = \underline{0}, \quad \underline{B}_M \underline{K}\, \underline{i}_G = \underline{0} \tag{16}$$

which are precisely Eqs. (3 a) and (3 b) of the multi-winding transformer.

2.2 Fundamental results:

Definition 2.1. A proper tree in an RCG network is a tree in which both branches of each gyrator are either in the tree or both in its co-tree.

Definition 2.2. A maximal proper tree in an RCG network is a proper tree having n_C a maximum, where n_C is the number of capacitor tree branches.

We will consider a special case of an RCG network N. The special case is specified in terms of a modified network $N_{(C)}$, that is, a subnetwork of N which is obtained from N by contracting (short circuit) all capacitor branches. We will study only the case where $N_{(C)}$ possesses a proper tree. Let us partition the branches of N into two classes, namely: the capacitor branches and the rest. Let us denote by subscript C the former and the subscript Σ the latter. Furthermore, subscript 1 represents cotree-branches and subscript 2 represents the tree branches. Then for a chosen maximal proper tree, KVL states

$$\begin{bmatrix} \underline{v}_{C1} \\ \underline{v}_{\Sigma 1} \end{bmatrix} = - \begin{bmatrix} \underline{F}_{CC} & \underline{0} \\ \underline{F}_{\Sigma C} & \underline{F}_{\Sigma\Sigma} \end{bmatrix} \begin{bmatrix} \underline{v}_{C2} \\ \underline{v}_{\Sigma 2} \end{bmatrix} \tag{17 a}$$

and KCL states

$$\begin{bmatrix} \underline{i}_{C2} \\ \underline{i}_{\Sigma 2} \end{bmatrix} = \begin{bmatrix} \underline{F}^t_{CC} & \underline{F}^t_{\Sigma C} \\ \underline{0} & \underline{F}^t_{\Sigma\Sigma} \end{bmatrix} \begin{bmatrix} \underline{i}_{C1} \\ \underline{i}_{\Sigma 1} \end{bmatrix} \tag{17 b}$$

Remarks: (a) By assuming that $N_{(C)}$ possesses a proper tree, we guarantee that a proper tree can always be drawn for N which includes the maximum number of tree capacitors, which is equal to the total number of capacitors less the number of independent capacitor loops. (b) The reason that there exists a zero in the upper right corner of $\underline{F}$ is obvious. For each fundamental loop which is defined by a capacitor link, the fundamental loop must contain only capacitors and no elements of the other kind, for otherwise the capacitor link should have been in the tree.

Let the branch relations be

$$\begin{bmatrix} \underline{i}_{C1} \\ \underline{i}_{C2} \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} \underline{C}_1 & 0 \\ \underline{0} & \underline{C}_2 \end{bmatrix} \begin{bmatrix} \underline{v}_{C1} \\ \underline{v}_{C2} \end{bmatrix} \tag{18 a}$$

$$\begin{bmatrix} \underline{v}_{\Sigma 1} \\ \underline{i}_{\Sigma 2} \end{bmatrix} = \begin{bmatrix} \underline{R}_1 & \underline{0} \\ \underline{0} & \underline{G}_2 \end{bmatrix} \begin{bmatrix} \underline{i}_{\Sigma 1} \\ \underline{v}_{\Sigma 2} \end{bmatrix} \tag{18 b}$$

The reason that there exist two zeros in (18 b) is because a proper tree is being used. Thus there are no gyrator couplings between tree branches and links.

By eliminating unwanted variables in (17) and (18) we arrive at the following explicit form of the state equation.

$$\frac{d}{dt}(\mathcal{C}\,\underline{v}_{C2}) = -\mathcal{G}\,\underline{v}_{C2} \tag{19a}$$

where

$$\mathcal{C} = [-\underline{F}^{t}_{CC} \quad \underline{1}] \begin{bmatrix} C_1 & \underline{0} \\ \underline{0} & \underline{C}_2 \end{bmatrix} \begin{bmatrix} -\underline{F}_{CC} \\ 1 \end{bmatrix} \tag{20a}$$

$$\mathcal{G} = \underline{F}^{t}_{\Sigma C}\, R_0^{-1}\, \underline{F}_{\Sigma C} \tag{20b}$$

where

$$\underline{R}_0 = \underline{R}_1 + \underline{F}_{\Sigma\Sigma}\underline{G}_2^{-1}\, \underline{F}^{t}_{\Sigma\Sigma} \tag{20c}$$

Remarks: (a) $\mathcal{C}$ is symmetric and positive definite. (b) It can be easily shown that $\underline{R}_0^{-1}$ exists if and only if $N_{(C)}$ possesses a unique solution. Furthermore the symmetric part of $\mathcal{G}$ is positive semi-definite. (c) In general $\underline{R}_0^{-1}$ exists. If it does not, the number of state variables is reduced by the nullity of $\underline{R}_0$. In other words the dimension of state vector or the order of complexity of the network at most equals the rank of $\underline{R}_0$. (d) If N has no pure gyrator cutsets, then $N_{(C)}$ has a proper tree.

In the following we give two conjectures for general RCG networks:

Theorem 2.1 (Conjectured): The maximum order of complexity of an RCG network is equal to the number of tree capacitors in any maximal proper tree.

Theorem 2.2 (Conjectured): If an RCG network has no proper tree, then it does not possess a unique solution.

2.3 Synthesis:

Theorem 2.3. A real matrix $\underline{A}$ is the $\underline{A}$-matrix of a passive network in the state equation $\dot{\underline{x}} = \underline{A}\ \underline{x}$, if and only if $\underline{A}$ is a stable matrix, that is, all eigenvalues of $\underline{A}$ lie in the closed LHP, and in its Jordan canonical form all imaginary eigenvalues are contained in 1×1 blocks.

