ON THE TRAJECTORIES OF A DIFFERENTIAL SYSTEM

by

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ABSTRACT

The so-called convexity assumption of Lee and Marcus, Fillipov, Roxin, and Warga is examined in detail. Under conditions less restrictive than Warga, it is shown that the set of relaxed trajectories is compact in the topology of uniform convergence, and that the set of trajectories is dense in the set of relaxed trajectories. It is also shown that the set of trajectories is compact if and only if the convexity assumption is satisfied, i.e., if and only if at each point in phase space, the set of permissible velocities is convex. Some interesting consequences of this result are derived.

INTRODUCTION

In this paper we investigate in detail the so-called convexity assumption made by Lee and Marcus [1], Fillipov [2], Roxin [3], and Warga [4] in their studies relating to the existence of optimal control. We first show (Theorem 2.1) that the set of relaxed trajectories is compact in the topology of uniform convergence. It is interesting to note that this result is true without a "Lipschitz condition" on the differential system. The next result (Theorem 2.2) shows that the set of trajectories is dense in the set of relaxed trajectories. In proving this result critical use is made of the Lipschitz condition. Finally in Theorem 2.3 we prove that the set of trajectories is closed if and only if the convexity assumption is satisfied, i.e., if and only if at each point in the phase space the set of permissible velocities form a convex set. Some of these assertions have been proved in a less general setting by Warga [4]. Also, for the main part, our proofs are different and simpler than those presented by Warga. We also remark that as an immediate consequence of Theorem 2.1 we obtain the results on the existence of optimal controls given in Reference [1] - [4].

In section 3 we derive some interesting consequences of Theorem 2.3 for the class of control systems for which the right-hand side of the differential equation is separable in the state and the control vectors (Eq. 3.1). We show that for such systems, if the convexity condition is not satisfied, then the set of trajectories and the set of limit points of the trajectories which are not themselves trajectories, are dense in each other. Furthermore, these two disjoint sets are pathwise connected if the initial set is pathwise connected. This result is interesting in the light of a result of Neustadt [5] which states that for a linear system the set of attainable sets is closed even if the convexity condition is not satisfied. We remark that the relations between the attainable sets and the convexity conditions are investigated in detail in Reference [6].

1. STATEMENT OF THE PROBLEM

We shall study the control system

$$\dot{x}(t) = f(x(t), t, u(t)) \tag{1.1}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control, and $t \in R$ is the time; f is a continuous mapping of $R^n \times R \times R^m$ into R^n; and $\dot{x}$ as usual denotes $\frac{dx}{dt}$. For each $(x, t) \in R^n \times R$ we are given a compact subset $U(x, t)$ of R^m such that the mapping $(x, t) \to U(x, t)$ is upper semicontinuous. Let

$$U = \bigcup_{(x,t) \in R^n \times R} U(x,t) \quad .$$

Finally, the following additional conditions are imposed on the function f: there exists a locally integrable function k and finite numbers M and N such that

$$(1.2) \qquad | f(x,t,u) - f(x',t,u) | \leq k(t) | x - x' | \quad ,$$

$$(1.3) \qquad | f(x,t,u) | \leq k(t) (M + N|x|)$$

for all x, x' in R^n, $u \in U$ and $t \in R$. Here and throughout, if $z \in R^{\ell}$ then $|z|$ denotes the Euclidean norm of z in R^{ℓ}.

We are also given a fixed compact subset X_0 of R^n and two finite numbers a and b with $a \leq b$. Let $I = [a,b]$.

Definition 1.1. For $x \in R^n$ and $t \in R$ let $F(x,t) = \{ f(x,t,u) | u \in U(x,t) \}$, and let $G(x,t)$ be the convex closure of $F(x,t)$.

Definition 1.2. An absolutely continuous function $x: I \to R^n$ is said to be a <u>trajectory</u> if

$$(1.4) \qquad x(a) \in X_0$$

(1.5) There is a measurable function $u: I \to R^m$ with $u(t) \in U(x(t),t)$ for $t \in I$, such that

$$\dot{x}(t) = f(x(t), t, u(t)) \text{ a. e. in } I.$$

$$U = \bigcup_{(x,t)\in R^n \times R} U(x,t) \quad .$$

Finally, the following additional conditions are imposed on the function f: there exists a locally integrable function k and finite numbers M and N such that

(1.2) $$| f(x,t,u) - f(x',t,u) | \leq k(t) | x - x' | \quad ,$$

(1.3) $$| f(x,t,u) | \leq k(t) (M + N|x|)$$

for all x, x' in R^n, $u \in U$ and $t \in R$. Here and throughout, if $z \in R^\ell$ then $|z|$ denotes the Euclidean norm of z in R^ℓ.

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Definition 1.1. For $x \in R^n$ and $t \in R$ let $F(x,t) = \{ f(x,t,u) \,|\, u \in U(x,t) \}$, and let $G(x,t)$ be the convex closure of $F(x,t)$.

