SYNTHESIS OF ACTIVE AND PASSIVE COMPATIBLE IMPEDANCES

by

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ABSTRACT

This report is concerned with the following problem: Given two rational functions $Z_1(s)$ and $Z_0(s)$, otherwise arbitrary but for which $R + Z_1(s)$ has no zeros in the right half plane, $Z_1(s)$ is to be realized as the driving point impedance of a lossless coupling two-port terminated in the impedance $Z_0(s)$. This problem had been previously considered and solved by Schoeffler and by Wohlers when $Z_1(s)$ and $Z_0(s)$ are positive real functions and the coupling network is reciprocal.

Necessary and sufficient conditions are given here for realizability in the contemplated form when neither of the two impedances are necessarily positive real and when the coupling network may be reciprocal or nonreciprocal, but still lossless. A realization procedure is described and examples are given to illustrate the approach.

I. INTRODUCTION

The problem of transforming one impedance by a lossless coupling network into another impedance is an interesting one since it has applications in impedance matching, filter design, and cascade synthesis. It is desirable that the coupling network be lossless so that signal power is transmitted to the load rather than dissipated in the coupling network.

Two impedances Z_0 and Z_1 are said to be compatible if one of them can be realized as the input impedance of a two-port, lossless network terminated in the other impedance, as shown in Fig. 1.

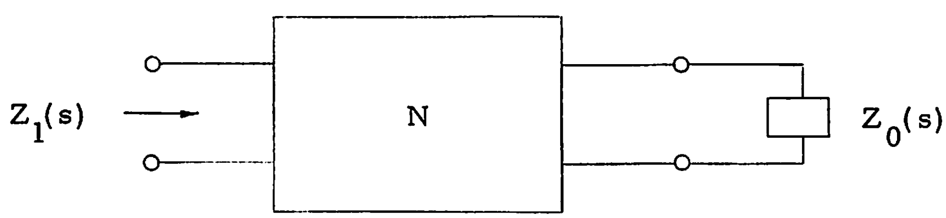


Fig. 1

The compatible impedance problem was first considered by Schoeffler[1] and recently by Wohlers.[2] In both cases, the compatible impedances were considered to be positive real functions and the coupling networks were assumed to be reciprocal.

It would be desirable to consider the problem of compatible impedances when one or both are not necessarily positive real, and when the coupling network is not necessarily reciprocal. The former condition will permit consideration of networks containing active devices, e.g., tunnel diodes. For the latter, the use of nonreciprocal coupling networks can enlarge the domain of compatible impedances and possibly simplify the coupling network.

Necessary and sufficient conditions for two, arbitrary, rational functions $Z_1(s)$ and $Z_0(s)$ to be compatible with respect to a reciprocal or a nonreciprocal, lossless, coupling network N are obtained in this paper. If two impedances are compatible, the canonical coupling network N can always be constructed by synthesizing a single positive real function. Hence, the calculations involved are relatively simpler than those required by the previous methods.

II. THE MAIN THEOREMS

A. THE MOTIVATION OF THE APPROACH

The problem to be considered is the following: if we start with two rational functions $Z_1(s)$ and $Z_0(s)$, necessary and sufficient conditions for them to be compatible are to be derived. From two compatible impedances, it is then necessary to develop a direct and simple synthesis procedure to construct a reciprocal or nonreciprocal, lossless, coupling network N such that when the output port is terminated in $Z_0(s)$, the input impedance is $Z_1(s)$.

One approach, which was used by earlier authors, is to use a scattering formalism. From the given $Z_1(s)$ and $Z_0(s)$, the scattering parameters S_{11} and S_{22} of the two-port N are determined. The unitary property of the scattering matrix is then applied in order to determine $S_{12} = S_{21}$. Realizability conditions on the scattering parameters are then determined by examining the admittance matrix of the augmented network. Wohlers does not concern himself with the actual synthesis of the coupling network but is satisfied with determing a set of realizable scattering parameters.

The approach we shall follow here will be somewhat different. It would be a formidable task to try to realize directly a non-pr function $Z_1(s)$ by a two-port N terminated in a prescribed $Z_0(s)$. However, consider the situation shown in Fig. 2 where a one-ohm resistor is placed across the input terminal. Let the impedance looking into the right-hand terminals of the two-port, when the left-hand terminals are terminated in 1 ohm, be $Z_2(s)$.

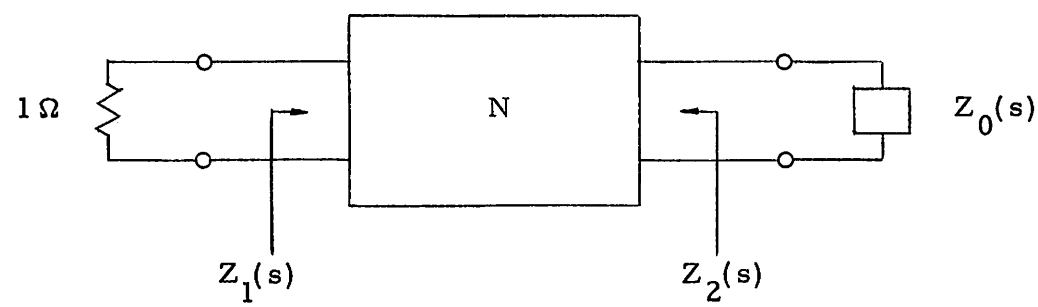


Fig. 2

The function $Z_2(s)$ is clearly pr and there are standard methods for realizing a pr function by a lossless, two-port terminated in a 1-Ω resistor.

Our approach will be the following:

(1) Given $Z_1(s)$, the input reflection coefficient is determined. Its $j\omega$-axis magnitude is identical to that of the output reflection coefficient.

(2) The output reflection coefficient is then found from its $j\omega$-axis magnitude. The flexibility inherent in the nonuniqueness of this determination is used to guarantee that $Z_2(s)$, determined from this reflection coefficient, will be pr and the network will be stable.

(3) $Z_2(s)$ is then synthesized as a lossless, reciprocal or nonreciprocal network terminated in R.

(4) When R is removed, the realized impedance $Z_1^*(s)$ may not necessarily equal the original impedance $Z_1(s)$ but the flexibility of (2) and the possibility of augmenting $Z_2(s)$ in (3) are used to guarantee this equality.

B. GENERAL PROPERTIES RELATED TO TRANSMISSION ZEROS AND CANCELLATIONS

We shall now study certain properties of compatible impedances. Given a rational function,

$$Z_1(s) = \frac{m_{11} + n_{11}}{m_{12} + n_{12}} , \qquad (1)$$

where the m's and n's stand for the even and odd parts of the relatively prime polynomials, respectively. Define a polynomial

$$T_{Z_1} = m_{11} m_{12} - n_{11} n_{12} \qquad (2)$$

Its zeros are called the transmission zeros of $Z_1(s)$ (corresponding polynomials for impedances $Z_0(s)$, $Z_2(s)$, etc., will be T_{Z_0}, T_{Z_2}, etc.) Note that the zeros of T_{Z_0} include the zeros of the even part of $Z_1(s)$ but also include some that may cancel in the even part.

Our first result concerns the relations between the transmission zeros of two impedances if they are compatible.

Lemma 1 If $Z_1(s)$ and $Z_0(s)$ as shown in Fig. 1 are compatible, then all the zeros of T_{Z_0} will be included among the zeros of T_{Z_1}.

Proof: The proof is straightforward and gives no insight into the problem, so it is placed in the appendix.

This lemma is a necessary condition for two impedances to be compatible. So, when verifying the compatibility of two given impedances $Z_1(s)$ and $Z_0(s)$, we first form T_{Z_1} and T_{Z_0} and check Lemma 1. If it is satisfied, we can proceed; if not, they are not compatible.

Since N is to be constructed by synthesizing $Z_2(s)$, and it is required that the resulting input impedance be the original $Z_1(s)$, the transmission zeros (abbr. t-zeros) naturally play an important role. Let us consider the configuration shown in Fig. 3

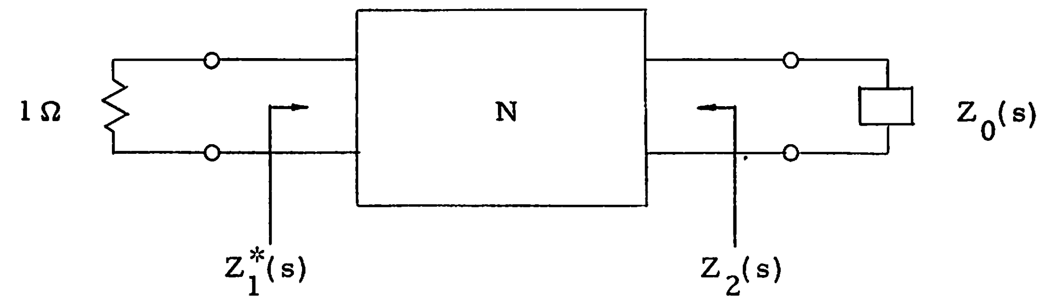


Fig. 3

where N is obtained by realizing $Z_2(s)$, a pr function. The realization may be obtained by using Youla's table.[3] $Z_1^*(s)$ is the resulting input impedance. $Z_2(s)$ must be so chosen and the realization of N so carried out that $Z_1^*(s) = Z_1(s)$.

We write $Z_0(s)$ as

$$Z_0(s) = \frac{m_1 + n_1}{m_2 + n_2} \quad . \tag{3}$$

A routine analysis of Fig. 3 will show that

$$Z_1^*(s) = \frac{A Z_0 + B}{C Z_0 + D} = \frac{A(m_1+n_1) + B(m_2+n_2)}{C(m_1+n_1) + D(m_2+n_2)} \tag{4}$$

$$Z_2(s) = \frac{D + B}{A + C} \tag{5}$$

where $$t = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{6}$$

is the polynomial chain matrix, consisting of the numerators of the usual chain parameters. Now form $1 + Z_1^*(s)$ and $Z_2(s) + Z_0(s)$ using Eqs. (3), (4), and (5). It will be found that the expressions for the numerators will be exactly the same. This means that the two equations

$$1 + Z_1^*(s) = 0 \tag{7}$$

$$Z_2(s) + Z_0(s) = 0 \tag{8}$$

will have the same roots, provided there is no cancellation with the denominator in either equation. The roots of these equations are the natural frequencies.

Suppose some cancellations have taken place in Eq. (4). Then $1 + Z_1^*(s) = 0$ will be missing some of the roots of $Z_2(s) + Z_0(s) = 0$. We say that these natural frequencies are not observable from the input port. The conditions under which such cancellations take place would be of great interest. They are given in the following lemma.

Lemma 2. If the two-port N in Fig. 3 is constructed by realizing $Z_2(s)$, (assuming $Z_2(s)$ and $Z_0(s)$ are non-Foster) then a factor $(s+s_0)^n$ will be cancelled from

$$Z_1^*(s) = \frac{A Z_0 + B}{C Z_0 + D} \tag{9}$$

if and only if (a) $(s+s_0)^n$ is a factor of the function $Z_2(s) + Z_0(s)$; (b) $(s+s_0)$ is a factor of T_{Z_2} at least to the nth order.

Proof. Necessity. Let us look at the t-zeros of $Z_1^*(s)$. Before cancellation, t-zeros of $Z_1^*(s)$ are zeros of

$$(Am_1+Bn_2)(Cn_1+Dm_2) - (An_1+Bm_2)(Cm_1+Dn_2) \tag{10}$$

or $$(AD-BC)(m_1m_2-n_1n_2) = T_{Z_2} \cdot T_{Z_0} \tag{11}$$

The last step follows from Eqs. (3) and (5). We see that the t-zeros of $Z_1^*(s)$ include the t-zeros of both $Z_0(s)$ and $Z_2(s)$. However, cancellations in $Z_1^*(s)$ will reduce the number of its t-zeros. The nature of this reduction remains to be determined.

Suppose a factor $(s+s_0)$ with $\text{Re}\, s_0 > 0$ has been cancelled from a function $Z_a(s)$ leaving the resulting function

$$Z_b(s) = \frac{m_{1b} + n_{1b}}{m_{2b} + n_{2b}} \tag{12}$$

Then, before cancellation,

$$Z_a(s) = \frac{m_{1b} + n_{1b}}{m_{2b} + n_{2b}} \cdot \frac{(s+s_0)}{(s+s_0)} = \frac{m_{1a} + n_{1a}}{m_{2a} + n_{2a}} \tag{13}$$

Clearly, $$m_{1a}\, m_{2a} - n_{1a}\, n_{2a} = (m_{1b}\, m_{2b} - n_{1b}\, n_{2b})(s^2 - s_0^2) \tag{14}$$

It follows that at the same point where a natural frequency is cancelled, a t-zero of the same multiplicity has also been cancelled. Hence, we can conclude;

(1) No cancellations in $Z_1^*(s)$ imply $1 + Z_1^*(s) = 0$ and $Z_0(s) + Z_2(s) = 0$ have exactly the same roots and the t-zeros of $Z_1^*(s)$ include the t-zeros of both $Z_2(s)$ and $Z_0(s)$.

(2) Cancellation of a factor in $Z_1^*(s)$ implies the presence of this factor in T_{Z_2} as well as in the equation $Z_2(s) + Z_0(s) = 0$.

Sufficiency.

