REPRESENTATION OF A DIFFERENTIAL SYSTEM

by

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1. INTRODUCTION

Starting with a general formulation of a differential system (DS), two other system representations are derived, and the relationships between the three represenations are investigated. The first, called a generalized dynamical system (GDS), studied by Roxin [1, 2, 3], is defined via the "attainability function" of the DS. Following Warga [4], a relaxed differential system (RDS) is defined by augmenting the set of "permissible" velocities of the DS. It is shown that the set of solutions of the derived GDS and the set of solutions of the derived RDS coincide with the closure of the set of trajectories of the DS in the topology of uniform convergence over finite intervals. In a less general and somewhat different setting, Warga [4], has obtained this relationship between a DS and its derived RDS. The most important difference between Warga's formulation and this one is the replacement of continuity with respect to the time parameter (required by Warga) by local integrability.

2. NOTATIONS AND DEFINITIONS

2.1. If x is a vector in n-dimensional real Euclidean space R^n, the norm of x is denoted by

$$|x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} .$$

2.2. $C[t_0, t_1]$ denotes the real Banach space of all continuous functions $x: [t_0, t_1] \to R^n$ where the norm of x is given by

$$\|x\| = \max_{t_0 \leq t \leq t_1} |x(t)| .$$

2.3. If A is a subset of a topological space, $\overline{A}$ denotes its closure.

2.4. If $x \in R^n$ and A is a subset of R^n, then

$$\rho(A, x) = \rho(x, A) = \inf\left\{|x - y| \;\middle|\; y \in A\right\} ;$$

and for $\mathcal{E} > 0$

$$S_{\mathcal{E}}(A) = \left\{x \in R^n \;\middle|\; \rho(x, A) < \mathcal{E}\right\}.$$

3. THE DIFFERENTIAL SYSTEM (DS)

The differential system (DS) under consideration is described by the vector differential equation

$$\dot{x}(t) = f\big(x(t), t, u(t)\big)$$

where the following conditions hold:

(i) $t \in R^1$, $x(t) \in R^n$ and $u(t) \in U$ for each t, where U is a fixed, bounded, nonempty subset of R^m.

(ii) For each fixed t, f is continuous in (x, u) for all (x, u) in $R^n \times R^m$.

(iii) There is a locally integrable function k, such that for every u in U and x, x' in R^n,

$$|f(x, t, u) - f(x', t, u)| \leq k(t) \; |x - x'| \; .$$

(iv) There are a locally integrable function ℓ and positive numbers M and N such that for every x in R^n and $u \in U$,

$$|f(x, t, u)| \leq \ell(t) \; (M + N|x|) \; .$$

Definition 3.1. A function $u : [t_0, t_1] \to R^m$ is called an <u>admissible control</u>, if it is measurable, and if $u(t) \in U$ for each $t \in [t_0, t_1]$.

Definition 3.2. A function $x : [t_0, t_1] \to R^n$ is called a <u>trajectory</u> of the DS, if there exists an admissible control u, such that x is absolutely continuous and $\dot{x}(t) = f(x(t), t, u(t))$ a.e. in $[t_0, t_1]$. The trajectory is said to start at $(x(t_0), t_0)$ and end at $(x(t_1), t_1)$.

Definition 3.3. (i) For x_0 in R^n, t_0, t_1 in R^1 with $t_0 \leq t_1$, let $T(x_0, t_0, t_1)$ be the set of those vectors x in R^n for which there exists a trajectory of the DS starting at (x_0, t_0) and ending at (x, t_1). In other words, $T(x_0, t_0, t_1)$ is the set of states attainable <u>at</u> time t_1 by the DS starting at (x_0, t_0) and using an admissible control.

(ii) Let $Q(x_0, t_0, t_1) = \overline{T(x_0, t_0, t_1)}$. Q is called the <u>attainability function</u> of the generalized dynamical system (GDS), derived from the DS. (See Section 5)

Definition 3.4. The set of all trajectories defined on $[t_0, t_1]$ and starting at (x_0, t_0) is denoted by $\mathcal{T}(x_0, t_0, t_1)$. $\mathcal{T}(x_0, t_0, t_1)$ is considered as a subset of $\mathcal{C}[t_0, t_1]$.

Lemma 3.1. Let $x_0 \in R^n$; $t_0, t_1 \in R^1$ with $t_0 \leq t_1$. Then

$$\bigcup_{t_0 \leq t \leq t_1} \left(T(x_0, t_0, t) \right)$$

is a bounded subset of R^n, and hence $\mathcal{T}(x_0, t_0, t_1)$ is a bounded subset of $\mathcal{C}[t_0, t_1]$.

Proof. Let $t \in [t_0, t_1]$. Each point of $T(x_0, t_0, t)$ is given by

$$x(t) = x_0 + \int_{t_0}^{t} f\left(x(s), s, u(s)\right) ds$$

for some admissible control u. From condition (iv) of the DS we have

$$|x(t)| \leq |x_0| + \int_{t_0}^{t} M\, \ell(s)\, ds + \int_{t_0}^{t} N\, \ell(s)\, |x(s)|\, ds$$

so that by Gronwall's lemma [5, p.11]

$$|x(t)| \leq |x_0| \exp\left(\int_{t_0}^{t} N\, \ell(s)\, ds \right) + \int_{t_0}^{t} \exp\left(\int_{s}^{t} N\, \ell(\tau)\, d\tau \right) M\, \ell(s)\, ds = \varphi(x_0, t_0, t).$$

Clearly $\varphi(x_0, t_0, t)$ is a continuous monotonic function of t and is independent of the admissible control u so that the result follows.

Lemma 3.2. (i) $\mathcal{T}(x_0, t_0, t_1)$ is an equicontinuous subset of $[t_0, t_1]$.