Proof: The necessity is obvious because a passive network is necessarily stable. The proof of sufficiency depends on a well-known Inertia Theorem,[4,5] that is, a stable matrix can be decomposed as

$$\underline{A} = -\mathcal{C}\mathcal{G}$$

where $\mathcal{C}$ is real, symmetric and positive definite, and the symmetric part of the real matrix $\mathcal{G}$ is positive semi-definite. The network

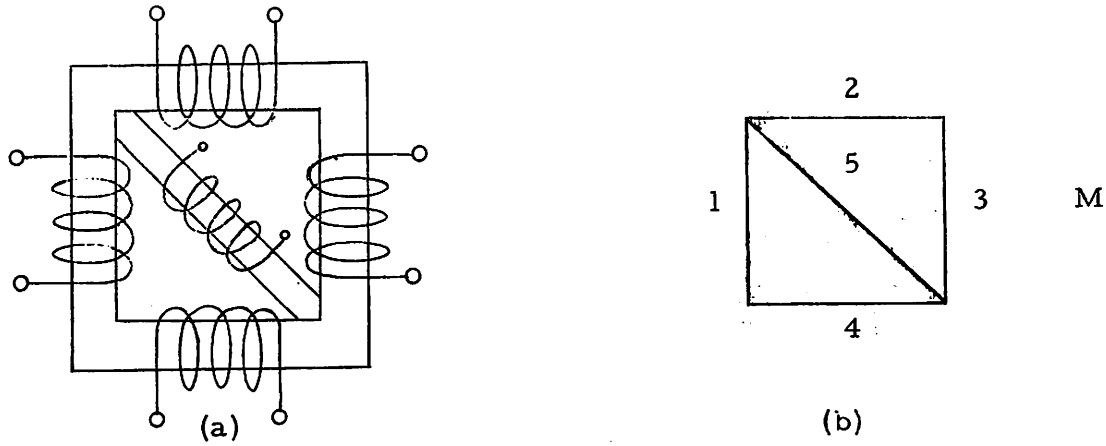

Fig. 1. Five-winding ideal transformer and its magnetic graph, M.

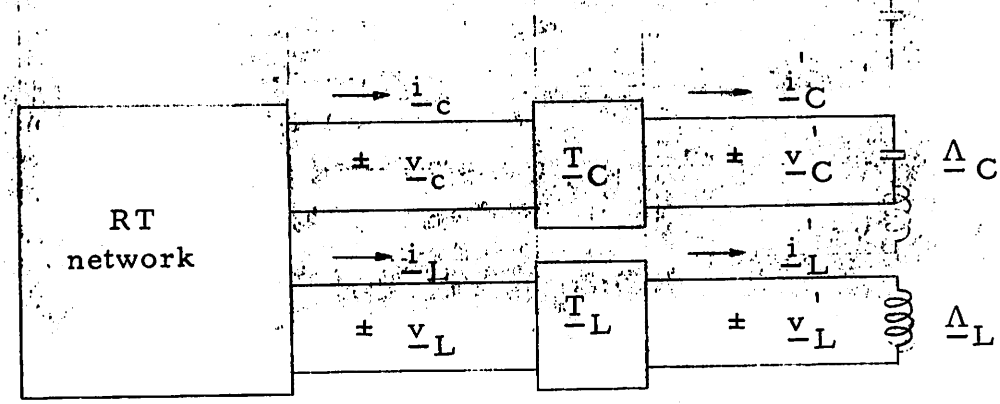

Fig. 2. Synthesis of the $\underline{A}$ - matrix.

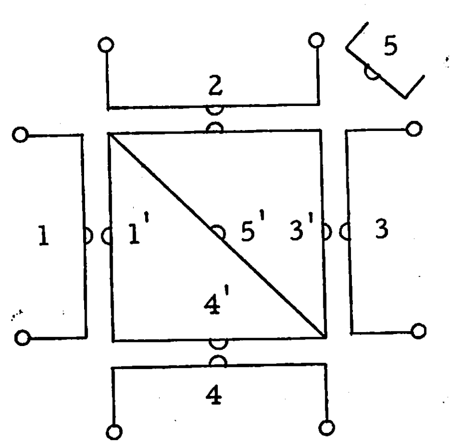

Fig. 3. Gyrator network which is equivalent to a 5 winding ideal transformer.

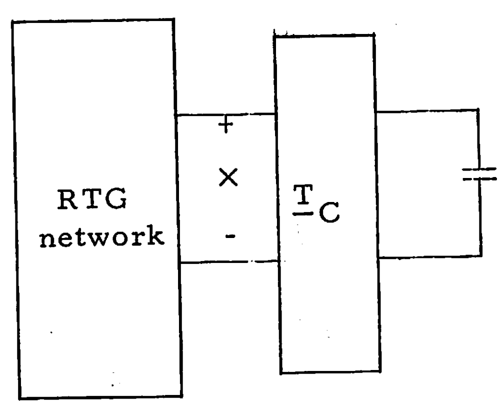

Fig. 4. Synthesis of the A-matrix for general passive networks.

realization is similar to the reciprocal case and is shown in Fig. 4.

PART II

Minimal Synthesis of a Positive Real Matrix

Given a positive real square matrix $\underline{Z}(p)$, it is desired to find a network containing passive elements for which the given matrix is the impedance matrix. We first consider the decomposition of $\underline{Z}(p)$ into a minimal set of state equations:

$$\begin{aligned} \dot{\underline{x}} &= \underline{A}\,\underline{x} + \underline{B}\,\underline{u} \\ \underline{y} &= \underline{C}\,\underline{x} + \underline{D}\underline{u} \end{aligned} \tag{1}$$

The matrices $\underline{A}$, $\underline{B}$, $\underline{C}$ and $\underline{D}$ are then shown to characterize a frequency independent network which when terminated in a minimal set of reactive elements at the secondary ports produces the desired impedance matrix $\underline{Z}(p)$ at the primary ports. In Section 1 we will present the general passive synthesis problem, thus our main concern is to investigate the passivity constraint. In Section 2 we will introduce, in addition, the reciprocity constraint.