Here, we want to show that if

$$Z_2(s) + Z_0(s) \tag{15}$$

$$T_{Z_2} \tag{16}$$

both have a common factor[Δ] $(s+s_0)^n$, this factor will be cancelled from $Z_1^*(s)$ defined by (4). That is, we want to prove that the polynomials:

$$A(m_1+n_1) + B(m_2+n_2) \tag{17}$$

$$C(m_1+n_1) + D(m_2+n_2) \tag{18}$$

have a common factor $(s+s_0)^n$. Now if we can show that one of these two polynomials has the factor $(s+s_0)^n$, then the other will also because the sum of these two polynomials equals

$$(A+C)\,(m_1+n_1) + (B+D)\,(m_2+n_2)\,, \tag{19}$$

which is the numerator of (15) and thus has the factor $(s+s_0)^n$ by hypothesis.

By the definition of t-zeros, $(A+C)$ and $(B+D)$ are not both zero at $s = -s_0$. Let us assume $B + D \neq 0$ at $s = -s_0$. It follows that B and D can not both be zero. We assume $B \neq 0$ at $s = -s_0$. (The opposite case, $D \neq 0$, can be handled in the same way). Multiply (19) by B.

$$B(A+C)\,(m_1+n_1) + B(B+D)\,(m_2+n_2)\,. \tag{20}$$

Δ If $Z_1(s) = \dfrac{N_1}{D_1}$ $Z_2(s) = \dfrac{N_2}{D_2}$, then by a factor of the function $Z_1(s) + Z_2(s)$, we mean a factor of the following polynomial $N_1D_2 + N_2D_1$.

It was assumed that

$$T_{Z_2} = AD - BC \tag{21}$$

has the factor $(s+s_0)^n$. By adding and subtracting AB, this becomes

$$T_{Z_2} = A(B+D) - B(A+C) . \tag{22}$$

If we make a Taylor series expansion of $B(A+C)$ and $A(B+D)$ around $s = -s_0$, the first n corresponding terms of these two series must be equal. Hence

$$B(A+C) = A(B+D) + (s+s_0)^n \; R(s) , \tag{23}$$

where $R(s)$ includes the higher power terms of the series. This can now be inserted into (20) to yield

$$B(A+C)\,(m_1+n_1) + B(B+D)\,(m_2+n_2)$$

$$= A(B+D)\,(m_1+n_1) + B(B+D)\,(m_2+n_2) + (s+s_0)^n \; R_1(s) \tag{24}$$

or $$(B+D)\left[A(m_1+n_1) + B(m_2+n_2)\right] + (s+s_0)^n \; R_1(s) \tag{25}$$

which by hypothesis has the factor $(s+s_0)^n$. But it was assumed that $(B+D) \neq 0$ at $s = -s_0$. We can therefore conclude that

$$A(m_1+n_1) + B(m_2+n_2) \tag{26}$$

has the factor $(s+s_0)^n$, as we were to prove. From (19), it follows that

$$C(m_1+n_1) + D(m_2+n_2) \tag{27}$$

also has the factor $(s+s_0)^n$. Thus, the factor $(s+s_0)^n$ will be cancelled in $Z_1^*(s)$ of Eq. (4).

The same kind of proof would show the same results if $B(-s_0) = 0$ but $D(-s_0) \neq 0$. A similar argument can be used for the remaining case:

$$B + D = 0 \text{ but } A + C \neq \text{ at } s = -s_0 . \qquad \text{Q.E.D.}$$

In the process of realizing $Z_2(s)$ to yield the lossless two-port, it sometimes becomes necessary to augment $Z_2(s)$, (augmenting means multiplying numerator and denominator by the same factor). These factors correspond to additional "sections" in the cascade synthesis, and they may or may not appear in $Z_1^*(s)$. Thus the augmentation of $Z_2(s)$ is a way of controlling the appearance of certain factors in the realized $Z_1^*(s)$ and in $T_{Z_1^*}$. Clearly, the conditions under which these factors appear in $T_{Z_1^*}$, or are cancelled, are of importance. They are given by the following lemma.

Lemma 3. For the same conditions as Lemma 2, if $Z_2(s)$ is augmented with the factor $(s+s_0)^n$ before realization, then a cancellation will result in $Z_1^*(s)$ of Eq. (4) only if:

$$Z_2(-s) - Z_0(s) \qquad (28)$$

has a factor $(s+s_0)^m$. If, (1) $0 < m < n$, the factor cancelled in $Z_1^*(s)$ will be $(s+s_0)^m$; (2) $m \geq n$, then the augmenting factor $(s+s_0)^n$ will be cancelled completely.

Proof: Let the augmenting factor $(s+s_0)^n$ be represented by the sum of its even and odd parts:

$$(s+s_0)^n = E_1 + O_1 \qquad (29)$$

Then, after this augmentation

$$Z_2(s) = \frac{(D+B)}{(A+C)} \frac{(s+s_0)^n}{(s+s_0)^n} = \frac{(BO_1+DE_1) + (BE_1 + DO_1)}{(AE_1 + CO_1) + (AO_1 + CE_1)} \tag{30}$$

Using this in Eq. (4), we obtain

$$Z_1^*(s) = \frac{(AE_1 + CO_1)\, Z_0(s) + (BE_1 + DO_1)}{(AO_1 + CE_1)\, Z_0(s) + (DE_1 + BO_1)} \tag{31}$$

We must now analyze both numerator and denominator of $Z_1^*(s)$ to determine the presence of factors $(s+s_0)^k$.

Let $N(s)$ be the numerator of $Z_1^*(s)$,

$$N(s) = (AE_1+CO_1)\; Z_0 + (BE_1+DO_1) \tag{32}$$

Substitute for E_1 in Eq. (29)

$$N(s) = (-AO_1+CO_1)\; Z_0 + (-BO_1+DO_1) + (A+B)\,(s+s_0)^n \tag{33}$$

or $$N(s) = O_1 \left[(D-B) - (A-C)\, Z_0\right] + (A+B)\,(s+s_0)^n \,. \tag{34}$$

Since O_1 is not equal to zero at $s = -s_0$, except if $s_0 = 0$ (which is a case of no interest), we must examine

$$(D-B) - (A-C)\, Z_0 \,. \tag{35}$$

Suppose that $$(D-B) - (A-C)\, Z_0 \Big|_{s = -s_0} \neq 0 \,. \tag{36}$$

Then $N(-s_0) \neq 0$, and no factor $(s+s_0)$ will cancel in $Z_1^*(s)$. On the other hand, suppose that

$$(D-B) - (A-C)\, Z_0 \tag{37}$$

has a factor $(s+s_0)^m$. The equation $(D-B) - (A-C)\, Z_0 = 0$ can be rewritten as

$$\frac{D(s) - B(s)}{A(s) - C(s)} - Z_0(s) = 0 \tag{38}$$

or

$$Z_2(-s) - Z_0(s) = 0 \,. \tag{39}$$

By assumption, the left side has the factor $(s+s_0)^m$. Hence, from (35), we see that $N(s)$ also has this factor $(s+s_0)^m$ for the case $m < n$. On the other hand, if $m \geq n$, the augmented factor $(s+s_0)^n$ can be completely factored from $N(s)$.

By a parallel approach, we can show that the denominator of $Z_1^*(s)$

$$D = (AQ_1+CE_1)\, Z_0 + (DE_1+BQ_1) \tag{40}$$

also has the factor $(s+s_0)$ with the same multiplicity as the numerator $N(s)$and thus the cancellation in $Z_1^*(s)$ amounts to m or n, whichever is smaller. Q.E.D.

With Lemmas 2 and 3 it is possible to determine whether t-zeros of $Z_2(s)$ will be cancelled from $Z_1^*(s)$ and whether $Z_2(s)$ can be augmented to supply any needed t-zeros.

C. THE REFLECTION COEFFICIENT

As mentioned previously, the coupling network is to be constructed by realizing $Z_2(s)$, which will itself be determined through the reflection coefficients. In Fig. 3, define the realized input reflection coefficient to be:

$$\rho_{11}^{*}(s) = \frac{Z_1^{*}(s) - 1}{Z_1^{*}(s) + 1} \qquad (41)$$

and the output reflection coefficient to be

$$\rho_{22}(s) = \frac{Z_2(s) - Z_0(-s)}{Z_2(s) + Z_0(s)} \quad . \qquad (42)$$

The zeros of the input reflection coefficient are at

$$Z_1^{*}(s) - 1 = 0 \qquad (43)$$

using Eq. (4) for $Z_1^{*}(s)$, we obtain

$$AZ_0 + B - (CZ_0 + D) = 0 \, , \qquad (44)$$

which, with (5) for $Z_2(s)$, becomes

$$Z_0(s) - Z_2(-s) = 0. \qquad (45)$$

Comparing (45) and (42), we can see that, except for those factors which are possibly cancelled in $Z_1^{*}(s)$, (and hence missing in $\rho_{11}^{*}(s)$) the zeros of the realized input reflection coefficient are the negatives of the zeros of the numerator of the output reflection coefficient.

Let us now observe how the zeros of $\rho_{11}^{*}(s)$ can be modified by such cancellations. A number of cases must be considered. We first assume that $Z_2(s) + Z_0(s) \neq 0$ at $s = -s_0$. In Lemma 3, we found the condition under which a cancellation in $Z_1^{*}(s)$ would take place when $Z_2(s)$ was augmented. This condition was the vanishing of $Z_2(-s) - Z_0(s)$ at $s = -s_0$ (condition (28)).

(a) If (28) does not vanish at $s = -s_0$, the factor $(s+s_0)$ will not be cancelled in $Z_1^*(s)$ and $Z_1^*(s)$ will have an extra t-zero according to Lemma 2.

But if $Z_2(s)$ is augmented by the factor $(s+s_0)$ Eq. (45) will become

$$Z_0(s) - Z_2(-s) \frac{(s-s_0)}{(s-s_0)} = 0 . \tag{46}$$

Thus, since the roots of this expression are the same as those of $Z_1^*(s) - 1 = 0$, this augmenting will produce an extra factor of $(s-s_0)$ in $\rho_{11}^*(s)$.

At the same time, the natural frequencies observed from the input port, which are zeros of the equation $1 + Z_1^*(s) = 0$, will be increased by the factor $(s+s_0)$, for those are now determined by the equation $Z_0(s) + Z_2(s) \frac{(s+s_0)}{(s+s_0)} = 0$. Hence, if the original reflection coefficient is called $\rho_{11}^*(s)$, and if the new reflection coefficient, after $Z_2(s)$ has been augmented by the factor $(s+s_0)$, is called $\rho_{11a}^*(s)$.
Using Eq. (41), we will have

$$\rho_{11a}^*(s) = \rho_{11}^*(s) \left(\frac{-s+s_0}{s+s_0} \right) \tag{47}$$

This relation can easily be extended to the case of an _n_th order augmenting factor.

(b) On the other hand, if (28) is equal to zero, the augmenting factor $(s+s_0)$ will be cancelled in $Z_1^*(s)$, as shown by Lemma 3. If we had realized $Z_2(s)$ without augmentation, the resulting $\rho_{11}^*(s)$ would have had a factor $(s+s_0)$ in the numerator, for (28) is equivalent to Eq. (45) and no cancellation would have been involved. But if $Z_2(s)$ is augmented by the factor $(s+s_0)$, an extra factor $(s-s_0)$ in the numerator of $\rho_{11}^*(s)$ will be produced. But the predicted cancellation of $(s+s_0)$ in $Z_1^*(s)$ will eliminate the factor $(s+s_0)$ from the numerator of $\rho_{11}^*(s)$. Hence, as a result of this augmentation, a zero of $\rho_{11}^*(s)$ will be shifted from the left-half plane to the right-half plane at the mirror image position. We will still have,

$$\rho_{11a}^*(s) = \rho_{11}^*(s) \left(\frac{-s+s_0}{s+s_0}\right) \tag{48}$$

but for this case a cancellation of $(s+s_0)$ is guaranteed to take place.

Let us now remove the earlier restriction that $Z_2(s) + Z_0(s) \neq 0$ at $s = -s_0$. Then, if the augmenting factor is of order n, the cancellation in $\rho_{11a}^*(s)$ depends on the factor $(s+s_0)^k$ in the numerator of $\rho_{11}^*(s)$. k can be determined by Lemma 2 and 3 as follows:

(1) If $Z_0(s) - Z_2(-s) \neq 0$ for $s = -s_0$ then $k = 0$.

(2) If $Z_0(s) - Z_2(-s)$ has a factor $(s+s_0)^m$,

$Z_0(s) + Z_2(s)$ has a factor $(s+s_0)^p$.

Then (a) if $p \geq m$, it follows that $k = 0$;

(b) if $p < m$, $k = m - p$.

The preceding discussion has thus established the following:

<u>Lemma 4.</u> If $Z_2(s)$ is augmented by a factor $(s+s_0)^n$ before realization, the resulting input reflection coefficient $\rho_{11a}^*(s)$ will be determined from the $\rho_{11}^*(s)$ which would have been obtained by realizing $Z_2(s)$ directly by the relation

$$\rho^*_{11a}(s) = \rho^*_{11}(s)\left(\frac{-s+s_0}{s+s_0}\right)^n , \tag{49}$$

where a possible factor $(s+s_0)^k$ in the numerator of $\rho^*_{11}(s)$ is determined by the above conditions (1) and (2).