(ii) $T(x_0, t_0, t_1)$ is jointly continuous in all its arguments, i.e., given x_0 in R^n; t_0, t_1 in R^1 with $t_0 \leq t_1$; and $\mathcal{E} > 0$; there is a $\delta > 0$ such that if $|x_0 - x'_0| < \delta$, $|t_0 - t'_0| < \delta$, $|t_1 - t'_1| < \delta$, and $t'_0 \leq t'_1$, then

$$T(x_0, t_0, t_1) \subset S_{\mathcal{E}}[T(x'_0, t'_0, t'_1)] ,$$

and

$$T(x'_0, t'_0, t'_1) \subset S_{\mathcal{E}}[T(x_0, t_0, t_1)] .$$

Proof. Consider two trajectories x and x' starting at (x_0, t_0) and (x'_0, t'_0), respectively, and determined by the same admissible control u defined for $t \geq \min\{t_0, t'_0\}$. Thus,

$$x(t) = x_0 + \int_{t_0}^{t} f\Big(x(s), s, u(s)\Big) ds ,$$

whereas

$$x'(t') = x'_0 + \int_{t'_0}^{t'} f\Big(x'(s), s, u(s)\Big) ds .$$

Let

$$\Delta x(t, t') = |x(t) - x'(t')| \le |x(t) - x(t')| + |x(t') - x'(t')|$$

$$= \Delta x_1(t, t') + \Delta x_2(t') \, . \tag{3.1}$$

(i): To prove (i), let B be a finite bound of $\mathscr{T}(x_0, t_0, t_1)$. Such a bound exists by Lemma 3.1. Then for $t_0 \le t \le t' \le t_1$,

$$\Delta x_1(t, t') = |x(t) - x(t')| \le \int_t^{t'} \left|f\Big(x(s), s, u(s)\Big)\right| ds$$

$$\le \int_t^{t'} (M + NB) \, \ell(s) \, ds \, . \tag{3.2}$$

since ℓ is locally integrable, given $\varepsilon > 0$, there is a $\delta_1 > 0$ such that if $|t - t'| < \delta_1$ then

$$\int_t^{t'} (M + NB) \, \ell(s) \, ds < \frac{\varepsilon}{2} \, . \tag{3.3}$$

Since the admissible control u was arbitrary, we see from (3.2) and (3.3) that, for every x in $\mathscr{T}(x_0, t_0, t_1)$ and for every t, t' in $[t_0, t_1]$ with $|t - t'| < \delta_1$,

$$|\Delta x_1(t, t')| < \frac{\varepsilon}{2} \, . \tag{3.4}$$

and (i) is proved.

To prove (ii), consider

$$\Delta x_2(t_1') = |x(t_1') - x'(t_1')| \le |x_0 - x_0'| + \Big|\int_{t_0}^{t_1'} f\big(x(s), s, u(s)\big)\, ds - \int_{t_0'}^{t_1'} f\big(x'(s), s, u(s)\big) ds\Big|$$

$$\le |x_0 - x_0'| + \int_{t_0'}^{t_1'} \Big|f\big(x(s), s, u(s)\big) - f\big(x'(s), s, u(s)\big)\Big| ds + \int_{t_0}^{t_0'} \Big|f\big(x(s), s, u(s)\big)\Big| ds$$

$$\le |x_0 - x_0'| + \int_{t_0'}^{t_1'} k(s)\, |x(s) - x'(s)|\, ds + \int_{t_0}^{t_0'} (M + NB)\, \ell(s)\, ds . \tag{3.5}$$

Let

$$\beta = \frac{1}{2}\mathcal{E}\left(\exp \int_{t_0}^{t_1'} k(s)\, ds\right)^{-1} .$$

Let $\delta_2 > 0$ be such that if $|t_0 - t_0'| < \delta_2$, then

$$\int_{t_0}^{t_0'} (M + NB)\, \ell(s)\, ds < \frac{1}{2}\beta .$$

Then if $|t_0 - t_0'| < \delta_2$ and $|x_0 - x_0'| < (1/2)\beta$ we have, from (3.5),

$$\Delta x_2(t_1') \le \beta + \int_{t_0'}^{t_1'} k(s)\, |x(s) - x'(s)|\, ds .$$

By Gronwall's lemma [5, p. 11],

$$\Delta x_2(t_1') \le \beta \exp\left(\int_{t_0'}^{t_1'} k(s)\, ds\right) \le \frac{1}{2}\mathcal{E} \;. \tag{3.6}$$

Substituting (3.6) and (3.4) into (3.1), we see that for $\delta = \min\{\delta_1, \delta_2, \frac{1}{2}\beta\}$, $\Delta x(t_1, t_1') \le \mathcal{E}$ and hence (ii) is proved.

4. THE GENERALIZED DYNAMICAL SYSTEM (GDS)

Following Roxin [3], we define a GDS by an "attainability function" $F(x_0, t_0, t)$, which for every x_0 in R^n; t_0, t_1 in R^1 with $t_0 \le t_1$, represents the set of states attainable by the system <u>at</u> time t_1, starting in state x_0 at time t_0. The attainability function F satisfies the following axioms:

4.A1. For x_0 in R^n; t_0, t_1 in R^1 with $t_0 \le t_1$, $F(x_0, t_0, t_1)$ is a non-empty, compact subset of R^n.

4.A2. For x_0 in R^n and t_0 in R^1, $F(x_0, t_0, t_0) = \{x_0\}$.

4.A3. $F(x_0, t_0, t_1)$ is jointly continuous in all its arguments, i.e., given x_0 in R^n; t_0, t_1 in R^1 with $t_0 \le t_1$; and $\mathcal{E} > 0$; there is a $\delta > 0$ such that if $|x_0 - x_0'| < \delta$, $|t_0 - t_0'| < \delta$, $|t_1 - t_1'| < \delta$, and $t_0' \le t_1'$, then

$$F\left(x_0, t_0, t_1\right) \subset S_{\mathcal{E}}\left[F\left(x_0', t_0', t_1'\right)\right],$$

and

$$F\left(x_0', t_0', t_1'\right) \subset S_{\mathcal{E}}\left[F\left(x_0, t_0, t_1\right)\right].$$

4.A4. For x_0 in R^n; t_0, t_1, t_2 in R^1 with $t_0 \le t_1 \le t_2$

$$F\left(x_0, t_0, t_2\right) = \bigcup_{x \in F(x_0, t_0, t_1)} F\left(x, t_1, t_2\right) .$$

Definition 4.1. A function $x : [t_0, t_1] \to R^n$ is called a motion of the GDS if for $t_0 \le \tau_0 \le \tau_1 \le t_1$

$$x(\tau_1) \in F\left(x(\tau_0), \tau_0, \tau_1\right) .$$

The motion is said to start at $\left(x(t_0), t_0\right)$ and end at $\left(x(t_1), t_1\right)$.