With no loss of generality we assume that the given $\underline{Z}(p)$ is regular on the $j\omega$-axis. We say that $\{\underline{A}, \underline{B}, \underline{C}, \underline{D}\}$ constitutes a minimal realization of $\underline{Z}(p)$ if

$$\underline{Z}(p) = \underline{D} + \underline{C}(p\underline{1} - \underline{A})^{-1}\,\underline{B} \tag{2}$$

and if the order of $\underline{A}$ is equal to the McMillan degree of $\underline{Z}(p)$, $\delta(\underline{Z})$. Following the system theory terminology we use the word realization to mean the determination of $\{\underline{A}, \underline{B}, \underline{C}, \underline{D}\}$ from the given $\underline{Z}(p)$. Recent work by Kalman and others give straightforward procedures of obtaining all equivalent minimal realizations of $\underline{Z}(p)$.[6-11] That is, if $\{\underline{A}, \underline{B}, \underline{C}, \underline{D}\}$ is a minimal realization, and $\underline{T}$ any non-singular real matrix,

$$\{\underline{T}^{-1}\underline{A}\,\underline{T},\ \underline{T}^{-1}\underline{B},\ \underline{C}\,\underline{T},\ \underline{D}\} \tag{3}$$

constitutes another minimal realization.

From conventional network synthesis, it is well-known that there exist methods of obtaining passive networks which have the specified impedance matrix, any positive real $\underline{Z}(p)$.[12,13] Furthermore passive networks with a minimum number of reactive elements equal to $\delta(\underline{Z})$, can be obtained, and we call such a synthesis minimal synthesis. The problem of finding equivalent networks has interests among network theorists for decades.[11,14] Our present approach to synthesis can be viewed as an attempt to find all equivalent minimum synthesis from a given positive real matrix, $\underline{Z}(p)$. The strategy is to start with any minimal realization of $\underline{Z}(p)$ and from this obtain a network realization, possibly non-passive, but employing the minimum number of reactive elements; we then introduce coordinate transformation in the state representation as indicated by the transformation in (3) to satisfy the constraints of passivity and reciprocity. Recent work by Youla and Tissi[9] was based on the scattering matrix characterization of an n-port network and depended on the synthesis method of Oono and Yasuura.[13] We will attack the problem directly in terms of impedance and hybrid matrices.

1. General Passive Network Synthesis

1.1 Reactance extraction and the impedance matrix of a frequency independent network:

Consider any minimal passive synthesis of a given $\underline{Z}(p)$. We assume with no loss of generality that the network is shown in Fig. 1 where $\hat{N}$ is frequency independent and all reactive elements are inductors of one henry. This can obviously be done since capacitors can be replaced by gyrators and inductors, while ideal transformers can be used to make all inductances unity. Let $\hat{N}$ be specified by the impedance matrix

$$\hat{\underline{Z}} = \begin{bmatrix} \underline{z}_{11} & \underline{z}_{12} \\ \underline{z}_{21} & \underline{z}_{22} \end{bmatrix} \tag{4}$$

then

$$\underline{Z}(p) = \underline{z}_{11} - \underline{z}_{12}(p\underline{1} + \underline{z}_{22})^{-1}\underline{z}_{21} \quad (5)$$

Comparing (5) with (2) or rather using the alternate, more general form of minimal realization in (3), we can make the following identifications

$$\begin{aligned} \underline{z}_{11} &= \underline{D} & \underline{z}_{12} &= \underline{C}\,\underline{T} \\ \underline{z}_{21} &= -\underline{T}^{-1}\underline{B} & \underline{z}_{22} &= -\underline{T}^{-1}\underline{A}\,\underline{T} \end{aligned} \quad (6)$$

More concisely, we have

$$\hat{\underline{Z}} = \begin{bmatrix} \underline{1} & \underline{0} \\ \underline{0} & \underline{T}^{-1} \end{bmatrix} \begin{bmatrix} \underline{D} & \underline{C} \\ -\underline{B} & -\underline{A} \end{bmatrix} \begin{bmatrix} \underline{1} & \underline{0} \\ \underline{0} & \underline{T} \end{bmatrix} \quad (7)$$

Thus if any minimal realization $\{\underline{A}, \underline{B}, \underline{C}, \underline{D}\}$ has been obtained, (7) gives the impedance matrix of a desired frequency independent network, which, when appropriately terminated in inductors, gives the synthesis of $\underline{Z}(p)$. The non-singular transformation matrix $\underline{T}$ gives flexibility to introduce passivity and reciprocity constraints. In the next section we will determine the passivity constraint in terms of the impedance matrix $\hat{\underline{Z}}$.

Remark: The proof that an impedance characterization for the network $\hat{N}$ exists is omitted.[10] However, it is important to point out that the fact depends on two assumptions, namely: (1) the given $\underline{Z}(p)$ is regular at infinity and (2) the synthesis is minimal.

1.2 Characterization of passivity:

If $\hat{\underline{Z}}$ is the impedance matrix of a frequency independent passive network, the symmetric part must be p.s.d. (positive semi-definite). The network $\hat{N}$ is then easily synthesized since the symmetric part of $\hat{\underline{Z}}$ can be synthesized by an RT network while the skew symmetric part of $\hat{\underline{Z}}$ represents a gyrator network. We will now use the p.s.d. property of $\hat{\underline{Z}}$ to derive the necessary and sufficient condition for a minimal passive realization of $\{\underline{A}, \underline{B}, \underline{C}, \underline{D}\}$. For simplicity we will state and prove the main theorem in terms of a scalar positive real function z(p). For a scalar z(p), we have

$$z(p) = d + \underline{c}^t (p\,\underline{1} - \underline{A})^{-1}\,\underline{b} \tag{8}$$

where d is a scalar, $\underline{c}$ and $\underline{b}$ are column vectors.