D. ANALYSIS OF FIGURE 2

We are now ready to discuss the procedure for determining the output reflection coefficient $\rho_{22}(s)$ and then determining $Z_2(s)$ from $\rho_{22}(s)$. In Fig. 2 it is assumed that $Z_1(s)$ and $Z_0(s)$ are compatible and non-Foster. The reflection coefficients are defined as follows:

$$\rho_{11}(s) = \frac{Z_1(s) - 1}{Z_1(s) + 1} \tag{50}$$

$$\rho_{22}(s) = \frac{Z_2(s) - Z_0(-s)}{Z_2(s) + Z_0(s)} \tag{51}$$

The two-port is lossless, so for all possibilities of $Z_0(s)$ and $Z_1(s)$,

$$|\rho_{11}(j\omega)| = |\rho_{22}(j\omega)| \quad \forall\ \omega . \tag{52}$$

The task now is to identify $\rho_{22}(s)$ from $\rho_{11}(s)$. If no cancellations are involved, then the roots of

$$Z_1(s) + 1 = 0 \tag{53}$$

and

$$Z_2(s) + Z_0(s) = 0 \tag{54}$$

will be the natural frequencies of the network, and they will be the same.

Also, $$1 - Z_1(-s) = 0 \tag{55}$$

and

$$Z_2(s) - Z_0(-s) = 0 \tag{56}$$

will have the same roots. Hence, by using these relations in Eq. (51), it might be expected that $\rho_{22}(s)$ will have the form

$$\frac{1 - Z_1(-s)}{1 + Z_1(s)} \; . \tag{57}$$

But the rational function, (57) has poles at the poles of $Z_1(-s)$ and zeros at the poles of $Z_1(s)$, whereas $\rho_{22}(s)$ does not necessarily have them, as can be seen from Eq. (51).

Also, $\rho_{22}(s)$ has poles at the poles of $Z_0(-s)$ and zeros at the poles of $Z_0(s)$ which are not present in (57). Hence, if no cancellation exists and if we let

$$Z_1(s) = \frac{m_{11} + n_{11}}{m_{21} + n_{21}} \tag{58}$$

and

$$Z_0(s) = \frac{m_1 + n_1}{m_2 + n_2} , \tag{59}$$

then we will have

$$\rho_{22}(s) = \frac{1 - Z_1(-s)}{1 + Z_1(s)} \cdot \frac{m_{21} - n_{21}}{m_{21} + n_{21}} \cdot \frac{m_2 + n_2}{m_2 - n_2} \; . \tag{60}$$

The right side is (57) with the inclusion and deletion of poles and zeros according to the preceding discussion. In the most general case, the right side should have additional factors to account for possible cancellations.

So,
$$\rho_{22}(s) = \frac{1 - Z_1(-s)}{1 + Z_1(s)} \cdot \frac{m_{21} - n_{21}}{m_{21} + n_{21}} \cdot \frac{m_2 + n_2}{m_2 - n_2} \cdot \frac{h(-s)}{h(s)} \quad (61)$$

$$= \frac{1 - Z_1(-s)}{1 + Z_1(s)} \quad \frac{P_d(s)}{P_d(-s)} , \quad (62)$$

where $h(s)$ is a Hurwitz polynomial to be determined according to the discussion surrounding Lemmas 2, 3, and 4 and $P_d(s)$ is defined in the equation.

$h(s)$ is not yet known. Suppose that it is determined by some means so that $\rho_{22}(s)$ becomes known. Then, solving Eq. (51) for $Z_2(s)$ in terms of $\rho_{22}(s)$, we will have

$$Z_2(s) = \frac{Z_0(s) + Z_0(-s)}{1 - \rho_{22}(s)} - Z_0(s) . \quad (63)$$

It is obvious that one necessary condition for $h(s)$ is that it must be so chosen as to make $Z_2(s)$ of Eq. (63) positive real.

E. CONDITIONS FOR COMPATIBILITY

In the following discussion, it will be assumed that the network of Fig. 2 is stable. The given impedances $Z_1(s)$ and $Z_0(s)$ will then be said to be compatible and stable. To establish our major theorem, we will require a previous result of Chan and Kuh.[4]

Theorem 1. (Chan and Kuh) Let $Z_0(s)$ be a given rational function which may or may not be pr, but it is non-Foster. Then

(a) $$Z_2(s) = \frac{Z_0(s) + Z_0(-s)}{1 - \rho(s)} - Z_0(s) \tag{64}$$

is a positive real function, and

(b) $Z_0(s) + Z_2(s) \neq 0$ for $\mathrm{Re}\ s \geq 0$ (except for degenerate cases) under certain conditions on $\rho(s)$ which are listed in the Appendix because they are so extensive.

It is our purpose now to establish necessary and sufficient conditions on two functions $Z_1(s)$ and $Z_0(s)$ to be compatible. Strong use will be made of Theorem 1. Sufficient conditions will be determined by construction. That is, the impedance $Z_2(s)$ in Eq. (64) will be synthesized as a lossless, two-port terminated in a resistor, as in Fig. 3. The realized impedance will be called $Z_1^*(s)$, and conditions will be determined under which $Z_1^*(s) = Z_1(s)$.

To start, we shall first concentrate on the transmission zeros.

Theorem 2. Given two non-Foster rational functions $Z_1(s)$ and $Z_0(s)$ which may or may not be positive real

$$Z_1(s) = \frac{m_{11} + n_{11}}{m_{21} + n_{21}}, \qquad Z_0(s) = \frac{m_1 + n_1}{m_2 + n_2} \tag{65}$$

and $\quad 1 + Z_1(s) \neq 0$ in $\mathrm{Re}\ s \geq 0$.

Form the reflection coefficient $\rho(s)$

$$\rho(s) = \frac{1 - Z_1(-s)}{1 + Z_1(s)} \cdot \frac{m_{21} - n_{21}}{m_{21} + n_{21}} \cdot \frac{m_2 + n_2}{m_2 - n_2} \cdot \frac{h(-s)}{h(s)} \tag{66}$$

and then form $Z_2(s)$ according to Eq. (64). Realize $Z_2(s)$ as a lossless, two-port terminated in a resistor as in Fig. 3. Let the input impedance of the realized network terminated in $Z_0(s)$ be called $Z_1^*(s)$.

Then the transmission zeros of $Z_1^*(s)$ can be made equal to those of the given function $Z_1(s)$, if

(a) the t-zeros of $Z_0(s)$ are included among those of $Z_1(s)$,

(b) a Hurwitz polynomial $h(s)$ exists such that $\rho(s)$ in Eq. (66) satisfies the conditions of Theorem 1, thus making $Z_2(s)$ positive real.

This theorem will be proved for the right-half plane and $j\omega$-axis transmission zeros separately. First we consider the right-half plane transmission zeros.

We start with the conditions (a) and (b) of the theorem as given. So $Z_2(s)$ is pr and can be realized in the desired form. By Lemma 1, all the t-zeros of $Z_0(s)$ will be included among those of $Z_1^*(s)$; hence, only the t-zeros of $Z_2(s)$ remain to be considered.

From Eq. (64), the even part of $Z_2(s)$ can be found; it is

$$\text{Ev } Z_2(s) = \frac{1}{2}\left[Z_2(s)+Z_2(-s)\right] = \frac{1}{2}\left[Z_0(s)+Z_0(-s)\right]\left[\frac{1}{1-\rho(s)}+\frac{1}{1-\rho(-s)}-1\right]$$

$$= \frac{1}{2}\left[Z_0(s)+Z_0(-s)\right]\frac{1-\rho(s)\,\rho(-s)}{(1-\rho(s))\,(1-\rho(-s))} \tag{67}$$

Equation (66) implies

$$1-\rho(s)\,\rho(-s) = \frac{2\,(Z_1(s)+Z_1(-s))}{(1+Z_1(s))\,(1+Z_1(-s))}\ . \tag{68}$$

Now combine these with Eq. (65) to get

$$\text{Ev } Z_2(s) = \frac{1}{2}\left[Z_2(s)+Z_2(-s)\right]$$

$$= \frac{1}{2}\left[\frac{Z_0(s)+Z_0(-s)}{1-\rho(s)}\right]\frac{m_{11}\,m_{21}-n_{11}\,n_{21}}{(m_{11}+m_{21}+n_{11}+n_{21})\,(m_{11}+m_{21}-n_{11}-n_{21})}$$

$$\cdot\ \frac{1}{1-\rho(-s)}\ . \tag{69}$$

According to condition (a) of the theorem, $T_{Z_0} = m_1 m_2 - n_1 n_2$ is a factor of $T_{Z_1} = m_{11} m_{21} - n_{11} n_{21}$. Let

$$P(s) = m_{11}' m_{21}' - n_{11}' n_{21}' = \frac{m_{11} m_{21} - n_{11} n_{21}}{m_1 m_2 - n_1 n_2} = \frac{T_{Z_1}}{T_{Z_0}} . \tag{70}$$

Thus,

$$\text{Ev } Z_2(s) = \frac{Z_2(s) + Z_2(-s)}{2} =$$

$$= \frac{1}{2} \left(\frac{Z_0(s) + Z_0(-s)}{1 - \rho(s)} \right) \frac{P(s)}{(m_{11}+m_{21}+n_{11}+n_{21})(m_{11}+m_{21}-m_{21}-n_{21})} \cdot \frac{T_{Z_0}}{1-\rho(-s)} \tag{71}$$

By the definition of $Z_2(s)$ in Eq. (64), the last fraction of Eq. (71) can be put in the form

$$\frac{T_{Z_0}}{1-\rho(-s)} \doteq (Z_2(-s) + Z_0(-s))(m_2^2 - n_2^2) . \tag{72}$$

Similarly, from Eq. (64), the first fraction becomes

$$\frac{Z_0(s) + Z_0(-s)}{1 - \rho(s)} = Z_2(s) + Z_0(s) . \tag{73}$$

Hence, Eq. (71) becomes

$$\text{Ev } Z_2(s) = \frac{1}{2} \left(Z_2(s) + Z_2(-s) \right)$$

$$= \frac{1}{2} \left[\left(Z_0(s) + Z_2(s) \right) \left(m_2 + n_2 \right) \right] \frac{(m_2 - n_2)(Z_2(-s) + Z_0(-s))\, P(s)}{(m_{11}+m_{21}+n_{11}+n_{21})(m_{11}+m_{21}-n_{11}-n_{21})} \tag{74}$$

Let us analyze Eq. (74) factor by factor for the right-half plane zeros. The first factor $(Z_2(s)+Z_0(s))(m_2+n_2) \neq 0$ for $\operatorname{Re} s > 0$ by Theorem 1, because $Z_2(s) + Z_0(s) \neq 0$ for $\operatorname{Re} s > 0$. Furthermore, since (m_2+n_2) is the denominator of $Z_0(s)$, the right-half plane zeros of the term (m_2+n_2) will cancel with the poles of $Z_0(s)$. Next consider (m_2-n_2) It is the denominator of $Z_0(-s)$ and its right-half plane factors will therefore cancel with the poles of $Z_2(-s) + Z_0(-s)$.

Next consider $(m_{11}+n_{11}+m_{21}+n_{21})$. It is the numerator of $1 + Z_1(s)$, which cannot vanish in the right-half plane because the network is assumed to be stable. What is left from $\operatorname{Ev} Z_2(s)$ is

$$R(s) = \left(\frac{Z_0(-s) + Z_2(-s)}{m_{11}+m_{21}-n_{11}-n_{21}} \right) \cdot P(s) \tag{75}$$

There are certainly cancellations in the first fraction since both numerator and denominator represent the negatives of the natural frequencies, one observed from the output, the other from the input. But any unobservable natural frequencies may not cancel. As a result, there may be some right-half plane zeros of $Z_2(-s) + Z_0(-s)$ which are also t-zeros of $Z_2(s)$ (for they are zeros of $\operatorname{Ev} Z_2(s)$). By Lemma 2, these zeros will be cancelled completely in $Z_1^*(s)$ (as they should be) and thus will not be t-zeros of $Z_1^*(s)$. On the other hand, polynomial $P(s)$ and $(m_{11}+m_{21}-n_{11}-n_{21})$ may have common factors and thus cancellations. It is precisely these cancellations which will cause the resulting $Z_1^*(s)$ to be missing some of the t-zeros of $Z_1(s)$.

To overcome the problem, we must augment $Z_2(s)$ before realization. The order of augmentation can be determined by inspecting the order of cancellation between $P(s)$ and $(m_{11}+m_{21}-n_{11}-n_{21})$ and the order of the corresponding factor in $Z_2(-s) - Z_0(s)$ as shown in

Lemma 3. The rest of the zeros of $P(s)$ will be included among the t-zeros of $Z_1^*(s)$.

Finally, we conclude that the t-zeros of $Z_1^*(s)$ can always be made equal to the t-zeros of $Z_1(s)$ in the right-half plane, if necessary, by augmenting $Z_2(s)$.

Next, we turn to the t-zeros on the $j\omega$-axis. In Lemma 2, only cancellation of non $j\omega$-axis factors were considered. Let us analyze briefly the cancellations in $Z_1^*(s)$ of factors on the $j\omega$-axis. Suppose a factor $(s^2+\omega_0^2)^n$ is cancelled in $Z_1^*(s)$ of Eq. (4). Then it must be a factor of

$$Z_2(s) + Z_0(s) \tag{76}$$

For stability reasons, (76) cannot have zeros on the $j\omega$-axis except for two degenerate cases, which are the only ones that need to be considered. These are the cases where $Z_2(s)$ and $Z_0(s)$ have (1) a common zero of the first order $(s^2+\omega_0^2)$ or (2) a common pole of the first order. For both cases, it can easily be shown that $Z_1^*(s)$ has exactly a first-order cancellation. This is the only cancellation that $Z_1^*(s)$ can possibly have for factors on the $j\omega$-axis.

Now turn to a consideration of the t-zeros of $Z_1^*(s)$. There are a number of cases to consider.