Remark 4.1. From axiom 4.A4 it is clear that every motion is a continuous function of time.

Definition 4.2. The set of motions of the GDS defined on $[t_0, t_1]$ and starting at (x_0, t_0) is denoted by $\mathcal{M}(x_0, t_0, t_1)$. $\mathcal{M}(x_0, t_0, t_1)$ is considered as a subset of $\mathcal{C}[t_0, t_1]$.

The following lemma is proved in Roxin [3].

Lemma 4.1. $\mathcal{M}(x_0, t_0, t_1)$ is a closed subset of $\mathcal{C}[t_0, t_1]$.

5. THE DERIVED GDS

The function Q of Definition 3.3(ii) is used to derive a GDS from the given DS.

Theorem 5.1. The function Q of Definition 3.3(ii) satisfies the axioms 4.A1 - 4.A4 of an attainability function for a GDS.

Proof. Conditions (i) - (iv) of Section 3 guarantee the existence of a solution to the DS for every initial condition (x_0, t_0) and every admissible control. Therefore $T(x_0, t_0, t_1)$ is nonempty. By Lemma 3.1, $T(x_0, t_0, t_1)$ is bounded so that $Q(x_0, t_0, t_1)$ is nonempty and compact. Hence Q satisfies 4.A1.

For a DS it is evident that $T(x_0, t_0, t_0) = \{x_0\}$ so that $Q(x_0, t_0, t_0) = \{x_0\}$.

By Lemma 3.2, T is jointly continuous in all its arguments. Since Q is the closure of T, it follows that Q is also jointly continuous and hence satisfies 4.A3.

From the definition of the function T it is clear that for x_0 in R^n, and t_0, t_1, t_2 In R^1 with $t_0 \leq t_1 \leq t_2$

$$T(x_0, t_0, t_2) = \bigcup_{x \in T(x_0, t_0, t_1)} T(x, t_1, t_2) . \tag{5.1}$$

By Lemma 3.2, T satisfies the hypothesis of Lemma A1 of the appendix, so that

$$\left\{ \overline{\bigcup_{x \in T(x_0, t_0, t_1)} T(x, t_1, t_2)} \right\} = \bigcup_{x \in \overline{T(x_0, t_0, t_1)}} \overline{T(x, t_1, t_2)} . \tag{5.2}$$

Then from (5.2) and Definition 3.3(ii)

$$Q(x_0, t_0, t_2) = \bigcup_{x \in Q(x_0, t_0, t_1)} Q(x, t_1, t_2) .$$

This completes the proof of Theorem 5.1.

6. RELATIONSHIP BETWEEN MOTIONS AND TRAJECTORIES

Let a, b be fixed numbers with $a \le b$ and x_0 a fixed element of R^n; let $I = [a, b]$; let $\mathcal{M} = \mathcal{M}(x_0, a, b)$ be the set of motions of the derived GDS defined on $[a, b]$ and starting at (x_0, a); and let $\mathcal{T} = \mathcal{T}(x_0, a, b)$.

Theorem 6.1. $\mathcal{M} = \overline{\mathcal{T}}$

Proof. From the definitions of a trajectory and a motion, and from the definitions of Q, it is evident that $\mathcal{M} \supset \mathcal{T}$. By Lemma 4.1 $\mathcal{M}$ is closed, so that

$$\mathcal{M} = \overline{\mathcal{M}} \supset \overline{\mathcal{T}} . \tag{6.1}$$

In order to show inclusion in the converse direction it will be demonstrated that for every motion x in $\mathcal{M}$, there is a trajectory $y \in \mathcal{T}$ arbitrarily close to x.

Let $\mathcal{E} > 0$ be given. By Lemma 3.2 $\mathcal{T}$ is equicontinuous so that, there is $\delta_1 > 0$ such that if t, t' are in I with $|t - t'| < \delta_1$, then

$$|y(t) - y(t')| < \frac{\mathcal{E}}{3} \qquad (6.2)$$

for every $y \in \mathcal{T}$.

Let $x \in \mathfrak{M}$ be given. Then x is uniformly continuous on I so that, there is $\delta_2 > 0$ such that if t, t' are in I with $|t - t'| < \delta_2$, then

$$|x(t) - x(t')| < \frac{\mathcal{E}}{3} . \qquad (6.3)$$

Letting $\delta = \min\{\delta_1, \delta_2\}$, we choose a finite sequence $a = t_0 < t_1 < \cdots < t_p = b$ such that $t_k - t_{k-1} < \delta$ for each k. Denote $x_k = x(t_k)$ for each k.

By 4.A3, there is $\eta_{p-1} > 0$ such that if $|z - x_{p-1}| < \eta_{p-1}$, then

$$Q(x_{p-1}, t_{p-1}, t_p) \subset S_{\mathcal{E}/3}[Q(z, t_{p-1}, t_p)] . \qquad (6.4)$$

For $0 \le k \le p - 2$, there is $\eta_k > 0$ such that if $|z - x_k| < \eta_k$, then

$$Q(x_k, t_k, t_{k+1}) \subset S_{r_k}[Q(z, t_k, t_{k+1})] , \qquad (6.5)$$

where $\quad r_k = \min\left\{\frac{\mathcal{E}}{3}, \eta_{k+1}\right\}$.

Now let $y_1 \in T(x_0, t_0, t_1)$ such that $|y_1 - x_1| < r_0$ and for $2 \le k \le p$, let $y_k \in T(y_{k-1}, t_{k-1}, t_k)$ such that $|y_k - x_k| < r_{k-1}$. This is possible because of (6.5), (6.4), and the fact that Q is the closure of T. Since each $y_k \in T(y_{k-1}, t_{k-1}, t_k)$, there is a trajectory v_k starting at (y_{k-1}, t_{k-1}) and ending at (y_k, t_k). Also for $t_{k-1} \leq t \leq t_k$,

$$|x(t) - v_k(t)| \leq |x(t) - x_k| + |x_k - y_k| + |y_k - v_k(t)|$$
$$\leq \mathcal{E}$$

from (6.2), (6.3) and because $|x_k - y_k| \leq r_k \leq (\mathcal{E}/3)$. Let $y \in \mathcal{J}$ be defined by $y(t) = v_k(t)$ for $t_{k-1} \leq t \leq t_k$. Then $\|x - y\| \leq \mathcal{E}$. Since $\mathcal{E} > 0$ and $x \in \mathcal{M}$ are arbitrary. $\mathcal{M} \subset \overline{\mathcal{J}}$, and the theorem is proved.