Theorem 1. z(p) is a positive real function if and only if there exists a positive definite symmetric matrix $\underline{P}$ and a matrix $\underline{L}$ such that

$$\underline{P}\,\underline{A} + \underline{A}^t\underline{P} + \frac{1}{2d}(\underline{P}\,\underline{b} - \underline{c})(\underline{P}\,\underline{b} - \underline{c})^t = -\underline{L}\,\underline{L}^t \tag{9}$$

Proof: We prove necessity first, that is, we assume that z(p) is positive real and derive (9). Given a positive real z(p), there exists a minimal, passive network realization. Since the network realization is minimal, the frequency independent part has an impedance matrix characterization. Since the network realization is passive, the impedance matrix is positive semi-definite. Thus given any minimal realization, there exists a non-singular $\underline{T}$ such that

$$\hat{\underline{Z}} = \begin{bmatrix} 1 & \underline{0}^t \\ \underline{0} & \underline{T}^{-1} \end{bmatrix} \begin{bmatrix} d & \underline{c}^t \\ -\underline{b} & -\underline{A} \end{bmatrix} \begin{bmatrix} 1 & \underline{0}^t \\ \underline{0} & \underline{T} \end{bmatrix} \tag{10}$$

and

$$\hat{\underline{Z}} + \hat{\underline{Z}}^t = \begin{bmatrix} 2d & \underline{k}^t \\ \underline{k} & \underline{Q} \end{bmatrix}$$

is positive semi-definite, where

$$\begin{aligned} \underline{k} &= \underline{T}^t \underline{c} - \underline{T}^{-1} \underline{b} \\ \underline{Q} &= -\underline{T}^{-1} \underline{A}\,\underline{T} - \underline{T}^t \underline{A}^t \underline{T}^{-1t} \end{aligned} \tag{11}$$

Thus

$$2d x_1^2 + 2x_1 \langle \underline{k}, \underline{x}_2 \rangle + \langle \underline{x}_2, \underline{Q}\,\underline{x}_2 \rangle \geq 0$$

for any real number x_1 and any real vector $\underline{x}_2$ (of appropriate dimension). By completing the square involving the first two terms, we obtain

$$\left[\sqrt{2d}\, x_1 + \frac{1}{\sqrt{2d}} \langle \underline{k}, \underline{x}_2 \rangle \right]^2 + \langle \underline{x}_2, \left[\underline{Q} - \frac{1}{2d} \underline{k}\, \underline{k}^t\right] \underline{x}_2 \rangle \geq 0$$

Hence we conclude that $\underline{Q} - \frac{1}{2d} \underline{k}\, \underline{k}^t$ must be positive semi-definite. Using (11) and defining

$$\underline{P} = \underline{T}^{-1t}\, \underline{T}^{-1} \tag{12}$$

we obtain the desired result in (9) where $\underline{L}$ is some real matrix since $\underline{L}\,\underline{L}^t$ is p.s.d.

We next prove sufficiency by assuming that if $\{\underline{A}, \underline{b}, \underline{c}, d\}$ is given along with a symmetric positive semi-definite $\underline{P}$ and a real $\underline{L}$ which satisfy (9), z(p) is positive real. First we factor $\underline{P} = \underline{T}^{-1t}\underline{T}^{-1}$ according to (12) and let $\underline{Q}$ and $\underline{k}$ be defined as in (11). Then (9) implies that

$$\langle \underline{x}_2, \underline{Q}\, \underline{x}_2 \rangle - \frac{1}{2d} \langle \underline{k}, \underline{x}_2 \rangle^2 \geq 0$$

hence tracing backwards, we find $\hat{\underline{Z}} + \hat{\underline{Z}}^t$ is positive semi-definite. Thus the matrix $\underline{Z}$ can be synthesized as a passive frequency independent network which when terminated in unit inductances produces the desired positive real z(p). Q.E.D.

Remarks: (a) When d = 0, (9) reduces to the familiar conditions for passivity[6,7]

$$\underline{P}\,\underline{A} + \underline{A}^t\underline{P} = -\underline{L}\,\underline{L}^t$$

$$\underline{P}\,\underline{b} = \underline{c}$$

(b) Results similar to (9) have appeared in Ref. 7 and 8. In Ref. 7, Kalman's result is a special case of (9) under the condition $\underline{L} = \underline{0}$, thus there exists $\underline{P}$ which satisfy (9) but is not included in Kalman's result. The result in Ref. 8 is again not as broad as ours for a similar reason.

(c) The above derivation can be generalized to yield the following theorem for a positive real matrix. The proof is given in Ref. 10.

Theorem 2. $\underline{Z}(p)$ is a positive real matrix with $\underline{Z}(\infty) < \infty$, and $\{\underline{A}, \underline{B}, \underline{C}, \underline{D}\}$ is an arbitrary minimal realization of $\underline{Z}(p)$. Let $\underline{U}$ be the orthogonal matrix which diagonalizes the symmetric part of $\underline{D}$, i.e.,

$$\underline{D} + \underline{D}^t = \underline{U}^t \begin{bmatrix} \underline{\Lambda} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} U, \text{ where } \underline{\Lambda} \text{ is nonsingular.}$$

Partition $\underline{U}$ as follows: $\underline{U} = \begin{bmatrix} \underline{U}_1 & \underline{U}_2 \\ \underline{U}_3 & \underline{U}_4 \end{bmatrix}$ where $\underline{U}_1$ is of the same order as $\underline{\Lambda}$. Then $\underline{Z}(p)$ is positive real if and only if there exists a positive definite symmetric matrix $\underline{P}$ and a real matrix $\underline{L}$ such that

(i) $[\underline{P}\underline{B} - \underline{C}^t] \begin{bmatrix} \underline{U}_3^t \\ \\ \underline{U}_4^t \end{bmatrix} = \underline{0}$

(ii) $\underline{P}\underline{A} + \underline{A}^t\underline{P} + [\underline{P}\underline{B} - \underline{C}^t]\underline{J}[\underline{P}\underline{B} - \underline{C}^t]^t = -\underline{L}\,\underline{L}^t$

where

$$\underline{J} = [\underline{U}_1\underline{U}_2]^t \underline{\Lambda}^{-1} [\underline{U}_1\underline{U}_2]$$

is the pseudo-inverse of $\underline{D} + \underline{D}^t$.