As for the right-half plane, any t-zeros of $Z_0(s)$ on the $j\omega$-axis will be included automatically among the t-zeros of $Z_1^*(s)$; so we only need to consider t-zeros of $Z_2(s)$.

From Eqs. (64), (67), and (68), the even part of $Z_2(s)$ can be put into the following convenient form:

$$\frac{1}{2}(Z_2(s)+Z_2(-s)) = \left(\frac{Z_1(s) + Z_1(-s)}{(1+Z_1(s))\,(1+Z_1(-s))}\right)\left(\frac{(Z_0(s)+Z_2(s))\,(Z_0(-s)+Z_2(-s))}{Z_0(s) + Z_0(-s)}\right) \tag{77}$$

For the network to be stable, we must have the condition that

$$1 + Z_1(s) \neq 0 \quad \text{for} \quad \text{Re } s \geq 0. \tag{78}$$

Thus, there is no cancellation of factors on the $j\omega$-axis in the first fraction; and its zeros are the t-zeros of $Z_1(s)$, with the same multiplicity. After we simplify the second fraction, the denominator will consist of all the t-zeros of $Z_0(s)$ plus póles of $Z_2(s)$ and $Z_2(-s)$.

If we temporarily exclude the degenerate cases in which $Z_0(s)$ and $Z_2(s)$ have a common $j\omega$-axis zero or pole, the numerator will not equal zero for $s = j\omega$. Thus, all t-zeros of $Z_0(s)$ on the $j\omega$-axis in the second fraction are cancelled with the corresponding t-zeros of $Z_1(s)$ in the first fraction, and the remaining t-zeros of $Z_1(s)$ will be precisely those of $Z_2(s)$. These will also be among those of $Z_1^*(s)$, since no cancellation is involved and the result is proved.

Now consider a special case in which a simple pole of $Z_2(s)$ on the $j\omega$-axis may coincide with a t-zero of $Z_1(s)$. In this case, there must be a second-order cancellation in Eq. (77), since $Z_2(-s)$ will also have this pole. This cancellation will cause the zero of $\text{Ev } Z_2(s)$ at $s = j\omega_0$ to be two orders less than the zero of $P(s) = \dfrac{T_{Z_1}}{T_{Z_0}}$. However, in this case, it is easy to show that the t-zero of $Z_2(s)$ at $s = j\omega_0$ is two orders higher than the zero of $\text{Ev } Z_2(s)$. Hence, the result follows for this special case as well. From this we can conclude that all the t-zeros of $Z_2(s)$ will be included among the t-zeros of $Z_1^*(s)$. Hence

$$\text{t-zeros of } Z_1(s) = \text{t-zeros of } Z_1^*(s) \text{ for } s = j\omega \tag{79}$$

There still remains to consider the degenerate cases in which $Z_2(s)$ and $Z_0(s)$ have a common $j\omega$-axis zero or pole. Suppose $Z_2(s)$ has a simple pole at $s = j\omega_0$ where $Z_0(s)$ has a pole of order γ. Under this assumption, if we simplify the second fraction of Eq. (77), the numerator will not be equal to zero for $s = j\omega_0$ and the denominator will have zeros to exactly the same order as the t-zeros of $Z_0(s)$. Equation (77) shows that $\text{Ev } Z_2(s)$ has zeros at $s = j\omega_0$ of the same order as the zero of $P(s)$. Since $Z_2(s)$ has a simple pole, that implies the t-zero of $Z_2(s)$ at $s = j\omega_0$ is two degrees higher than the zero of $P(s)$. But because $Z_2(s)$ and $Z_0(s)$ have a common simple pole, there will be a cancellation of the factor $(s^2+\omega_0^2)$ in the realized $Z_1^*(s)$, as was pointed out. So Eq. (79) is still true for this case.

Finally, suppose $Z_2(s)$ has a simple zero at $s = j\omega_0$ where $Z_0(s)$ has a zero of order γ. It is easy to see this time that the t-zero of $Z_2(s)$ is two degrees higher than the zero of $P(s)$. Once again a cancellation in the realized $Z_1^*(s)$ will cause Eq. (79) to be true.

This completes the proof that we can realize a $Z_1^*(s)$ which has exactly the same t-zeros as those of $Z_1(s)$ over the whole complex plane. Q.E.D.

Our next task is to turn to the input reflection coefficient and to discover how the realized reflection coefficient $\rho_{11}^*(s)$, given in terms of $Z_1^*(s)$ by (41) is related to the original one $\rho_{11}(s)$, given in terms of $Z_1(s)$ by (50).

First, let us consider the magnitudes. Suppose the two-port N is synthesized using the $\rho(s)$ of (66) as $\rho_{22}(s)$ yielding an input reflection coefficient $\rho_{11}^*(s)$. Then, from (52),

$$|\rho_{11}^{*}(j\omega)| = |\rho(j\omega)| \quad \forall\, \omega \tag{80}$$

But, from its definition in Eq. (66),

$$|\rho(j\omega)| = \left|\frac{1 - Z_1(-j\omega)}{1 + Z_1(j\omega)}\right| \left|\frac{P_d(-j\omega)}{P_d(j\omega)}\right| = \left|\frac{1-Z_1(j\omega)}{1+Z_1(j\omega)}\right| = |\rho_{11}(j\omega)| \tag{81}$$

Hence, the result

$$|\rho_{11}^{*}(j\omega)| = |\rho_{11}(j\omega)| \quad \forall\, \omega \tag{82}$$

Next, let us show that if the magnitudes of $\rho_{11}^{*}(j\omega)$ and $\rho_{11}(j\omega)$ are equal and if the t-zeros of $Z_1^{*}(s)$ and $Z_1(s)$ are equal (both of which hypotheses have been shown to be the case), then the poles of $\rho_{11}(s)$ and $\rho_{11}^{*}(s)$ will be the same and the zeros of $\rho_{11}^{*}(s)$ will be equal either to those of $\rho_{11}(s)$ or their mirror images with respect to the $j\omega$-axis.

In fact, for two arbitrary rational functions having the same $j\omega$-axis magnitudes, poles and zeros are either equal or are mirror images of each other; in addition, the two may differ by an all-pass function. In the present case, stability requires the poles of both $\rho_{11}(s)$ and $\rho_{11}^{*}(s)$ to be in the left-half plane, and so they cannot be mirror images. The only remaining thing to prove is that $\rho_{11}(s)$ does not have an extra all-pass function.

Now Eq. (82) implies

$$\rho_{11}(s) = \pm\, \rho_{11}^{*}(s) \prod_{k} \left(\frac{s_k - s}{s_k + s}\right) \tag{83}$$

where $\operatorname{Re} s_k > 0$ in the regular all-pass function because of stability.

Since the <u>poles</u> of $\rho_{11}^{*}(s)$ and the <u>zeros</u> of the all-pass are in opposite half planes, no cancellation between them can take place. But consider the zeros of $\rho_{11}^{*}(s)$ and the poles of the all-pass. Suppose a pole of the all-pass does not cancel with a zero of $\rho_{11}^{*}(s)$. That is, suppose

$$\rho_{11}(s) = \pm \rho_{11}^{*}(s) \frac{s_0 - s}{s_0 + s} \tag{84}$$

with no cancellation. Then we can show that T_{Z_1} will have one more factor, $(s_0^2 - s^2)$, than T_{Z_1}. This contradicts the assumption of identical transmission zeros. Hence, all factors in the denominator of the all-pass in Eq. (83) must cancel with numerator factors of $\rho_{11}^{*}(s)$. To sum up: the poles of $\rho_{11}(s)$ = the poles of $\rho_{11}^{*}(s)$; the zeros of $\rho_{11}(s)$ are equal either to the zeros of $\rho_{11}^{*}(s)$ or to their mirror images with respect to the $j\omega$-axis.

Here we have a problem. The poles of $\rho_{11}(s)$ and $\rho_{11}^{*}(s)$ are the same but the zeros may be mirror images. But if $Z_1^{*}(s)$ is to equal $Z_1(s)$, $\rho_{11}^{*}(s)$ should equal $\rho_{11}(s)$. We will now show that any zeros of $\rho_{11}^{*}(s)$ which are mirror images of corresponding ones in $\rho_{11}(s)$ can be shifted to their mirror images positions without influencing the transmission zeros of $Z_1^{*}(s)$, by properly augmenting $Z_2(s)$.

In Eq. (66), let $Z_1(s)$ be replaced by $\frac{(m_{11}+n_{11})}{(m_{21}+n_{21})}$ and let the first two fractions be multiplied together. The result will be,

$$\rho(s) = \frac{m_{21} - n_{21} - m_{11} + n_{11}}{m_{21} + n_{21} + m_{11} + n_{11}} \cdot \frac{m_2 + n_2}{m_2 - n_2} \cdot \frac{h(-s)}{h(s)} . \tag{85}$$

Since the $\rho(s)$ in Eq. (85) is to be the $\rho_{22}(s)$ in Eq. (42) and the network is to be constructed by realizing $Z_2(s)$, the zeros of the resulting reflection coefficient $\rho_{11}^{*}(s)$ will be determined by the factors augmenting

$Z_2(s)$ as well as by the zeros of $\rho(s)$. How the augmenting factors of $Z_2(s)$ will influence the resulting reflection coefficient was thoroughly discussed in Lemma 4. So we will now concentrate only on the manner in which the zeros of $\rho(s)$ in Eq. (85) determine corresponding zeros of $\rho_{11}^{*}(s)$.

Consider the network in Fig. 3 which is the same as that of Fig. 2 except that $Z_1(s)$ is replaced by $Z_1^{*}(s)$. As a result, Sec. II-B for $Z_1(s)$ can be applied to $Z_1^{*}(s)$. We can tell immediately from Eqs. (55) and (56) and also from Eqs. (53) and (54) that, except for possible cancellations in $Z_1^{*}(s)$:

[1] If the factor $m_2 + n_2$ is excluded, zeros of $\rho_{11}^{*}(s)$ are the negatives of the zeros of $\rho(s)$.

[2] If the factor $m_2 - n_2$ is excluded, poles of $\rho_{11}^{*}(s)$ are the same as the poles of $\rho(s)$.

A special case which was not considered in Sec. II-D is the possibility that there may be common factors between the functions $Z_2(s)+Z_0(s)$ and $Z_0(s) + Z_0(-s)$, the even part of $Z_0(s)$. It follows that the same common factor will be present in $Z_2(s) - Z_0(-s)$ and $Z_2(s) + Z_0(s)$. A straightforward approach will show that the following additional relation is true.

[3] Common factors between $Z_2(s) - Z_0(-s)$ and $Z_2(s) + Z_0(s)$ will cancel in $\rho_{22}(s)$. But even though cancelled in $\rho_{22}(s)$, such a factor, say $(s+s_0)^n$, will automatically appear in the denominator of the realized $\rho_{11}^{*}(s)$ and the corresponding mirror image factor $(s-s_0)^n$ will appear in the numerator of $\rho_{11}^{*}(s)$.

Having determined these conditions, we can easily determine the zeros of $\rho_{11}^{*}(s)$. $\rho(s)$ of Eq. (85) will be used as $\rho_{22}(s)$; and Eq. (85) will be analyzed.

In the first place, note that there may be possible common factors between the second fraction of Eq. (85) and other fractions. Because of conditions [1] and [2] just stated, factors (m_2+n_2) and (m_2-n_2) will be excluded automatically from the numerator and denominator of $\rho_{11}^*(s)$, respectively. Hence, they will not be considered further.

In the remaining two fractions in Eq. (85), the all-pass function $\frac{h(-s)}{h(s)}$ is next considered. A factor in $h(s)$, $(s+s_0)$, will be in the denominator of $\rho_{11}^*(s)$ by [2], but the corresponding factor $(-s+s_0)$ in $h(-s)$ will also become $(s+s_0)$ in the numerator of $\rho_{11}^*(s)$ by [1]. So a cancellation in $\rho_{11}^*(s)$ will result. It thus follows that except for possible cancellations between $h(s)$ and the polynomial $(m_{21}-n_{21}-m_{11}+n_{11})$, no factor of $h(s)$ or $h(-s)$ will appear in $\rho_{11}^*(s)$.

Let us initially assume no cancellation of factors in numerator and denominator of Eq. (85). From [1] we see that the factors in the numerator of the first fraction which are actually in $1 - Z_1(-s)$ will change to the mirror image positions in the numerator of $\rho_{11}^*(s)$, (i.e., $Z_1(s)-1$). From [2], factors in the denominator of the first fraction, (i.e., factors of $1+Z_1'(s)$), will remain unchanged in the denominator of $\rho_{11}^*(s)$. It follows that zeros and poles of ρ_{11}^* will equal those of ρ_{11} in this case of no cancellations in $\rho(s)$. So,

$$\rho_{11}^*(s) = \pm\ \rho_{11}(s). \tag{86}$$

Now suppose some cancellations do take place in $\rho(s)$. These will be of two types:

(1) No factors of $h(-s)$ can cancel with a factor of $(m_{11}+m_{21}+n_{11}+n_{21})$ since the former is anti Hurwitz while the latter is Hurwitz. (The last is true because this polynomial is the numerator of $1 + Z_1(s)$ whose zeros are natural frequencies.) But there may be a common factor

$(s+s_0)^n$ between $h(s)$ and $(m_{21}-n_{21}-m_{11}+n_{11})$, which is Hurwitz. Without this cancellation, the factor $(s+s_0)^n$ in $(m_{21}-n_{21}-m_{11}+n_{11})$ would be the negative of a corresponding factor $(-s+s_0)^n$ needed in $\rho_{11}^*(s)$ because it is a factor that $\rho_{11}(s)$ has. In other words, because of this cancellation, $\rho_{11}^*(s)$ would be missing a factor $(-s+s_0)^n$. At the same time, $h(-s)$ in the numerator will introduce a corresponding factor $(-s+s_0)^n$ in $\rho(s)$. By [1], there would be introduced into $\rho_{11}^*(s)$ an extra factor $(s+s_0)^n$ which $\rho_{11}(s)$ does not have. To overcome this difficulty, suppose $Z_2(s)$ is augmented by the factor $(s+s_0)^n$. As shown in Lemma 4, $\rho_{11}^*(s)$ will be multiplied by the all-pass function

$\left(\frac{-s+s_0}{s+s_0}\right)^n$. This will shift the extra factor causing the difficulty in in $\rho_{11}^*(s)$ to its image position without influencing the t-zeros of $Z_1^*(s)$.