7. THE RELAXED DIFFERENTIAL SYSTEM (RDS)

Following Warga [4], we derive a relaxed differential system from the DS of Section 3. The class of differential systems under study is more general than that investigated by Warga. The difference lies in the more general boundedness and Lipschitzian conditions on f (see Section 3) and, more critically, the replacement of continuity in t by local integrability.

Definition 7.1. For the DS of Section 3, let

$$F(x, t) = \left\{ y \in R^n \;\middle|\; y = f(x, t, u), u \in U \right\}.$$

Thus F(x, t) represents the set of "permissible velocities" of the DS at the phase (x, t).

Definition 7.2. For x in R^n and t in R^1 let $G(x, t)$ be the convex closure of $F(x, t)$.

Definition 7.3. (i) A function $x:[t_0, t_1] \to R^n$ is called a relaxed trajectory if x is absolutely continuous and

$$\dot{x}(t) \in G(x(t), t) .$$

a.e. in $[t_0, t_1]$. The relaxed trajectory is said to start at $(x(t_0), t_0)$ and end at $(x(t_1), t_1)$.

(ii) The set of relaxed trajectories starting at (x_0, t_0) and defined on $[t_0, t_1]$ is denoted by $\mathcal{R}(x_0, t_0, t_1)$. $\mathcal{R}(x_0, t_0, t_1)$ is considered as a subset of $\mathcal{C}[t_0, t_1]$.

Let a, b with $a \leq b$ be fixed finite numbers and x_0 a fixed vector in R^n; let $I = [a, b]$; let $\mathcal{R} = \mathcal{R}\left(x_0, a, b\right)$, and let $\mathcal{T} = \mathcal{T}(x_0, a, b)$.

Lemma 7.1 is standard, and Lemma 7.2 is proved in Cullum [6].

Lemma 7.1. Let A be a closed convex subset of R^n. Then $x^* \in A$ if and only if for each $y \in R^n$

$$\inf_{x \in A} \langle y, x \rangle \leq \langle y, x^* \rangle \leq \sup_{x \in A} \langle y, x \rangle .$$

Lemma 7.2. Let $\varphi_m : I \to R^n$ for $m = 1, 2, 3, \ldots,$ and $\tilde{\varphi} : I \to R^n$ be measurable functions, uniformly bounded by an integrable function, and suppose that for each $y \in R^n$ and each measurable subset E of I,

$$\int_E \langle y, \varphi_m(t) \rangle \, dt \to \int_E \langle y, \tilde{\varphi}(t) \rangle \, dt .$$

Then, for each $y \in R^n$,

$$\overline{\lim_n} \langle y, \tilde{\varphi}_m(t) \rangle \geq \langle y, \tilde{\varphi}(t) \rangle \geq \underline{\lim_n} \langle y, \tilde{\varphi}_m(t) \rangle$$

a.e. in I.

Lemma 7.3. Let $x \in \mathcal{R}$ be fixed. Then for each $\mathcal{E} > 0$, there exist measurable functions α_i, u_i for each $i = 1, \ldots, n+1$ such that for each t in I, $\alpha_i(t) \geq 0$ with

$$\sum_i \alpha_i(t) = 1 ,$$

$u_i(t) \in U$ and

$$\int_a^t \left| \dot{x}(s) - \sum_i \alpha_i(s) \, f\Big(x(s), \; s, \; u_i(s)\Big) \right| ds \leq \mathcal{E} . \tag{7.1}$$

Proof. Let $q : I \to R^1$ be a measurable function. A point t in I is said to be a regular point of q if

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} |q(t) - q(s)| \, ds = 0$$

we note that if q is in L_1, almost all points of I are regular points of q.

For t in I and $u \in U$, let $g(t, u) = f(x(t), t, u)$. Let U_0 be a countable dense subset of U. For each u in U_0 let T_u be the set of regular points of the function $\dot{x}$. Let

$$T = T_x \bigcap \left\{ \bigcap_{u \in U_0} T_u \right\} .$$

Then measure (T) = measure (I).

Let γ be any positive number. Then for each $t \in T$, there exists numbers $\beta_i(t) \geq 0$, $1 \leq i \leq n+1$ with

$$\sum_i \beta_i(t) = 1 ,$$

and vectors $v_i(t) \in U_0$, $1 \leq i \leq n+1$ such that

$$\left| \dot{x}(t) - \sum_i \beta_i(t) \, g\Big(t, \; v_i(t)\Big) \right| < \gamma . \tag{7.2}$$

Furthermore, there is a number $h(t) > 0$ such that for $0 < h < h(t)$,

$$\frac{1}{h} \int_t^{t+h} \left| \dot{x}(t) - \dot{x}(s) \right| ds < \gamma , \tag{7.3}$$

and

$$\frac{1}{h} \int_t^{t+h} \left| g\Big(t, \; v_i(t)\Big) - g\Big(s, \; v_i(t)\Big) \right| ds < \gamma . \tag{7.4}$$

Thus from (7.2), (7.3), (7.4), and the triangle inequality, we see that for $0 < h < h(t)$,

$$\frac{1}{h}\int_t^{t+h} \left|\dot{x}(s) - \sum_i \beta_i(t)\, g\Big(s,\ v_i(t)\Big)\right| ds < 3\gamma \tag{7.5}$$

Let $I_t = \Big(t,\ t + h(t)\Big)$; then $J = \bigcup_{t\in T} I_t$ covers I except for a set of measure zero. Let E be an open set containing this null set such that

$$\int_E \Big(|\dot{x}(s)| + \ell(s)\,(M + N\,\|x\|)\Big)\, dx < \gamma . \tag{7.6}$$