Remark: The above theorems are existence theorems. In synthesis, we need to determine $\underline{P}$. Kalman[7] and Meyer[8] gave constructive techniques for obtaining $\underline{P}$. An alternate approach is given in Ref. 10.

2. Passive Reciprocal Network Synthesis

In this section we are dealing with positive real matrix $\underline{Z}(p)$ which is symmetric and we restrict the passive elements to be RLCT.

2.1 The hybrid matrix of a frequency independent network:

Consider the network in Fig. 2a where N is a RT network while its terminations contain k_1 inductors of one henry and k_2 capacitors of one farad, and $k_1 + k_2 = \delta(Z)$. Similar to the treatment in Section 1 we can think of the k_2 capacitors being replaced by gyrators terminated at k_2 inductors, then the frequency independent network contains k_2 gyrators in addition. This is shown in Fig. 2b, where $\hat{N}$ includes the gyrators. In Section 1 we indicated the existence of impedance characterization of network $\hat{N}$. Introducing equations for the gyrators, we can easily show that there exists a hybrid characterization for the network N. Then open-circuit ports for N are the input ports and the inductively terminated ports while the short circuit ports are the capacitively terminated ports. Let the hybrid matrix of N be

$$\underline{H} = \begin{bmatrix} \underline{h}_{11} & \underline{h}_{12} \\ \underline{h}_{21} & \underline{h}_{22} \end{bmatrix}$$

Reciprocity of N requires that $\underline{h}_{11}$ and $\underline{h}_{22}$ be symmetric and $\underline{h}_{21} = -\underline{h}_{12}^t$. Passivity of N requires that $\underline{h}_{11}$ and $\underline{h}_{22}$ be p.s.d. If both requirements are satisfied, N can be synthesized as shown in Fig. 3.

An alternate way of expressing reciprocity is to introduce a diagonal matrix $\underline{\Sigma}$ which contain +1's and -1's only. We can then state the following Lemma on reciprocity.

Lemma 1: A square matrix of real numbers, $\underline{H}$, is the hybrid matrix of a reciprocal network if and only if there exists a diagonal matrix $\underline{\Sigma}$, such that $\underline{\Sigma}\,\underline{H}$ is symmetric.

2.2. Characterization of Reciprocity

Again we start with a minimal realization $\{\underline{A}, \underline{B}, \underline{C}, \underline{D}\}$ of the given symmetric, positive real $\underline{Z}(p)$. Since from conventional network synthesis we know that there exist passive, reciprocal minimal networks which have the given $\underline{Z}(p)$ as the impedance matrix, we assert that there exist transformations such that

$$\underline{M} = \begin{bmatrix} \underline{1} & \underline{0} \\ \underline{0} & \underline{T}^{-1} \end{bmatrix} \begin{bmatrix} \underline{D} & \underline{C} \\ -\underline{B} & -\underline{A} \end{bmatrix} \begin{bmatrix} \underline{1} & \underline{0} \\ \underline{0} & \underline{T} \end{bmatrix} \tag{13}$$

is the hybrid matrix of a passive, reciprocal, frequency independent network. From the reciprocity requirement in Lemma 1, we know that the matrix $\underline{\Sigma}\,\underline{M}$ is symmetric. For convenience we partition $\underline{\Sigma}$ as follows

$$\underline{\Sigma} = \left[\begin{array}{c|c} \underline{1} & \underline{0} \\ \hline \underline{0} & \underline{\Sigma}' \end{array}\right]$$

where $\underline{\Sigma}'$ is of order $\delta(\underline{Z})$ and contains k_1 $+1$'s and k_2 -1's. Writing $\underline{\Sigma}\,\underline{M} = (\underline{\Sigma}\,\underline{M})^t$, we have

$$\begin{bmatrix} \underline{1} & \underline{0} \\ \underline{0} & \underline{\Sigma}' \end{bmatrix} \begin{bmatrix} \underline{1} & \underline{0} \\ \underline{0} & \underline{T}^{-1} \end{bmatrix} \begin{bmatrix} \underline{D} & \underline{C} \\ -\underline{B} & -\underline{A} \end{bmatrix} \begin{bmatrix} \underline{1} & \underline{0} \\ \underline{0} & \underline{T} \end{bmatrix} = \begin{bmatrix} \underline{1} & \underline{0} \\ \underline{0} & \underline{T}^t \end{bmatrix} \begin{bmatrix} \underline{D}^t & -\underline{B}^t \\ \underline{C}^t & -\underline{A}^t \end{bmatrix} \begin{bmatrix} \underline{1} & \underline{0} \\ \underline{0} & \underline{T}^{-1t} \end{bmatrix} \begin{bmatrix} \underline{1} & \underline{0} \\ \underline{0} & \underline{\Sigma}' \end{bmatrix}$$

Multiplying on the left by $\begin{bmatrix} \underline{1} & \underline{0} \\ \underline{0} & \underline{T}^{-1t} \end{bmatrix}$ and on the right by $\begin{bmatrix} \underline{1} & \underline{0} \\ \underline{0} & \underline{T}^{-1} \end{bmatrix}$, and setting

$$\underline{S} = \underline{T}^{-1t}\, \underline{\Sigma}'\, \underline{T}^{-1} \tag{14}$$

we obtain an equivalent statement of reciprocity in terms of the symmetric matrix $\underline{S}$:

$$\begin{bmatrix} \underline{1} & \underline{0} \\ \underline{0} & \underline{S} \end{bmatrix} \begin{bmatrix} \underline{D} & \underline{C} \\ -\underline{B} & -\underline{A} \end{bmatrix} \text{ is symmetric} \tag{15}$$

which implies

$$\begin{aligned} \underline{S}\,\underline{B} &= -\,\underline{C}^t \\ \underline{S}\,\underline{A} &= \underline{A}^t\,\underline{S} \end{aligned} \tag{16}$$

We now state and prove the following:

Lemma 2: Let $\underline{Z}(p)$ be a symmetric rational matrix and $\{\underline{A}, \underline{B}, \underline{C}, \underline{D}\}$ any minimal realization; then the symmetric nonsingular matrix $\underline{S}$ defined in (16) is uniquely determined.