(2) In $\left(\frac{m_{21} - n_{21} - m_{11} + n_{11}}{m_{21} + n_{21} + m_{11} + n_{11}}\right)$, there may be common factors which are Hurwitz between numerator and denominator. The common factors can be shown to be the same as those between $1 + Z_1(s)$ and $Z_1(s) + Z_1(-s)$. The latter function has the same zeros as $T_{Z_0} \cdot P(s)$. The above mentioned condition [3] tells us that factors between T_{Z_0} and $1 + Z_1(s)$ cancelled in $\rho(s)$ will not cause any problem because they will be present in $\rho_{11}^*(s)$ anyway. On the other hand, possible factors between $P(s)$ and $1 + Z_1(s)$ need to be taken care of. But, it was shown in the proof of Theorem 2 that proper augmentation of $Z_2(s)$ with a factor common to $P(s)$ and $1 + Z_1(s)$ will make the t-zeros of $Z_1^*(s)$ the same as those of $Z_1(s)$. This same augmentation will introduce into $\rho_{11}^*(s)$ the zero factor $(s_0-s)^n$ which $\rho_{11}(s)$ has. (It will also introduce a pole factor $(s_0+s)^n$, as discussed in Theorem 2.)

We can conclude that Eq. (86) can be made true whether or not there are any cancellations in $\rho(s)$.

There remains only the question of sign. It is well known that if $\rho_{11}(s) = -\rho^*_{11}(s)$, then

$$Z_1(s) = \frac{1}{Z^*_1(s)} \tag{87}$$

If it turns out that $\rho^*_{11}(s)$ will equal $-\rho_{11}(s)$, it will be necessary to use a gyrator at the input port, as shown in Fig. 4, in order to invert the realized impedance.

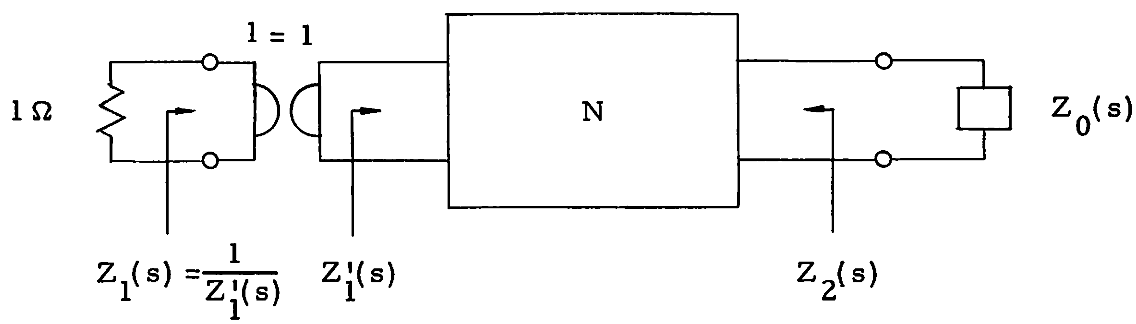


Fig. 4

The sign of $\rho^*_{11}(s)$ is determined by $h(s)$ which is selected to satisfy Eq. (66). If a minus sign must be introduced there, then $\rho^*_{11}(s)$ will equal to $-\rho_{11}(s)$ and a gyrator will be required.

With Theorem 2 and the results following it, we have established the following theorem:

Theorem 3. Given two non-Foster functions $Z_1(s)$ and $Z_0(s)$ which may or may not be pr, with $1 + Z_1(s) \neq 0$ for $\operatorname{Re} s \geq 0$, necessary and sufficient conditions for them to be compatible and stable are

(i) The t-zeros of $Z_0(s)$ are included among those of $Z_1(s)$.

(ii) A Hurwitz polynomial $h(s)$ exists such that

$$\rho(s) = \frac{1 - Z_1(-s)}{1 + Z_1(s)} \cdot \frac{m_{21} - n_{21}}{m_{21} + n_{21}} \cdot \frac{m_2 + n_2}{m_2 - n_2} \cdot \frac{h(-s)}{h(s)}$$

satisfies the conditions of Theorem 1,

where $$Z_1(s) = \frac{m_{11} + n_{11}}{m_{21} + n_{21}},$$

$$Z_0(s) = \frac{m_1 + n_1}{m_2 + n_2} .$$

III. METHOD OF SYNTHESIS

A. GENERAL COUPLING NETWORKS

Although the proof of Theorems 2 and 3 are quite lengthy, the actual synthesis of the coupling network is fairly simple and straightforward. It will be discussed in this section. We will first consider the general case where the coupling network N may contain both reciprocal and nonreciprocal elements.

Since the sufficient conditions for compatibility were actually established by realization, we have already discussed each of the steps in the synthesis. However, it would be useful to organize the synthesis procedure in a step-by-step form for clarity.

Given two non-Foster rational functions

$$Z_1(s) = \frac{m_{11} + n_{11}}{m_{21} + n_{21}} \quad , \quad Z_0(s) = \frac{m_1 + n_1}{m_2 + n_2}$$

(1) Determine a positive number R such that $R + Z_1(s) \neq 0$ for $\text{Re } s \geq 0$. Permissible $Z_1(s)$ functions are limited to those for which such a number R exists.

(2) Form the polynomials

$$T_{Z_1} = m_{11}m_{21} - n_{11}n_{21}$$

$$T_{Z_0} = m_1 m_2 - n_1 n_2$$

$$P(s) = \frac{T_{Z_1}}{T_{Z_0}}$$

T_{Z_0} must be contained in T_{Z_1} ; if it is not, Z_0 and Z_1 are not compatible.

(3) Determine a Hurwitz polynomial $h(s)$ such that

$$\rho(s) = \frac{1 - Z_1(-s)/R}{1 + Z_1(s)/R} \cdot \frac{m_{21} - n_{21}}{m_{21} + n_{21}} \cdot \frac{m_2 + n_2}{m_2 - n_2} \cdot \frac{h(-s)}{h(s)} \qquad (88)$$

satisfies the conditions of Theorem 1. $h(s)$ is not unique, even when it exists. A straightforward approach is to use the interpolation technique of Wohlers.[2] For a "canonical" coupling network, one having the fewest elements, it is necessary to choose an $h(s)$ of lowest possible order. The $h(s)$ selected will influence the factors by which $Z_2(s)$ is to be augmented in the next step and, hence, this selection should be guided by that consideration.

(4) Form $Z_2(s)$ from

$$Z_2(s) = \frac{Z_0(s) + Z_0(-s)}{1 - \rho(s)} - Z_0(s) \tag{89}$$

and reduce it to simplest form by cancelling common factors. Then augment Z_2 with the following factors:

(a) Common factors between $P(s)$ and $1 + Z_1(s)/R$ but not $1 - Z_1(s)/R$,

(b) Common factors between $h(s)$ and $1 - Z_1(-s)/R$ but not $1 + Z_1(-s)/R$.

(5) Realize Z_2 as a lossless, two-port N terminated in a R ohm resistance. The two-port N can be reciprocal if all the transmission zeros of $Z_2(s)$ are of even order. If not, $Z_2(s)$ cannot be augmented further but must be realized as a nonreciprocal two-port.

B. RECIPROCAL COUPLING NETWORK

If it is required that N be reciprocal, then the t-zeros of the augmented Z_2 must be of even order.

Except for certain cancellations, t-zeros of $Z_2(s)$ consist of right-half plane zeros of $P(s)$ and zeros of $h(-s)$. Thus, for a reciprocal coupling network N, the regular all-pass $h(-s)/h(+s)$ in (92) will contain: (1) as first-order factors those zeros of $P(s)$ which are of odd multiplicity; and (2) another all-pass function, say $b^2(-s)/b^2(s)$, which is a perfect square. Class (1) factors are to make all t-zeros of $Z_2(s)$ of even order. Class (2) factors may be necessary to make $\rho(s)$ satisfy Theorem 1. The factors used in augmenting $Z_2(s)$ are still the same as before. Since those account for factors that have cancelled, they are restored by augmenting $Z_2(s)$ in order to make t-zeros of $Z_2(s)$ even. Again, a canonical network will result if $b^2(-s)/b^2(s)$ has the least possible order.

There is a limitation on the realizability of reciprocal coupling networks. If there is a factor in $P(s)$ which is also among the t-zeros of $Z_0(s)$, then this factor in $P(s)$ must be of even order. For if it is of odd order, we would expect that it must be cancelled in the realized $Z_1^*(s)$. Cancellations can take place under the following two circumstances:

(1) There has been no augmentation in $Z_2(s)$. In this case, if there is to be a cancellation, Lemma 2 tells us that

$$Z_2(s) + Z_2(-s) \tag{90}$$

and

$$Z_0(s) + Z_2(s) \tag{91}$$

must have at least a common factor $(s+s_0)$. Since this is also a t-zero of $Z_0(s)$, then

$$Z_0(s) + Z_0(-s) \tag{92}$$

has this factor. Add (90) and (92) and then compare with (91); we see that

$$Z_0(-s) + Z_2(-s) \tag{93}$$

has this factor $(s+s_0)$ also. This means that

$$Z_0(s) + Z_2(s) \tag{94}$$

has the factor $(s-s_0)$, which contradicts the stability assumption.

(2) $Z_2(s)$ has been augmented by a factor $(s+s_0)$. For cancellation in $Z_1^*(s)$ in this case, Lemma 3 tells us that

$$Z_0(s) - Z_2(-s) \tag{95}$$

and

$$Z_0(s) + Z_0(-s) \tag{96}$$

have a common factor $(s+s_0)$. This implies that

$$Z_0(-s) + Z_2(-s) \tag{97}$$

also has the factor. This contradicts the stability assumption also. In other words, we cannot successfully find any $h(-s)/h(+s)$ to make $\rho(s)$ satisfy the conditions of Theorem 1 in this case.

IV. EXAMPLES

We shall now illustrate the procedure with a number of examples. They will be chosen to show a number of the possible variations. The numbered steps correspond to those given in Sec. III.

Example 1. The following example was given by Wohlers[2]. It will be considered here by our method for comparison purposes. Let

$$Z_1(s) = \frac{s+\frac{1}{17}}{s+17} \qquad Z_0(s) = \frac{s+\frac{1}{3}}{s+3} \tag{98}$$

Solution. (1) Since $Z_1(s)$ is pr, $1 + Z_1(s) \neq 0$ for $\text{Re } s \geq 0$.

(2) $T_{Z_1} = (s^2-1) = T_{Z_0}$

Hence, Z_0 and Z_1 can be compatible and

$$P(s) = 1 . \tag{99}$$

(3) $$\rho(s) = \frac{1 - \dfrac{\frac{1}{17} - s}{17 - s}}{1 + \dfrac{\frac{1}{17} + s}{17 + s}} \cdot \frac{17 - s}{17 + s} \cdot \frac{3 + s}{3 - s} \cdot \frac{h(-s)}{h(s)} \tag{100}$$

$$= \frac{144}{17s + 145} \cdot \frac{3 + s}{3 - s} \cdot \frac{h(-s)}{h(s)} \, . \tag{101}$$

Condition 1 of Theorem 1 requires $\rho(1) = 1$ and so $h(-1)/h(1) = 9/16$.

Condition 2 and 3 are satisfied automatically.

We conclude that the given functions are compatible.

Here if we choose $h(s)$ to be of first order, the coupling network will be nonreciprocal; if $h(s)$ is a perfect square, N will be reciprocal.

Let us do both.

A first order $h(s)$ is $(a+s)$. Hence,

$$\left. \frac{a - s}{s + a} \right|_{s = 1} = \frac{9}{16} \, ; \qquad a = \frac{25}{7} \tag{102}$$

$$\rho(s) = \frac{144}{17s + 145} \cdot \frac{3 + s}{3 - s} \cdot \frac{\frac{25}{7} - s}{\frac{25}{7} + s} \tag{103}$$

On the other hand, for a reciprocal network, $h(s) = (a'+s)^2$

$$\left. \left(\frac{a'-s}{a'+s} \right)^2 \right|_{s = 1} = \frac{9}{16} \qquad a' = 7 \tag{104}$$

$$\rho'(s) = \frac{144}{17s + 145} \cdot \frac{3 + s}{3 - s} \cdot \left(\frac{7-s}{7+s} \right)^2 \, . \tag{105}$$

(4) $Z_2(s)$ is calculated from (64) as

$$Z_2(s) = Z_0(s) \frac{\frac{Z_0(-s)}{Z_0(s)} + \rho(s)}{1 - \rho(s)} . \tag{106}$$

For nonreciprocal N

$$Z_2(s) = \frac{17s + \frac{7225}{21}}{17s + \frac{75}{7}} . \tag{107}$$

For reciprocal N

$$Z_2'(s) = \frac{51s^2 + 700s + 14161}{51s^2 + 1428s + 441} . \tag{108}$$

No augmentation is necessary for both cases, since there exists no such common factor as stated in the 4th step in Sec. III - A.