Let E, $I_{t_1}, \ldots, I_{t_n}$ be a finite subcovering of I. It is assumed that $t_k < t_{k+1}$ for each k. Now let $J_k = \Big(t_k,\ t_k'\Big)$ where $t_k' = \min\Big\{h(t_k),\ t_{k+1}\Big\}$ and let $J_0 = I - \Big(\bigcup_k J_k\Big)$. Then $J_0 \subset E$. Now define the function $\alpha_i(t)$, $u_i(t)$ as follows:

$$\alpha_i(t) = \beta_i(t_k) \quad \text{for} \quad t \in J_k \qquad k = 1, \ldots, N;\ i = 1, \ldots, n+1,$$

$$u_i(t) = v_i(t_k) \quad \text{for} \quad t \in J_k \qquad k = 1, \ldots, N;\ i = 1, \ldots, n+1,$$

$$\alpha_1(t) = 1, \quad \alpha_2(t) = \cdots = \alpha_{n+1}(t) = 0 \quad \text{for} \quad t \in J_0,$$

$$u_i(t) = u_i \qquad i = 1, \ldots, n+1, \quad \text{for} \quad t \in J_0 ,$$

where the u_i are arbitrary fixed elements of U. Then,

$$\int_I \left|\dot{x}(s) - \sum_i \alpha_i(s)\, g\left(s,\ u_i(s)\right)\right| ds \le \int_{J_0}\left(|\dot{x}(s)| + \sum_i{}' \alpha_i(s)\, \left|g\left(s,\ u_i(s)\right)\right|\right) ds$$

$$+ \sum_k \int_{J_k} \left|\dot{x}(s) - \sum_i \alpha_i(s)\, g\left(s,\ u_i(s)\right)\right| ds$$

$$\le \gamma + 3\gamma \sum_k \text{mes}\ |J_k| \le \gamma + 3\gamma\ \text{mes}\ (I)$$

by (7.5) and (7.6). By taking $\gamma < \varepsilon / \left(1 + 3\text{mes}\ (I)\right)$, we obtain (7.1).

Theorem 7.1. $\mathcal{R}$ is closed in $\mathcal{C}$.

Proof. Let $x \in \mathcal{R}$ so that $\dot{x}(t) \in G\left(x(t),\ t\right)$ a.e. in I. But then from the fact that G(x, t) is the convex closure of F(x, t), and condition (iv) of Section 3, we see that

$$|\dot{x}(t)| \le \ell(t)\ (M + N\ |x(t)|) \quad \text{a.e. in } I. \tag{7.7}$$

Therefore the same argument as in Lemma 3.1 shows that there exist $B < \infty$ such that $\|x\| \le B$ for $x \in \mathcal{R}$.

Now let $\{x_n\}$ be a sequence in $\mathcal{R}$ such that $x_n \to \tilde{x}$ forsome $\tilde{x}$ in $\mathcal{C}$. It must be shown that (i) $\tilde{x}$ is absolutely continuous and (ii) $\dot{\tilde{x}}(t) \in G(\tilde{x}(t),\ t)$ a.e. in I. Because of (7.7), given any $\varepsilon > 0$, there is a $\delta > 0$ such that for every x in $\mathcal{R}$ and every finite sequence $t_1 < t_1' \le t_2 < t_2' < \cdots \le t_m < t_m'$ in I with $\sum_i |t_i' - t_i| < \delta$,

$$\sum_i |x(t_i') - x(t_i)| \leq \sum_i \int_{t_i}^{t_i'} |\dot{x}(s)|\, ds \leq \sum_i \int_{t_i}^{t_i'} \ell(s)\,(M + NB)\, ds \leq \varepsilon . \tag{7.8}$$

Therefore,

$$\sum_i |\tilde{x}(t_i') - \tilde{x}(t_i)| \leq \sum_i |\tilde{x}(t_i') - x_n(t_i')| + \sum_i |x_n(t_i') - x_n(t_i)| + \sum_i |x_n(t_i) - \tilde{x}(t_i)| .$$

The second term on the right-hand side is less than ε by (7.8), and the remaining terms can be made arbitrarily small by choosing n large since $\|x_n - \tilde{x}\| \to 0$. Therefore

$$\sum_i |\tilde{x}(t_i') - \tilde{x}(t_i)| \leq \varepsilon ,$$

so that $\tilde{x}$ is absolutely continuous.

Next, it will be shown that the functions $\dot{x}_n$ and $\dot{\tilde{x}}$ satisfy the hypothesis of Lemma 7.2. Clearly, from (7.7) these functions are uniformly bounded by an integrable function. Let E be any measurable subset of I. For $\gamma > 0$ there is $\delta > 0$ such that

$$\int_A \left(|\dot{\tilde{x}}(s)| + |\dot{x}_n(s)| \right) ds < \frac{\gamma}{3}$$

for every n and every measurable set A of I of measure less than δ. Let J be a finite union of intervals (t_k, t_k'), $k = 1, \ldots, p$, such that measure

$(E - J) < \delta$, and let N be so large that $\|\tilde{x} - x_n\| < (\gamma/3p)$ for $n > N$. Then for $n > N$,

$$\left|\int_E \left(\dot{x}_n(s) - \dot{\tilde{x}}(s)\right) ds\right| \leq \int_{E-J} \left|\dot{x}_n(s) + \dot{\tilde{x}}(s)\right| ds$$

$$+ \sum_k \left|\int_{t_k}^{t'_k} \left(\dot{x}_n(s) - \dot{\tilde{x}}(s)\right) ds\right|$$

$$\leq \frac{\gamma}{3} + \sum_k \left(|x_n(t'_k) - \tilde{x}(t'_k)| + |x_n(t_k) - \tilde{x}(t_k)|\right)$$

$$\leq \gamma .$$

Hence

$$\int_E \dot{x}_n(s)\, ds \to \int_E \dot{\tilde{x}}(s)\, ds$$

for every measurable subset E of I, and therefore for each $y \in R^n$,

$$\int_E \left\langle y,\ \dot{x}_n(s)\right\rangle ds \to \int_E \left\langle y,\ \dot{\tilde{x}}(s)\right\rangle ds .$$

By Lemma 7.2, for each $y \in R^n$,

$$\overline{\lim_n} \left\langle y,\ \dot{x}_n(t)\right\rangle \geq \left\langle y,\ \dot{\tilde{x}}(t)\right\rangle \geq \underline{\lim_n} \left\langle y,\ \dot{x}_n(t)\right\rangle \tag{7.9}$$

a.e. in I.