Proof: Let $\underline{S}_1$ and $\underline{S}_2$ be two symmetric matrices which both satisfy (16), let $\underline{Y} = \underline{S}_1 - \underline{S}_2$, then

$$\underline{Y}\,\underline{B} = \underline{0}$$

$$\underline{Y}\,\underline{A} = \underline{A}^t\,\underline{Y}$$

Thus we have

$$\underline{Y}\,\underline{B} = \underline{0}$$

$$\underline{Y}\,\underline{A}\,\underline{B} = \underline{A}^{t}\,\underline{Y}\,\underline{B} = \underline{0}$$

$$\underline{Y}\,\underline{A}^{2}\,\underline{B} = \underline{A}^{t}\,\underline{Y}\,\underline{A}\,\underline{B} = \underline{0}$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

In particular, if the characteristic polynomial of $\underline{A}$ has degree $k = \delta(\underline{Z})$ we have $\underline{Y}[\underline{B}, \underline{A}\underline{B}, \underline{A}^{2}\underline{B}, \ldots, \underline{A}^{k-1}\underline{B}] = \underline{0}$. Since minimal realization implies complete controllability, that is, columns of $[\underline{B}, \underline{A}\,\underline{B}, \underline{A}^{2}\underline{B}, \ldots, \underline{A}^{k-1}\underline{B}]$ span a k-dimensional space; hence $\underline{Y} = \underline{0}$, $\underline{S}_1 = \underline{S}_2$.
Q. E. D.

The determination of $\underline{S}$ for a given minimal realization is straightforward and is obtained by solving the linear equations in (16). Alternately, $\underline{S}$ can be determined as follows: from (16),

$$\underline{S}\,\underline{B} = -\underline{C}^{t}$$

$$\underline{S}\,\underline{A}\,\underline{B} = \underline{A}^{t}\,\underline{S}\,\underline{B} = -\underline{A}^{t}\,\underline{C}^{t}$$

$$\underline{S}\,\underline{A}^{2}\,\underline{B} = -\underline{A}^{t^{2}}\,\underline{C}^{t}$$

$$\underline{S}\,\underline{A}^{k-1}\,\underline{B} = -\underline{A}^{t\ k-1}\,\underline{C}^{2}$$

Thus $\underline{S}[\underline{B}, \underline{A}\underline{B}, \ldots, \underline{A}^{k-1}\underline{B}] = -[\underline{C}^{t}, \underline{A}^{t}\underline{C}^{t}, \ldots, \underline{A}^{t\ k-1}\underline{C}^{t}]$

Hence $\underline{S} = -[\underline{C}^{t}, \underline{A}^{t}\,\underline{C}^{t}, \ldots, \underline{A}^{t\ k-1}\,\underline{C}^{t}][\underline{B}, \underline{A}\underline{B}, \ldots, \underline{A}^{k-1}\underline{B}]^{-1}$ (17)

Once $\underline{S}$ is found, we only need to factor it according to (14) to obtain a transformation matrix $\underline{T}$. It is seen that the class of transformations $\underline{T}$ is precisely that class which reduces the quadratic form $\langle \underline{x}, \underline{S}\,\underline{x} \rangle$ to a sum of squares. Thus by the invariance of the number of positive and negative signs in any such reduction, we conclude that the number of inductors k_1 and the number of capacitors k_2 are fixed in any minimal reciprocal synthesis. This agrees with Ref. 9.

2.3. Reciprocity and Passivity

It should be clear that even though $\underline{S}$ is uniquely determined from a given minimal realization, its decomposition into $T^{-1t}\,\Sigma'\,T^{-1}$ is not unique. The flexibility allows us to introduce the passivity constraint to guarantee a passive reciprocal synthesis. The problem can be illustrated with the diagram in Fig. 4, where $\mathcal{M}$ represents the class of all minimal realization $\{\underline{A}, \underline{B}, \underline{C}, \underline{D}\}$, $\mathcal{P}$ is the class of all passive minimal realization and $\mathcal{R}$ is the class of all reciprocal realization. More specifically,

$$\mathcal{P} = \{\underline{M} \mid \underline{M} \in \mathcal{M},\ \underline{M} \text{ is p.s.d.}\}$$

$$\mathcal{R} = \{\underline{M} \mid \underline{M} \in \mathcal{M},\ \Sigma\,\underline{M} \text{ is symmetric}\}$$

Given a symmetric positive real $\underline{Z}(p)$ it is known that $\mathcal{P} \cap \mathcal{R}$ is non-empty. Starting with any minimal realization, we can determine all passive minimal realizations as in Section 1. Similarly, starting with any minimal realization we can obtain all reciprocal realization by decomposing the matrix $\underline{S}$ according to (14). What we wish to find is a transformation which maps $\mathcal{M}$ into $\mathcal{P} \cap \mathcal{A}$. There are two approaches. The first is to start with a passive realization and introduce a transformation which maintains passivity but satisfies recipocity. The second is to start with a reciprocal realization and introduce a transformation which maintains reciprocity but satisfies passivity. The following interesting properties are noted:

$$\text{Let } \underline{M} \in \mathcal{A}, \text{ thus } \underline{\Sigma}\,\underline{M} = \begin{bmatrix} \underline{1} & \underline{0} \\ \underline{0} & \underline{\Sigma}' \end{bmatrix} \underline{M} \text{ is symmetric.}$$