(5) We synthesize $Z_2(s)$ first as a lossless two port terminated in a 1-Ω resistor; then we remove the resistor. The result is shown in Fig. 5

The reciprocal coupling network found by synthesizing $Z_2'(s)$ is shown in Fig. 6.

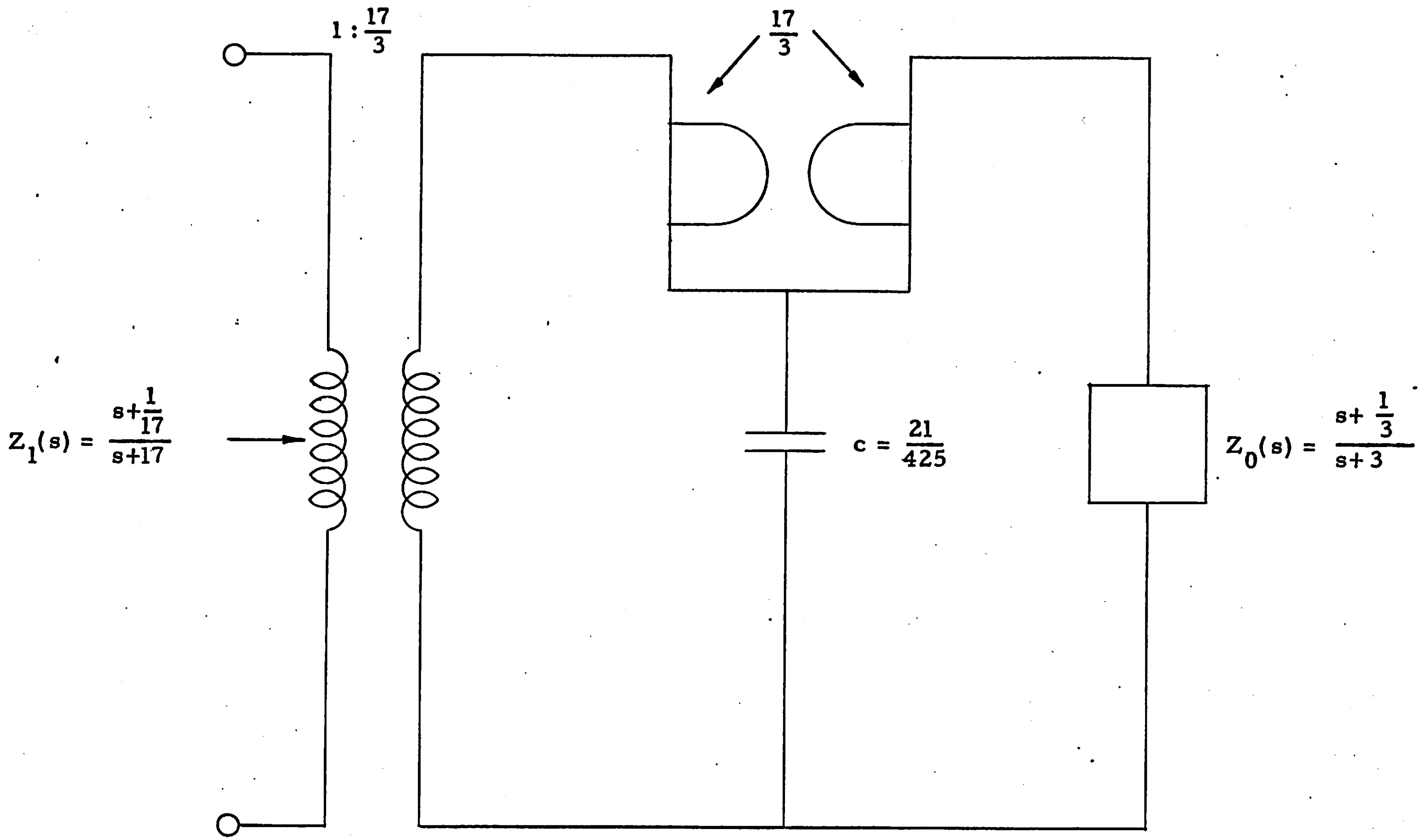


Fig. 5

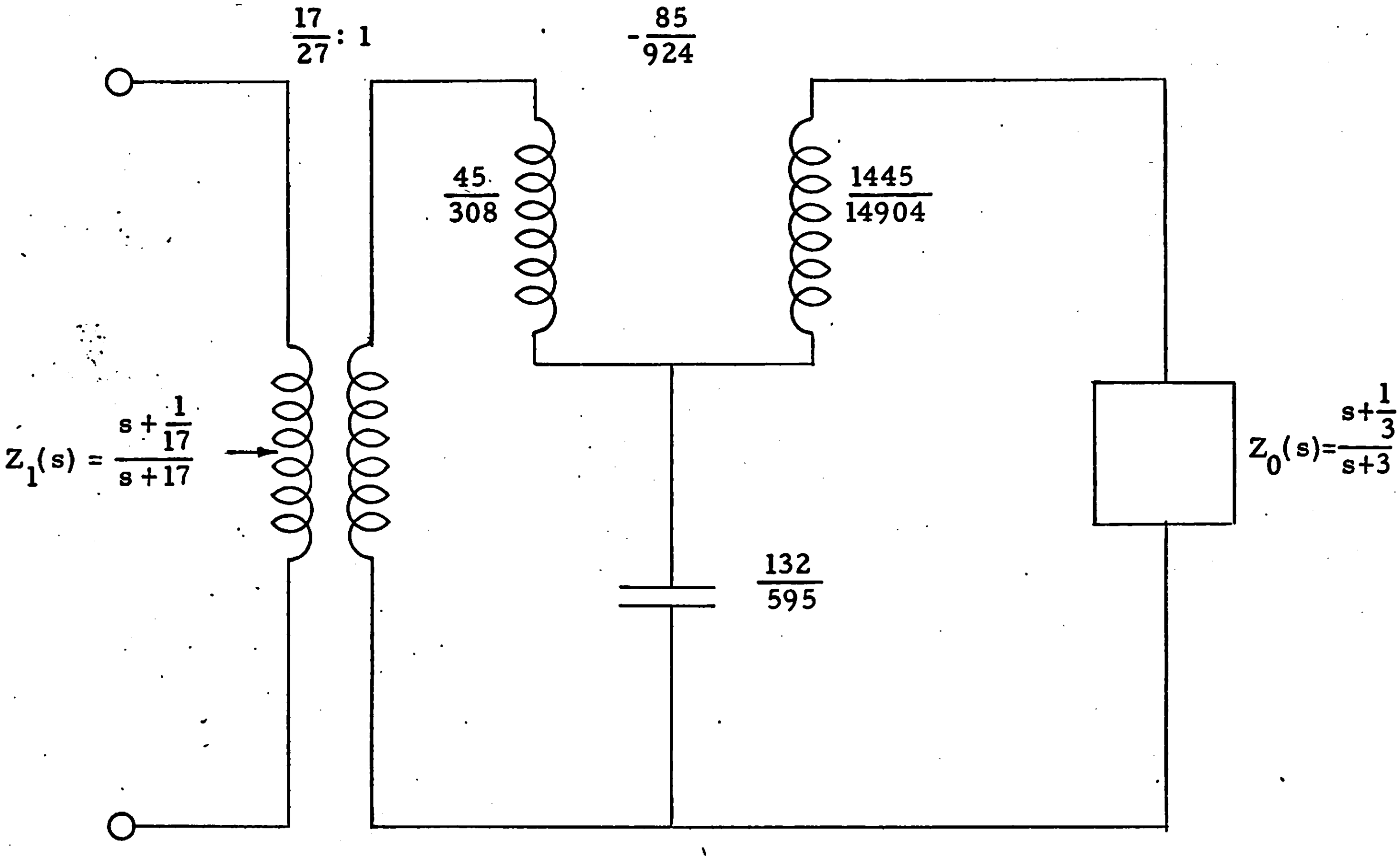


Fig. 6

The same result as in Fig. 6 would be obtained by Wohlers' approach. In that case, the coupling network would be realized from its two-port parameters.

Example 2. Given $$Z_1(s) = \frac{s^3 + 8s^2 + 56s + 64}{s^3 + 16s^2 + 32s + 32} \tag{109}$$

$$Z_0(s) = \frac{-s^2 - 3s + 2}{s + 4} \tag{110}$$

where $Z_1(s)$ is pr, $Z_0(s)$ is obviously not.

(1) $1 + Z_1(s) \neq 0$ for Re $s \geq 0$.

(2) $$T_{Z_1} = (s^2 - 8)(s^2 - 16)^2 \tag{111}$$

$$T_{Z_0} = (s^2 - 8) \tag{112}$$

Thus, T_{Z_0} is a factor of T_{Z_1}.

$$P(s) = (s^2 - 16)^2 = (s+4)^2(s-4)^2 \tag{113}$$

(3) $$\rho(s) = \frac{1 - \dfrac{-s^3 + 8s^2 - 56s + 64}{-s^3 + 16s^2 - 32s + 32}}{1 + \dfrac{s^3 + 8s^2 + 56s + 64}{s^3 + 16s^2 + 32s + 32}} .$$

$$\vdots \quad \frac{-s^3 + 16s^2 - 32s + 32}{s^3 + 16s^2 + 32s + 32} \cdot \frac{s + 4}{-s + 4} \cdot \frac{h(-s)}{h(s)} . \tag{114}$$

A straightforward derivation shows a possible $h(s)$ to be $h(s) = 1$.

(4) $Z_2(s)$ is found from (64) as

$$Z_2(s) = \frac{s^2 + 9s + 2}{s + 8} \tag{115}$$

If we compute $1 + Z_1(s)$, there results

$$(s+4)(s+2)(s+6) \tag{116}$$

Comparing this with $P(s)$ in (113) shows that there is a common factor $(s+4)$, we thus have to augment $Z_2(s)$ by this factor

$$Z_2(s) = \frac{s^2 + 9s + 2}{s + 8} \cdot \frac{s + 4}{s + 4} \qquad (117)$$

In synthesizing $Z_2(s)$, either the Darlington-c section can be removed first, or the pole at $s = \infty$ can be removed first. If we do the latter, the result is shown in Fig. 7.

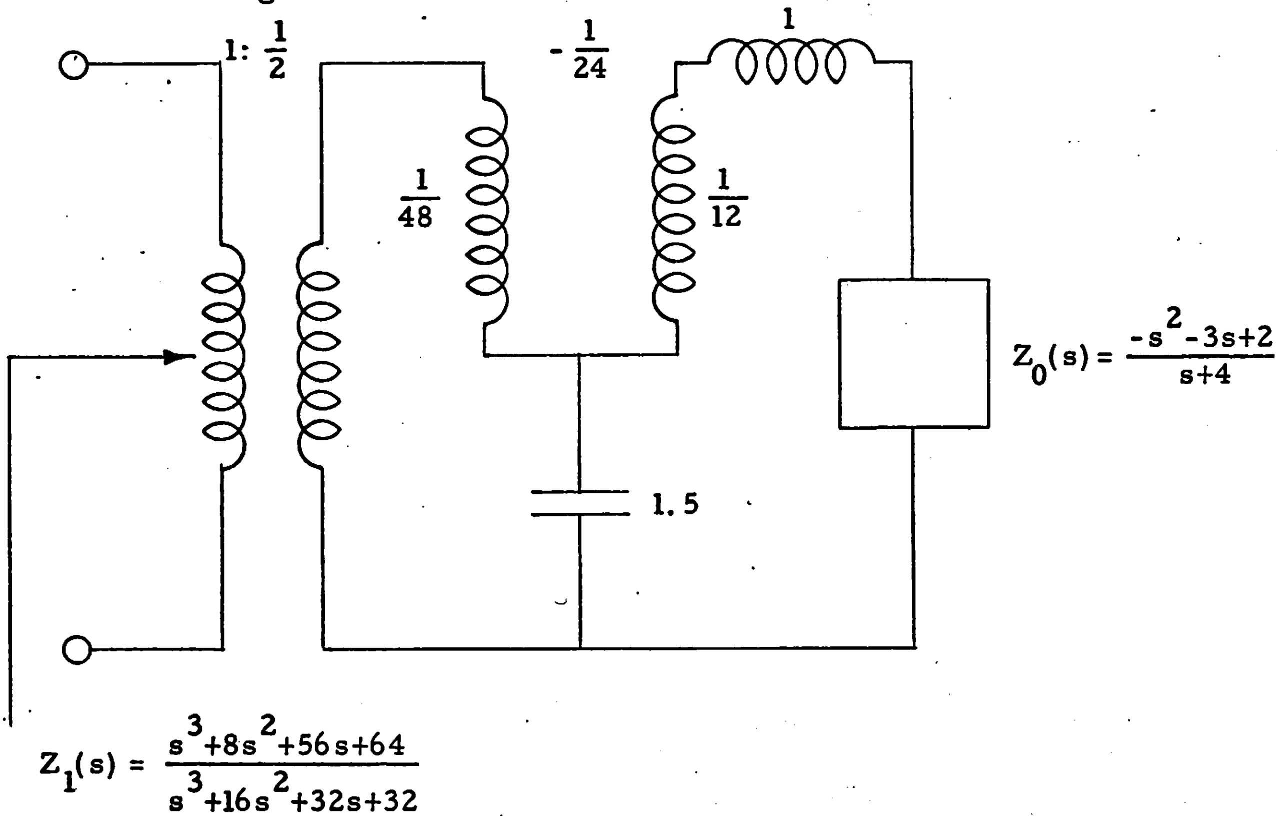


Fig. 7

<u>Example 3.</u> Let us have $Z_1(s) = \frac{s^2 + 4s + 6}{s^2 + 6s + 4}$ $\qquad Z_0(s) = \frac{s + 3}{s + 4}$ (118)

Both $Z_1(s)$ and $Z_0(s)$ are pr.