Since the function $f(x, t, u)$ is continuous in x for fixed u, t, the set function $G(x, t)$ is continuous in x for fixed t. Hence for every fixed t and every $y \in R^n$, since $\|x_n - \tilde{x}\| \to 0$,

$$\max_{z \in G(x_n(t), t)} \langle y, z \rangle \to \max_{z \in G(\tilde{x}(t), t)} \langle y, z \rangle ,$$

and (7.10)

$$\min_{z \in G(x_n(t), t)} \langle y, z \rangle \to \min_{z \in G(\tilde{x}(t), t)} \langle y, z \rangle .$$

Hence from (7.9) and (7.10), for each $y \in R^n$ we have

$$\max_{z \in G(\tilde{x}(t), t)} \langle y, z \rangle \geq \langle y, \dot{\tilde{x}}(t) \rangle \geq \min_{z \in G(\tilde{x}(t), t)} \langle y, z \rangle$$

a.e. in I. By Lemma 7.1, $\dot{\tilde{x}}(t) \in G(\tilde{x}(t), t)$ a.e. in I and the theorem is proved.

Lemma 7.4. Let u be an admissible control, X a compact subset of R^n, and $\mathcal{E} > 0$. Then there is a continuous function $h : X \times I \to R^n$ such that for every measurable function $x : I \to X$,

$$\int_I \left| f\Big(x(t), t, u(t)\Big) - h\Big(x(t), t\Big) \right| dt < \mathcal{E} . \tag{7.11}$$

Proof. From Section 3,

$$|f(x, t, u) - f(x', t, u)| \leq k(t) \, |x - x'|$$

for u in U and x, x' in R^n.

Let

$$\delta = \frac{1}{2}\,\mathcal{E}\left(|I| \int_I k(t)\,dt\right)^{-1}$$

where $|I|$ = measure (I). Let $V_1, \ldots, V_p$ be a finite covering of X such that the diameter of these sets is less than δ. Therefore, if x, x' are in V_j then $|x - x'| < \delta$, so that

$$\int_I \left|f\left(x,\ t,\ u(t)\right) - f\left(x',\ t,\ u(t)\right)\right|\,dt \le \delta \int_I K(t)\,dt = \frac{\mathcal{E}}{2|I|}\ . \tag{7.12}$$

For each j, let x_j be a fixed point in V_j and let $h_j : I \to R^n$ be continuous functions such that

$$\int_I \left|f\left(x_j,\ t,\ u(t)\right) - h_j(t)\right|\,dt < \frac{\mathcal{E}}{2|I|}\ . \tag{7.13}$$

Now let $\varphi_1(x), \ldots, \varphi_p(x)$ form the partition of unity on X with respect to the $\{V_j\}$ i.e., the φ_j are continuous real-valued functions on X with the property that

$$0 \le \varphi_j(x) \le 1;\quad \varphi_j(x) = 0\quad x \notin V_j\ ,\quad \sum_j \varphi_j(x) \equiv 1\ . \tag{7.14}$$

Then define,

$$h(x,\ t) = \sum_j h_j(t)\,\varphi_j(x)\ .$$

Now let $x: I \to X$ be any measurable function and define the measurable sets

$$E_j = \left\{ t \in I \mid x(t) \in V_j \right\}. \tag{7.15}$$

Then, by (7.12), (7.14), (7.15), and Schwartz inequality,

$$\begin{aligned}
\beta_j &= \int_{E_j} \left| \sum_j \varphi_j(x(t)) \left(f\left(x(t),\ t,\ u(t)\right) - f\left(x_j,\ t,\ u(t)\right)\right) \right| dt \\
&\leq \left\{ \int_{E_j} \sum_j \varphi_j(x(t))\, dt \right\} \left\{ \int_{E_j} \left| f\left(x(t),\ t,\ u(t)\right) - f\left(x_j,\ t,\ u(t)\right)\right| dt \right\} \\
&\leq \left(\int_{E_j} dt \right) \frac{\varepsilon}{2|I|} = \text{mes}\,(E_j)\, \frac{\varepsilon}{2|I|} .
\end{aligned}$$

Hence

$$\sum_j \beta_j = \sum_j \text{mes}\,(E_j)\, \frac{\varepsilon}{2|I|} = \frac{\varepsilon}{2} .$$

Similarly by (7.13), (7.14) and (7.15),

$$\int_I \left| \sum_j \varphi_j(x(t)) \left\{ f\left(x_j,\ t,\ u(t)\right) - h_j(t) \right\} \right| dt < \frac{\varepsilon}{2} .$$

Combining the last two results, an application of the triangle inequality yields (7.10).

Lemma 7.5. For $i = 1, \ldots, n+1$, let u_i be admissible controls and let $\alpha_i : I \to R^1$ be measurable functions with $\alpha_i(t) \geq 0$,

$$\sum_i \alpha_i(t) \equiv 1.$$

Let $x : I \to R^n$ be any continuous function. Then, for each $\mathcal{E} > 0$, there is an admissible control $u_{\mathcal{E}}$ such that for every t_1, t_2 in I,

$$\left| \int_{t_1}^{t_2} \left(\sum_i \alpha_i(s)\, f\Big(x(s),\ s,\ u_i(s)\Big) - f\Big(x(s),\ s,\ u_{\mathcal{E}}(s)\Big) \right) ds \right| < \mathcal{E} .$$

Proof. Let the range of the function x be the compact subset X of R^n, and let $\gamma > 0$ be fixed. By Lemma 7.4, for each i there exists a continuous function $h_i : X \times I \to R^n$ such that

$$\int_I \left| f\Big(x(t),\ t,\ u_i(t)\Big) - h_i(x(t),\ t) \right| dt < \gamma . \tag{7.16}$$