The mapping $\underline{T}: \mathcal{A} \to \mathcal{A}$ has the property $\underline{T}^{-1t}\underline{\Sigma}'\underline{T}^{-1} = \underline{S} = \underline{\Sigma}'$. In particular, if the given $\underline{Z}(p)$ corresponds to the impedance matrix of an RL or RC network, $\underline{\Sigma}'$ is either $\underline{1}$ or $-\underline{1}$. Thus $\underline{T}^{-1t}\underline{T}^{-1} = \underline{1}$, that is, $\underline{T}$ is an orthogonal transformation. Since an orthogonal transformation preserves the p.s.d. property, starting with a reciprocal realization which is passive, we always end up with a passive realization. Thus it can be shown that all reciprocal realizations for RL or RC impedance matrices are passive.

In general, the transformation which maintains reciprocity has the property $\underline{T}^{-1t}\underline{\Sigma}'\underline{T}^{-1} = \underline{\Sigma}'$, or $\underline{T}^{-1}\underline{\Sigma}'\underline{T}^{-1t} = \underline{\Sigma}'$, which implies $\underline{T}^{-1t}\underline{T}^{-1}\underline{\Sigma}'\underline{T}^{-1t}\underline{T}^{-1} = \underline{\Sigma}'$. Thus we can introduce the reciprocity constraint in terms of the symmetric positive definite matrix $\underline{P} = \underline{T}^{-1t}\underline{T}^{-1}$:

$$\underline{P}\,\underline{\Sigma}'\underline{P} = \underline{\Sigma}' \qquad (18)$$

Eq. (18) can be used in conjunction with the passivity constraint in Theorem 1 or 2 to determine $\underline{P}$.[10] It should be noted that the matrix $\underline{P}$ satisfying (18) and the passivity constraint will not, in general, produce directly a network which is reciprocal. Since by factoring $\underline{P} = \underline{T}^{-1t}\underline{T}^{-1}$, a transformation matrix $\underline{T}$ may not satisfy the reciprocity constraint $\underline{T}^{-1t}\underline{\Sigma}'\underline{T}^{-1} = \underline{\Sigma}'$. However, we are guaranteed that we are within an orthogonal transformation to yield a reciprocal and passive network realization. This is clear since (18) implies

$$\underline{T}^{-1t}\underline{T}^{-1}\underline{\Sigma}'\underline{T}^{-1t}\underline{T}^{-1} = \underline{\Sigma}'$$

or

$$\underline{T}^{-1}\underline{\Sigma}'\underline{T}^{-1t} = \underline{T}^{t}\underline{\Sigma}'\underline{T} \tag{19}$$

which is orthogonal. Introducing $\hat{\underline{T}} = \underline{T}\underline{U}$ where $\underline{U}$ is the orthogonal matrix which diagonalizes $\underline{T}^{t}\underline{\Sigma}'\underline{T}$, we obtain

$$\underline{U}^{t}\underline{T}^{t}\underline{\Sigma}'\underline{T}\,\underline{U} = \underline{\Sigma}'$$

or

$$\hat{\underline{T}}^{t}\hat{\underline{\Sigma}}'\underline{T} = \underline{\Sigma}'$$

or

$$\hat{\underline{T}}^{-1t}\underline{\Sigma}'\hat{\underline{T}}^{-1} = \underline{\Sigma}'$$

which is the reciprocity constraint.

We will not discuss the determination of $\underline{P}$ which satisfies these constraints. Our main purpose is to derive the constraints of passivity and reciprocity in terms of a minimal realization of a symmetric and positive real matrix.

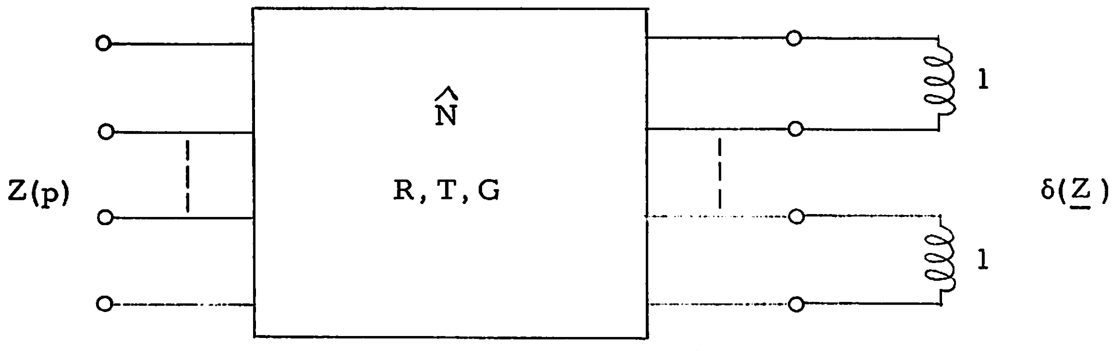

Fig. 1. Passive synthesis of a positive real $\underline{Z}(p)$.