(1) $1 + Z_1(s) \neq 0$ for $\text{Re } s \geq 0$

(2) $T_{Z_1} = (s^2-12)(s^2-2)$ (119)

$T_{Z_0} = (s^2-12)$ (120)

T_{Z_0} is contained in T_{Z_1}

$P(s) = (s^2-2)$ (121)

(3) $$\rho(s) = \frac{1 - \dfrac{s^2 - 4s + 6}{s^2 - 6s + 4}}{1 + \dfrac{s^2 + 4s + 6}{s^2 + 5s + 5}} \cdot$$

$$\cdot \frac{s^2 - 6s + 4}{s^2 + 6s + 4} \cdot \frac{4 + s}{4 - s} \cdot \frac{h(-s)}{h(s)} \quad (122)$$

$$= \frac{-(s+1)}{s^2 + 5s + 5} \cdot \frac{4 + s}{4 - s} \cdot \frac{h(-s)}{h(s)} \quad (123)$$

$\rho(s)$ will satisfy the conditions of Theorem 1 if $h(s) = 1 + s$.

$$\rho(s) = \frac{-(s+1)}{s^2 + 5s + 5} \cdot \frac{4 + s}{4 - s} \cdot \frac{1 - s}{1 + s} \quad (124)$$

Note there is a common factor $(s+1)$ between $h(s)$ and $1 - Z_1(-s)$. Hence, ρ_{11}^* will have a zero at $s = 1$ in the wrong half plane.

(4) $Z_2(s)$ is found to be

$$Z_2(s) = \frac{s + 1}{s + 2} \quad (125)$$

We have to augment $Z_2(s)$ by the factor $(s+1)$ in order to shift the zero of $\rho_{11}^*(s)$ into the correct position. A cancellation in $Z_1^*(s)$ is guaranteed by Lemma 4.

$$Z_2(s) = \frac{s+1}{s+2} \cdot \frac{s+1}{s+1} \tag{126}$$

The realization is shown in Fig. 8.

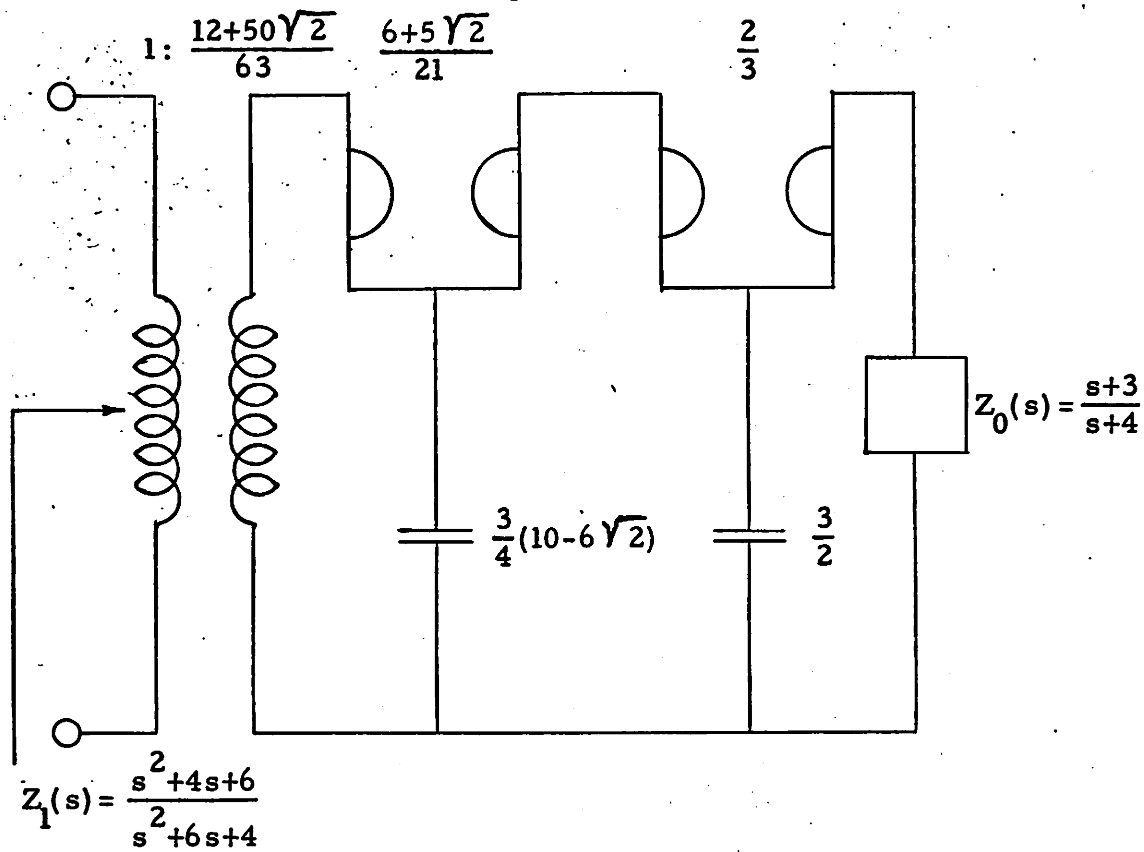


Fig. 8

Example 4. Given $$Z_1(s) = \frac{0.9s^2+2.1s-1}{1.65s^3+3.85s^2+15.9s+7.6} \tag{127}$$

$$Z_0(s) = 0.8+0.15s+\frac{1}{s-1} = \frac{0.15s^2+0.65s+0.2}{s-1} \tag{128}$$

$Z_0(s)$ is the impedance of a tunnel diode, taking all its parameters into account. Neither $Z_1(s)$ nor $Z_0(s)$ is pr.

Solution. (1) $1 + Z_1(s) = \frac{1}{D(s)}(1.65s^3+4.75s^2+18s+6.6) \neq 0$ for $\text{Re } s \geq 0$ (129)

(2) $$T_{Z_0} = 0.8(s^2+\frac{1}{4}) \tag{130}$$

$$T_{Z_1} = 30.40(s^2+\frac{1}{4}) \tag{131}$$

T_{Z_0} is contained in T_{Z_1} with $P(s) = 1$. (132)

(3) Form the reflection coefficient $\rho(s)$

$$\rho(s) = \frac{-1.65s^3+2.95s^2-13.8s+8.6}{1.65s^3+4.75s^2+18s+6.6} \cdot \frac{s-1}{-s-1} \cdot \frac{h(-s)}{h(s)} \tag{133}$$

From Theorem 1, at $s = \pm \frac{j}{2}$, the t-zeros of $Z_0(s)$

$$\rho(\pm \frac{j}{2}) = 1.$$

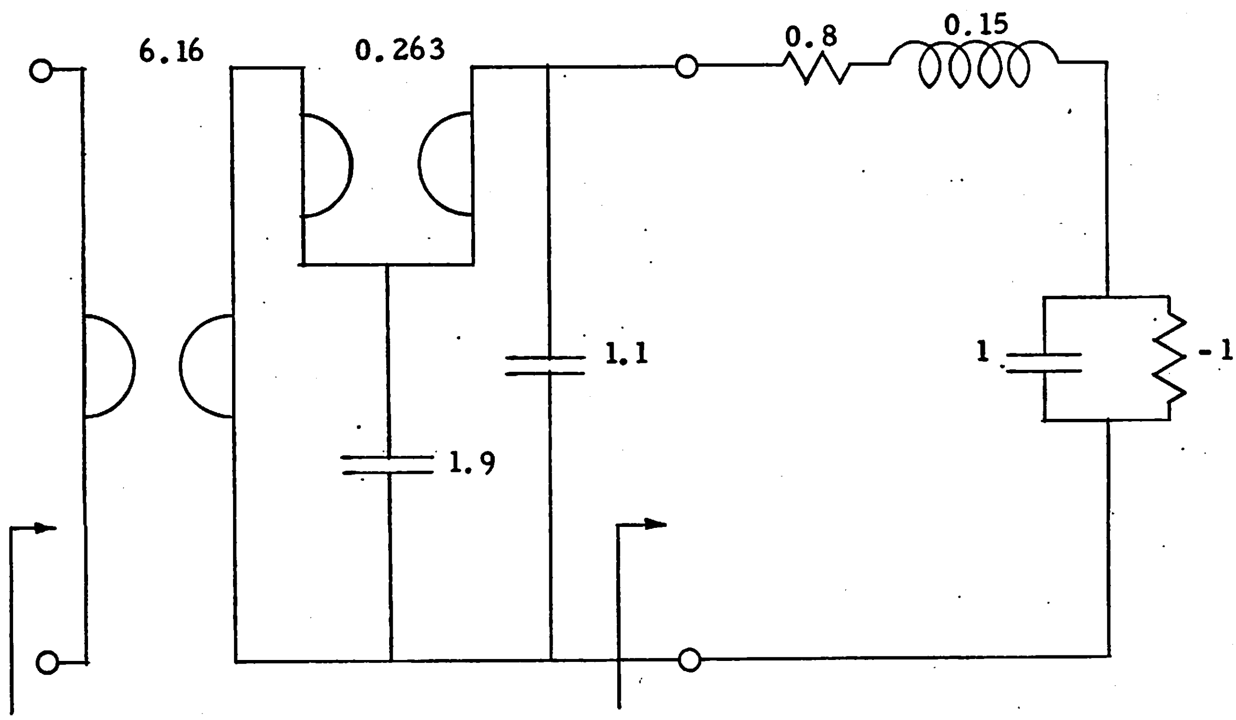
$h(s) = -(s+2.1)$ satisfies this condition as well as other conditions.

Hence $$\rho(s) = -\frac{1.65s^3-2.95s^2+13.8s-8.6}{1.65s^3+4.75s^2+18s+6.6} \cdot \frac{s-1}{s+1} \cdot \frac{2.1-s}{2.1+s} \tag{134}$$

(4) $Z_2(s)$ is found to be

$$Z_2(s) = \frac{s+0.2}{1.1s^2+0.6s+7.6} \tag{135}$$

Note, a -1 is introduced to $\rho(s)$, hence when $Z_2(s)$ is realized a gyrator will have to be used at the input port. The realization is shown in Fig. 9.



$$Z_1(s) = \frac{0.9s^2+2.1s-1}{1.65s^3+3.85s^2+15.9s+7.6} \qquad Z_0(s) = 0.8+0.15s+\frac{1}{s-1}$$

Fig. 9

If we were to realize $Z_1'(s) = \frac{1}{Z_1(s)}$, the gyrator at the input port would have been saved.

ACKNOWLEDGMENT

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APPENDIX A

I. Proof of Lemma 1

Lemma 1. If $Z_1(s)$ and $Z_0(s)$, as shown in Fig. 1, are compatible, then all the zeros of T_{Z_0} will be included among the zeros of T_{Z_1}.

Proof. If $Z_0(s)$ and the lossless two port are defined by

$$Z_0(s) = \frac{m_1 + n_1}{m_2 + n_2} \tag{A1}$$

and

$$t(s) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{A2}$$

respectively where $t(s)$ is the same as in (8), then $Z_1(s)$ will be

$$Z_1(s) = \frac{A(m_1+n_1) + B(m_2+n_2)}{C(m_1+n_1) + D(m_2+n_2)} \tag{A3}$$

If there is no cancellation in Ev Z_1, then

$$T_{Z_1} = (m_1 m_2 - n_1 n_2)(AD - BC) = T_{Z_0}(AD - BC) \tag{A4}$$

In this case the zeros of T_{Z_0} are included among those of T_{Z_1}. It must be shown that if there are cancellations, the cancelled factors must be only those of $AD - BC$.

The proof is carried out in two steps.

(1) T_{Z_0} has a zero of nth order at $s = -s_0$ but

$$AD - BC \neq 0 \text{ at } s = -s_0 . \tag{A5}$$

To prove that $Z_1(s)$ also has a t-zero at $s = -s_0$ of order n is equivalent to proving that there is no cancellation of the factor $(s+s_0)$ between the numerator and the denominator of $Z_1(s)$ or equivalently, to prove that at $s = -s_0$, the numerator and the denominator of $Z_1(s)$ are not simultaneously equal to zero. At $s = -s_0$, m_1, m_2, n_1 and n_2 are not all zero. Let $n_1 \neq 0$. (By a similar approach, each of the others in turn can be assumed $\neq 0$.)

In (A3) replace n_2 by the equation

$$n_2 = \frac{m_1 m_2}{n_1} \qquad \text{for } s = -s_0 \tag{A6}$$

(A3) becomes

$$Z_1(-s_0) = \frac{(An_1+Bm_2)(m_1+n_1)}{(Cn_1+Dm_2)(m_1+n_1)} \tag{A7}$$

We will prove by contradiction that both the numerator and the denominator will not simultaneously be zero.