Let $\delta_1 > 0$ be such that for every x in X and t, t' in I with $|t - t'| < \delta_1$

$$|h_i(x,\ t) - h_i(x,\ t')| < \gamma \tag{7.17}$$

for each i. Let $\delta_2 > 0$ be such that for every measurable subset A of I with measure less than δ_2,

$$\int_A (M + NB)\, \ell(t)\, dt < \gamma . \tag{7.18}$$

Here B is the uniform bound of $\mathcal{R}$. Let $\delta = \min\{\delta_1, \delta_2\}$, and let $a = t_0 < t_1 < \cdots < t_p = b$ with $\left(t_{i+1} - t_i\right) < \delta$ for each i. Let $I_k = \left[t_{k-1}, t_k\right]$, and subdivide each interval I_k into n+1 subinterval $I_{k,1}, I_{k,2}, \ldots, I_{k,n+1}$ such that the measure of $I_{k,j}$,

$$|I_{k,j}| = \int_{I_k} \alpha_j(t)\, dt \ . \tag{7.19}$$

Now define the admissible control $\tilde{u}$, by

$$\tilde{u}(t) = u_j(t) \text{ whenever } t \in I_{kj} \text{ for some } j,\ k\ . \tag{7.20}$$

For convenience, denote $\tilde{f}_i(t) = f_i\left(x(t),\ t,\ u_i(t)\right)$; $\tilde{f}(t) = f\left(x(t),\ t,\ \tilde{u}(t)\right)$; $\tilde{h}_i(t) = h_i(x(t),\ t)$. Then from (7.20),

$$\int_{I_k} \tilde{f}(t)\, dt = \sum_j \int_{I_{kj}} \tilde{f}_j(t)\, dt \ . \tag{7.21}$$

Let r_1, r_2 be any integers with $1 \le r_1 \le r_2 \le p$. Then from (7.21),

$$\Delta(r_1, r_2) = \sum_{k=r_1}^{r_2} \left| \int_{I_k} \sum_i \alpha_i \tilde{f}_i - \tilde{f} \right| = \sum_{k=r_1}^{r_2} \left| \int_{I_k} \sum_i \alpha_i \tilde{f}_i - \sum_i \int_{I_{k,i}} \tilde{f}_i \right|$$

$$= \sum_{k=r_1}^{r_2} \left| \int_{I_k} \sum_i \alpha_i \left(\tilde{f}_i - \tilde{h}_i\right) + \int_{I_k} \sum_i \alpha_i \tilde{h}_i - \sum_i \int_{I_{k,i}} \tilde{h}_i \right.$$

$$\left. - \sum_i \int_{I_{k,i}} \left(\tilde{f}_i - \tilde{h}_i\right) \right|$$

$$\leq \left(\sum_{k=r_1}^{r_2} \sum_i \int_{I_k} \alpha_i \left| \tilde{f}_i - \tilde{h}_i \right| \right) + \left(\sum_{k=r_1}^{r_2} \sum_i \int_{I_{k,i}} \left| \tilde{f}_i - \tilde{h}_i \right| \right)$$

$$+ \left(\sum_{k=r_1}^{r_2} \left| \int_{I_k} \sum_i \alpha_i \tilde{h}_i - \sum_i \int_{I_{k,i}} \tilde{h}_i \right| \right) . \tag{7.22}$$

Consider the first term of the last bound in (7.22); by (7.16),

$$\sum_{k=r_1}^{r_2} \sum_i \int_{I_k} \alpha_i \left| \tilde{f}_i - \tilde{h}_i \right| \leq \sum_i \sum_{k=r_1}^{r_2} \int_{I_k} \left| \tilde{f}_i - \tilde{h}_i \right|$$

$$\leq \sum_i \int_I \left| \tilde{f}_i - \tilde{h}_i \right| \leq (n+1)\,\gamma . \tag{7.23}$$

Similarly by (7.16), the second term is bounded by $(n+1)\gamma$. The third term can be written as

$$\sum_{k=r_1}^{r_2} \left| \int_{I_k} \sum_i \alpha_i \tilde{h}_i - \sum_i{}' \int_{I_{ki}} \tilde{h}_i \right|$$

$$= \sum_{k=r_1}^{r_2} \left| \int_{I_k} \sum_i \alpha_i(t) \left(\tilde{h}_i(t) - \tilde{h}_i(t_k) \right) - \sum_i \int_{I_{ki}} \left(\tilde{h}_i(t) - \tilde{h}_i(t_k) \right) \right|$$

$$\le \sum_{k=r_1}^{r_2} \left(\int_{I_k} \sum_i \alpha_i(t) \left| \tilde{h}_i(t) - \tilde{h}_i(t_k) \right| + \sum_i \int_{I_{ki}} \left| \tilde{h}_i(t) - \tilde{h}_i(t_k) \right| \right)$$

$$\le \sum_{k=r_1}^{r_2} \left(\gamma \int_{I_k} \sum_i \alpha_i(t)\, dt + \gamma \sum_i \int_{I_{ki}} dt \right) \le \sum_{k=r_1}^{r_2} 2\gamma \operatorname{mes}(I_k)$$

$$\le 2\gamma \operatorname{mes}(I)\,. \tag{7.24}$$

The third-last bound in (7.24) follows from (7.17). Using the above bound, we see from (7.22) that,

$$\Delta(r_1,\ r_2) \le 2\gamma(n+1 + |I|)\,. \tag{7.25}$$

Now let t_1 and t_2 be arbitrary points in I with $t_1 \le t_2$. Let r_1 and r_2 be integers such that

$$\delta(t_1, t_2) = \left| \int_{t_1}^{t_2} \left(\sum_i \alpha_i \tilde{f}_i - \tilde{f} \right) \right| \le \sum_{k=r_1} \left| \int_{I_k} \left(\sum_i \alpha_i \tilde{f}_i - \tilde{f} \right) \right|$$

$$+ \left| \int_{\tilde{I}_1} \left(\sum_i \alpha_i \tilde{f}_i - \tilde{f} \right) \right| + \left| \int_{\tilde{I}_2} \left(\sum_i \alpha_i \tilde{f}_i - \tilde{f} \right) \right| \tag{7.26}$$

where $\tilde{I}_1$ is a subinterval of I_{r_1} or I_{r_1+1} and $\tilde{I}_2$ is a subinterval of I_{r_2} or I_{r_2+1}. Then by (7.18),

$$\left| \int_{\tilde{I}_1} \left(\sum_i \alpha_i \tilde{f}_i - \tilde{f} \right) \right| \le \int_{\tilde{I}_1} \left(\sum \alpha_i |\tilde{f}_i| + |\tilde{f}| \right)$$

$$\le \int_{\tilde{I}_1} (M + NB)\, \ell(t) \left(\sum \alpha_i + 1 \right) < 2\gamma \tag{7.27}$$

Similarly,

$$\left| \int_{\tilde{I}_2} \left(\sum_i \alpha_i \tilde{f}_i - \tilde{f} \right) \right| < 2\gamma \; . \tag{7.28}$$

From (7.26), (7.27), (7.28) and (7.25) we see that

$$\delta(t_1, t_2) \le \Delta(r_1, r_2) + 4\gamma \le 2\gamma(n+1 + |I|) + 4\gamma$$

$$= 2\gamma(n+3 + |I|) .$$

Therefore for $\gamma = (\mathcal{E}/2)(n+3 + |I|)^{-1}$, the function $\tilde{u}$ defined in (7.2)) satisfies the assertion of the lemma.