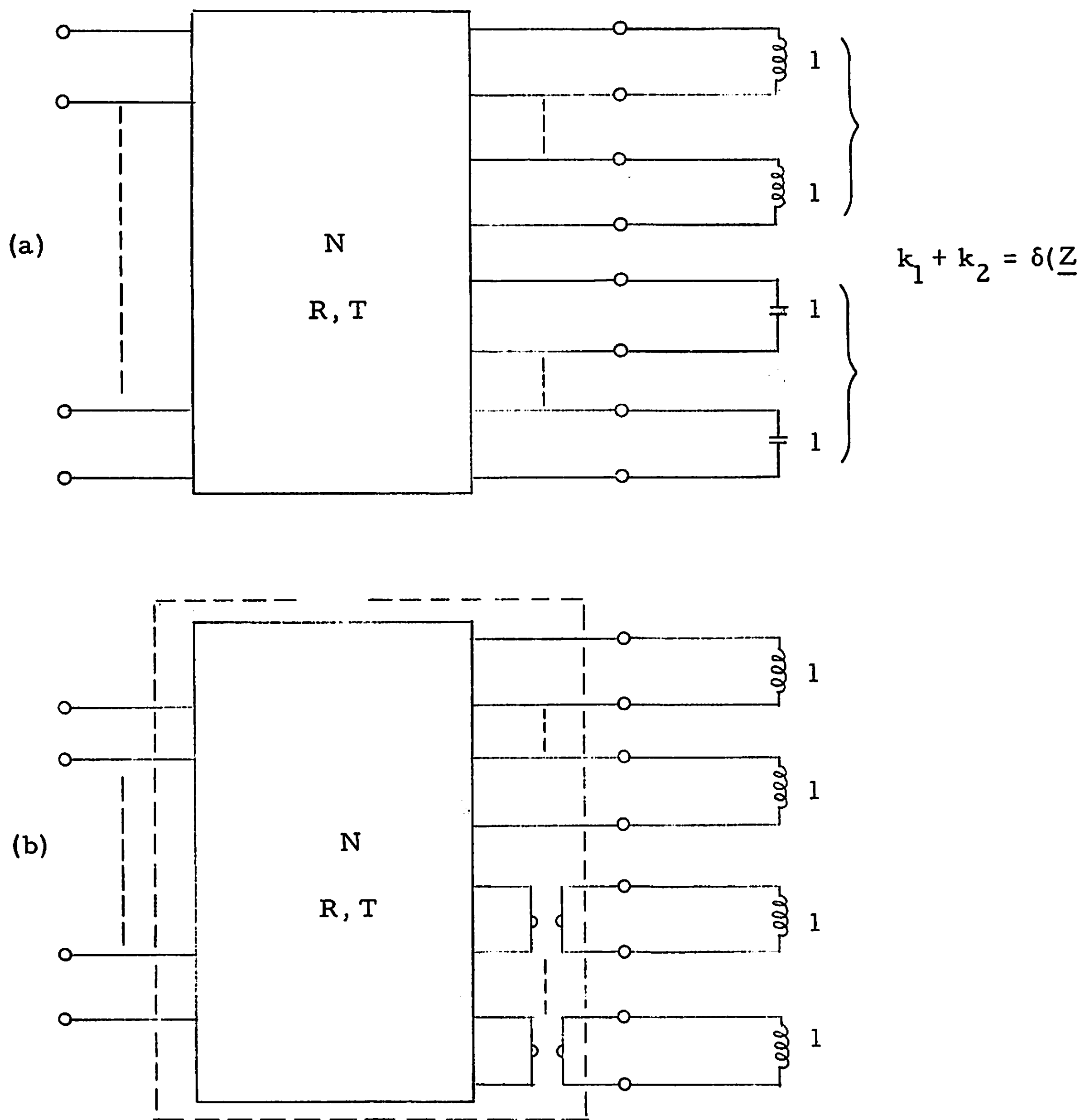

Fig. 2. Passive reciprocal synthesis of a symmetric positive real $\underline{Z}(p)$.

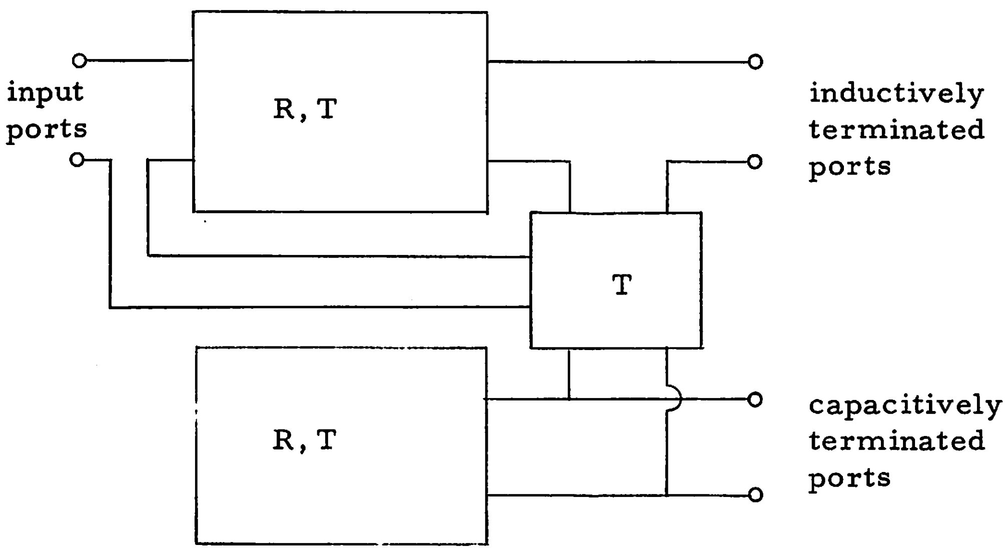

Fig. 3. Synthesis of passive, reciprocal, frequency-independent network.

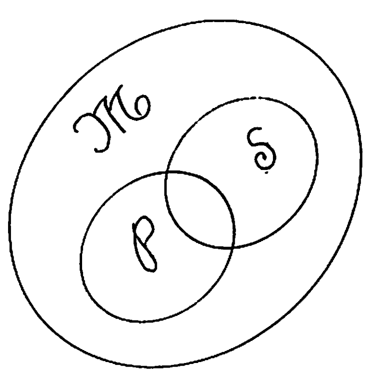

Fig. 4. Diagram illustrating synthesis of passive reciprocal networks.

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