At $s = -s_0$, if $(m_1+n_1) = 0$ then from (A3), $(m_2+n_2) = 0$ also. This means that there must be a common factor $(s+s_0)$ between the numerator and the denominator of $Z_0(s)$ which contradicts the assumption. Next, if

$$An_1 + Bm_2 = 0 \tag{A8}$$

$$Cn_1 + Dm_2 = 0$$

(Note that $n_1 \neq 0$.) For this set to have a solution requires that

$$AD - BC = 0 \tag{A9}$$

which contradicts (A5). Hence, no cancellation can take place for this case and Z_1 will have an nth order to zero at $s = -s_0$.

(2) T_{Z_0} has a zero of the nth order at $s = -s_0$ but AD - BC also has a zero at $s = -s_0$ to mth order. In this case, cancellations are certainly possible. What will be shown is that T_{Z_1} must always have a zero at $s = -s_0$ at least to the nth order.

In the proof, we need to decompose the lossless two-port N into m + 1 cascaded sections. To justify the scheme, a previous result by Oono[5] is given below with slight modifications.

"A lossless two-port N is completely determined up to within a possible gyrator at the output and a twist of the output leads by $Z_2(s) = \frac{D + B}{C + A}$ and the strict right-half plane zeros of AD - BC."

In the statement, $Z_2(s)$ is defined as in Fig. 2. To determine N, we realize $Z_2(s)$ as a cascade of Darlington-E sections, one for each factor $(s+s_0)$ of AD - BC. Since there is a factor $(s+s_0)^m$, we end up with m + 1 sections as shown in Fig. A1, where the first one contains all the other transmission zeros.

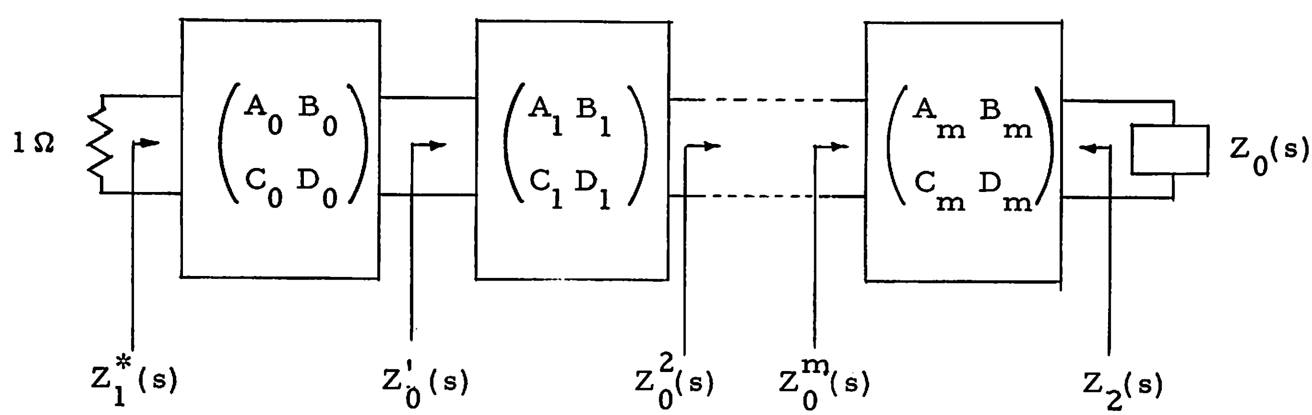

Fig. A 1

In Fig. A1 ,

$$t_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \quad i = 1, 2, \ldots m \tag{A10}$$

is the corresponding polynomial chain matrix of the ith section. $Z_0^i(s)$ is defined in Fig. A1 ; also

$$A_i D_i - B_i C_i = (s+s_0)(s+\bar{s}_0)(s-s_0)(s-\bar{s}_0) \tag{A11}$$

and

$$A_0 D_0 - B_0 C_0 \neq 0 \text{ at } s = -s_0 \tag{A12}$$

By Oono's theorem, either

$$Z_1^*(s) = K\, Z_1(s) \tag{A13}$$

or

$$Z_1^*(s) = \frac{1}{K\, Z_1(s)} \tag{A14}$$

Hence, $T_{Z_1^*}$ and T_{Z_1} have the same zeros.

Let us first consider $Z_0^m(s)$ as shown in Fig. A2.

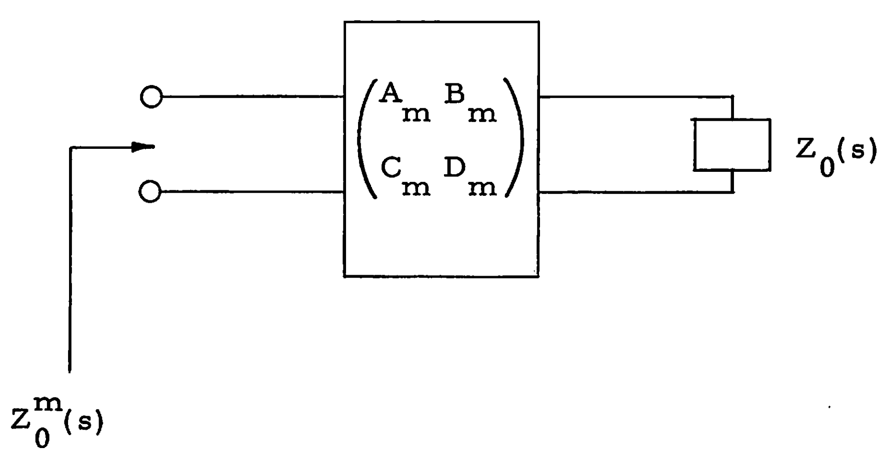

Fig. A2

From (A11), $A_m D_m - B_m C_m = 0$ at $s = -s_0$ (A15)

but $$\frac{d}{ds}(A_m D_m - B_m C_m)\Big|_{s=-s_0} \neq 0. \tag{A16}$$

Now we prove that the cancellation of the factor $(s+s_0)^k$ in

$$Z_0^m(s) = \frac{A_m Z_0 + B_m}{C_m Z_0 + D_m} \tag{A17}$$

can at most be of the first order.

For both the numerator and the denominator of Z_0^m to equal zero, it is necessary that

$$Z_0 + \frac{B_m}{A_m}\Big|_{s=-s_0} = 0 \tag{A18}$$

$$Z_0 + \frac{D_m}{C_m}\Big|_{s=-s_0} = 0 \tag{A19}$$

since $A_m(-s_0) \neq 0$ and $C_m(-s_0) \neq 0$. If the numerator and denominator were to have a second order zero at $-s_0$, it would be necessary to have

$$\frac{d}{ds}(Z_0(s)) + \frac{d}{ds}\left(\frac{B_m}{A_m}\right)\Big|_{s=-s_0} = 0 \tag{A20}$$

$$\frac{d}{ds}(Z_0(s)) + \frac{d}{ds}\left(\frac{D_m}{C_m}\right)\Big|_{s=-s_0} = 0 \tag{A21}$$

By subtracting these two, there results,

$$\frac{d}{ds}\left(\frac{B_m}{A_m} - \frac{D_m}{C_m}\right)\Bigg|_{s=-s_0} = 0 \tag{A22}$$

which implies

$$\frac{d}{ds}\left(A_m D_m - B_m C_m\right)\Big|_{s=-s_0} = 0 \tag{A23}$$

which contradicts (A16). So, a second order zero is not possible.

We can conclude that for each section, at most a first order cancellation can take place, no matter what $Z_0(s)$ is. It follows that for m cascaded sections in Fig. A1, at most an $\underline{m}$th order cancellation can take place. For the last section,

$$A_0 D_0 - B_0 C_0 \Big|_{s=-s_0} \neq 0 \tag{A24}$$

we then can use the result established in part I to see that no further cancellation can take place for $s = -s_0$.

Since the maximum possible order of the t-zero of Z_1^* at $s = -s_0$ equals $n + m$, and at most an $\underline{m}$th order factor is cancelled, T_{Z_1} has at least an $\underline{n}$th order zero at $s = -s_0$. Q.E.D.

APPENDIX B

Conditions of Theorem 1.

The Laurent series expansions around $s_i(s_r, s_p$ or other frequencies) for the following functions are given by

$$Z_0(s) = a_{-m}(s-s_i)^{-m} + a_{-(m-1)}(s-s_i)^{-(m-1)} + \ldots + a_0 + a_1(s-s_i) + \ldots$$

$$Z_0(-s) = b_{-n}(s-s_i)^{-n} + b_{-(n-1)}(s-s_i)^{-(n-1)} + \ldots + b_0 + b_1(s-s_i) + \ldots$$

$$-\frac{Z_0(-s)}{Z_0(s)} = c_{-k}(s-s_i)^{-k} + c_{-(k-1)}(s-s_i)^{-(k-1)} + \ldots + c_0 + c_1(s-s_i) + \ldots$$

$$\rho(s) = d_{-\ell}(s-s_i)^{-\ell} + d_{-(\ell-1)}(s-s_i)^{-(\ell-1)} + \ldots + d_0 + d_1(s-s_i) + \ldots$$

$Z_2(s) = k_{-1}(s-j\omega_i)^{-1} + k_0 + \ldots$ where k_{-1} is real and positive.

<u>Theorem 1.</u>

Let $Z_0(s)$ be a given rational impedance which may be either active or passive but non-Foster. Then

$$Z_2(s) = \frac{Z_0(s) + Z_0(-s)}{1 - \rho(s)} - Z_0(s)$$

is a positive real function and $Z_0(s) + Z_2(s) \neq 0$ for $\operatorname{Re} s \geq 0$, except for degenerate cases, if and only if $\rho(s)$ satisfies the following three conditions

Condition 1(a) : In the RHP, $1 - \rho(s)$ is not zero except at a Ev $Z_0(s)$ zero s_r or order r or possibly at poles of $Z_0(s)$. The latter case will be included in Condition 3. In the former case,

$$d_0 = c_0 = 1$$

and if $Z_0(s_r)$ is regular,

$$d_i = 0, \; i = 1, 2, \ldots, r-1$$

And if $Z_0(s_r)$ is a pole of order m

$$d_i = c_i = 0, \quad i = 1, 2, \ldots, r+m-1$$

and

$$d_i = c_i ; \qquad i = r+m, \ldots, r+2m-1$$

Condition 1(b) : On the $j\omega$-axis, the function $1 - \rho(s)$ may have a first order zero at $j\omega_0$, which is neither a Ev Z_0 zero nor a pole of $Z_0(s)$. Then

$$\rho(s) = 1 + d_1 (s - j\omega_0) + \ldots$$

where d_1 is real and

$$\frac{\mathrm{Re}\, Z_0(j\omega)}{-d_1} > 0.$$

At a zero of Ev $Z_0(s)$, $j\omega_r$, of order r,

$$d_0 = c_0 = 1,$$

and if $Z_0(j\omega_r)$ is regular,

$$d_i = 0, \; i = 1, 2, \ldots, r-1$$

and $$d_{r-1} \neq 0 \text{ (if degenerate)}$$

And $Z_0(j\omega_r)$ is a pole of order m,

$$d_i = c_i = 0\,, \quad i = 1, 2, \ldots, r + m - 1$$

and $$d_i = c_i\,, \qquad i = r + m, r + m + 1, \ldots, r + 2m - 1$$

and $$d_{r+2m-1} = c_{r+2m-1} - k_{-1}c_{r+m/a-m} \quad \text{(if degenerate)}$$

where k_{-1} is real and positive.

Condition 2: On the $j\omega$-axis, $|\rho(j\omega)|$ satisfies the following

$$|\rho(j\omega)| > 1 \text{ if } \operatorname{Re}(Z_0(j\omega)) < 0$$

$$|\rho(j\omega)| \leq 1 \text{ if } \operatorname{Re}(Z_0(j\omega)) \geq 0$$

Condition 3(a): In the open RHP, $\rho(s)$ is analytic except at s_p, which is a pole of $Z_0(s)$ of order m and is a pole of $Z_0(-s)$ of order n, $m, n \geq 0$. Then

$$d_i = 0, \quad i < m - n - 1$$

and $$d_i = c_i\,, \quad i = n - n,\ m - n + 1 \ldots \begin{cases} 2m-n+q & \text{if } m \geq n,\ (m-n=1) \text{ and } c_{m-n+i}=0,\ i=1, 2, \ldots q \\ m - n & \text{(if } m \geq n) \\ 2m - n - 1 & \text{(if } m < n) \end{cases}$$

and if $m = 0$, the second equation is not needed.

Condition 3(b): On the $j\omega$-axis, $\rho(s)$ is analytic. At $s = j\omega_p$ which is a pole of $Z_0(s)$ and $Z_0(-s)$ of order m, we have

$$d_i = c_i \, , \quad i = 0 \, , \, 1, \, . \, . \, . \, m - 1$$

and
$$d_{m-1} = c_{m-1} + k_{-1} (1-c_0)/a_{-m} \quad \text{(if degenerate)}$$

and if
$$c_0 = 1 \quad \text{and} \quad c_i = 0, \quad i = 1, \, 2, \, . \, . \, . \, q$$

$$d_i = c_i \, , \quad i = 0, \, 1, \, . \, . \, . \, m + q$$

$$d_{m+q} = c_{m+q} - k_{-1} \, c_{1+q} \, / \, a_{-m}$$

and if $m = 1$

$$d_0 = \frac{c_0 \, a_{-1} + k_{-1}}{a_{-1} + d_{-1}} \qquad \text{(degenerate)}$$

and if $m = 1$ and $c_0 = 1$, $d_0 = c_0 = 1$ and

$$d_1 = \frac{a_{-1} c_1}{a_{-1} + k_{-1}}$$

where k_{-1} is real and positive .

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