Theorem 7.2. $\mathcal{R} = \overline{\mathcal{T}}$.

Proof. By definition, $\mathcal{R} \supset \mathcal{T}$ and by Theorem 7.1 $\mathcal{R}$ is closed so that it suffices to show that $\mathcal{R} \subset \overline{\mathcal{T}}$. To this end, let x in $\mathcal{R}$ and $\mathcal{E} > 0$ be given. Let

$$\gamma = \frac{\mathcal{E}}{2}\left(\int_I k(t)\, dt\right)^{-1}, \qquad (7.29)$$

where k is the function of condition (iii) of Section 3. By Lemma 7.3, there exist measurable functions α_i, $i = 1, \ldots, n+1$ with $\alpha_i(t) \ge 0$ and $\sum \alpha_i(t) \equiv 1$ and there exist admissible controls $u_1, \ldots, u_{n+1}$ such that for all t in I

$$\int_a^t \left|\dot{x}(s) - \sum_i \alpha_i(s)\, f\bigl(x(s),\ s,\ u_i(s)\bigr)\right| ds < \gamma . \qquad (7.30)$$

By Lemma 7.5, there exists an admissible control u_γ such that for all t in I

$$\left| \int_a^t \left(\sum_i \alpha_i(s)\, f\big(x(s),\ s,\ u_i(s)\big) - f\big(x(s),\ s,\ u_\gamma(s)\big) \right) ds \right| < \gamma . \tag{7.31}$$

Let y in $\mathfrak{F}$ be defined by

$$\dot{y}(t) = f\big(y(t),\ t,\ u_\gamma(t)\big) \tag{7.32}$$

with $$y(a) = x(a) .$$

Then from (7.30), (7.31) and condition (iii) of Section 3,

$$|x(t) - y(t)| = \left| \int_a^t \dot{x}(s)\, ds - \int_a^t f\big(y(s),\ s,\ u_i(s)\big)\, ds \right|$$

$$\leq \left| \int_a^t \left(\dot{x}(s) - \sum_i \alpha_i(s)\, f\big(x(s),\ s,\ u_i(s)\big) \right) ds \right|$$

$$+ \left| \int_a^t \left(\sum_i \alpha_i(s)\, f\big(x(s),\ s,\ u_i(s)\big) - f\big(x(s),\ s,\ u_\gamma(s)\big) \right) ds \right|$$

$$+ \left| \int_a^t \left(f\big(x(s),\ s,\ u_\gamma(s)\big) - f\big(y(s),\ s,\ u_\gamma(s)\big) \right) ds \right|$$

$$\leq 2\gamma + \int_a^t k(s)\, |x(s) - y(s)|\, ds .$$

By Gronwall's lemma, the last estimate implies that for each t in I,

$$|x(t) - y(t)| \le 2\gamma \int_a^t k(s)\, ds < \mathcal{E} .$$

This completes the proof of Theorem 7.2.

Combining Theorems 6.1 and 7.2 gives

Theorem 7.3. $\mathcal{R} = \overline{\mathcal{F}} = \mathcal{M}$.

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APPENDIX

Lemma A1. For each x in R^n let $T(x)$ be a subset of R^n such that the mapping $x \to T(x)$ is continuous, i.e., for x in R^n and $\varepsilon > 0$ there is a $\delta > 0$ such that for $|x - y| < \delta$

$$T(x) \subset S_\varepsilon [T(y)] \tag{A-1}$$

and

$$T(y) \subset S_\varepsilon [T(x)] \ . \tag{A-2}$$

Then for every bounded set A in R^n

$$\overline{\bigcup_{x \epsilon A} T(x)} = \bigcup_{x \epsilon \overline{A}} \overline{T(x)} \ . \tag{A-3}$$

Proof. Let $y \epsilon \overline{\bigcup_{x \epsilon A} T(x)}$; then there is a sequence $y_n \epsilon \bigcup_{x \epsilon A} T(x)$ such that $y_n \to y$. For each n let x_n in A be such that $y_n \epsilon T(x_n)$. Taking subsequences if necessary, it can be assumed that x_n converges to x in $\overline{A}$. Given $\varepsilon > 0$ let N be so large that, for $n > N$

$$T(x_n) \subset S_\varepsilon [T(x)]$$

and

$$|y_n - y| < \varepsilon \ ,$$

so that,

$$\rho\left(y, T(x)\right) \leq 2\mathcal{E} .$$

Since $\mathcal{E}$ is arbitrary, this implies that $y \in \overline{T(x)}$ and hence

$$\overline{\bigcup_{x \in A} T(x)} \subset \bigcup_{x \in \overline{A}} \overline{T(x)} .$$

On the other hand let $y \in \bigcup_{x \in \overline{A}} \overline{T(x)}$; then there is $x \in \overline{A}$ such that $y \in \overline{T(x)}$. Let x_n be a sequence in A converging to x, and let y_n be a sequence in T(x) converging to y. Let $\mathcal{E} > 0$, and let N be so large that,

$$T(x) \subset S_{\mathcal{E}}\left[T(x_n)\right]$$

and

$$|y - y_n| < \mathcal{E}$$

for $n > N$. Therefore $\rho\left(y, \overline{\bigcup_{x \in A} T(x)}\right) < 2\mathcal{E}$ and since $\mathcal{E}$ is arbitrary $y \in \overline{\bigcup_{x \in A} T(x)}$ and the lemma is proved.