Definition 1.2. An absolutely continuous function $x : I \rightarrow R^n$ is said to be a <u>trajectory</u> if

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(1.5) There is a measurable function $u : I \rightarrow R^m$ with $u(t) \in U(x(t),t)$ for $t \in I$, such that

$$\dot{x}(t) = f(x(t),t,u(t)) \text{ a. e. in } I.$$

Note. By Fillipov's lemma [1], (1.5) is equivalent to

(1.6) $$x(t) \in F(x(t), t) \quad \text{a.e. in } I.$$

Let $\mathcal{T}$ denote the set of all trajectories. Following Warga [4], we make

Definition 1.3. An absolutely continuous function $x: I \to R^n$ is said to be a <u>relaxed trajectory</u> if

(1.7) $$x(a) \in X_0$$

(1.8) $$\dot{x}(t) \in G(x(t), t) \quad \text{a.e. in } I.$$

Let $\mathcal{R}$ denote the set of all relaxed trajectories. It is clear that $\mathcal{T} \subset \mathcal{R}$. Let $\mathcal{C}$ denote the real Banach space of all continuous functions $x: I \to R^n$ with the norm of x given by

$$\|x\| = \max_{t \in I} |x(t)| .$$

We will consider $\mathcal{T}$ and $\mathcal{R}$ as subsets of $\mathcal{C}$. Our purpose is to investigate the relationship between $\mathcal{T}$ and $\mathcal{R}$. In particular, we will show that $\mathcal{R}$ is a compact subset of $\mathcal{C}$; $\mathcal{R}$ is equal to the $\mathcal{C}$-closure of $\mathcal{T}$; $\mathcal{T}$ is closed in $\mathcal{C}$ if $F(x,t) = G(x,t)$ for every (x,t), and if the mapping $(x,t) \to U(x,t)$ is continuous, the converse statement is also true.

2. THE RELATION BETWEEN $\mathcal{R}$ AND $\mathcal{T}$

Using (1.3), we make an elementary application of Gronwall's lemma to obtain

Lemma 2.1. $\mathcal{R}$ and a fortiori $\mathcal{T}$ are bounded subsets of $\mathcal{C}$.

From (1.3) and the preceding lemma we immediately have

Corollary 2.1. There is an integrable function μ defined on I such that for every x in $\mathcal{R}$

$$(1.9) \qquad |\dot{x}(t)| \leq \mu(t) \quad \text{a.e. in I.}$$

Corollary 2.2. $\mathcal{R}$ is an equicontinuous family of functions.

Proof. Let $\epsilon > 0$. Since the function μ in (1.9) is integrable, there is a $\delta = \delta(\epsilon) > 0$ such that if t_1, t_2 in I and $|t_1 - t_2| < \delta$ then $\int_{t_1}^{t_2} \mu(t)\, dt \leq \epsilon$. Hence for any x in $\mathcal{R}$,

$$|x(t_1) - x(t_2)| \leq \int_{t_1}^{t_2} |\dot{x}(t)|\, dt \leq \int_{t_1}^{t_2} \mu(t)\, dt \leq \epsilon .$$

Theorem 2.1. $\mathcal{R}$ is a compact subset of $\mathcal{C}$.

Proof. By Lemma 2.1, Corollary 2.2, and the Arzelà-Ascoli Theorem, it suffices to show that $\mathcal{R}$ is closed in $\mathcal{C}$. To this end, let $\{x_n\}$ be a sequence in $\mathcal{R}$ converging to an element x in $\mathcal{C}$.

We first prove that x is absolutely continuous. Indeed let $\epsilon > 0$, and let $\delta = \delta(\epsilon) > 0$ be such that for every finite increasing sequence

$a \leq t_1 \leq t_1' \leq \dots \leq t_m \leq t_m' \leq b$ with $\sum_{i=1}^{m} |t_i' - t_i| < \delta$ we have

$$\sum_{i=1}^{m} \int_{t_i}^{t_i'} \mu(t)\, dt \leq \frac{\epsilon}{2}$$

where μ is the function given in Corollary 2.1. Now given such a sequence, let n be sufficiently large so that $\| x - x_n \| \leq 1/4m\epsilon$. Then,

$$\sum_{i=1}^{m} | x(t_i') - x(t_i) | \leq \sum_{i=1}^{m} \Big\{ | x(t_i) - x_n(t_i) | + | x(t_i') - x_n(t_i') |$$

$$+ | x_n(t_i) - x_n(t_i') | \Big\} \leq \epsilon .$$

Hence x is absolutely continuous. It remains to show that $\dot{x}(t) \in G(x(t), t)$ a.e. in I. We first show that $\dot{x}_n$ converges to $\dot{x}$ in the weak topology of $L_1(I)$. To this end, let E be any subset of I with positive measure. Then for any $\epsilon > 0$, there is a finite disjoint union

$$H = \bigcup_{i=1}^{m} (t_i, t_i')$$

of internals such that the measure of the difference $E - H$ is less than ϵ.

Then

$$\left|\int_E (\dot{x}_n(t) - \dot{x}(t))\,dt\right| \le \left|\int_H (\dot{x}_n(t) - \dot{x}(t))\,dt\right| + \int_{E-H} (|\dot{x}_n(t)| + |\dot{x}(t)|)\,dt$$

$$\le \sum_{i=1}^{m} \{|x_n(t_i') - x(t_i')| + |x_n(t_i) - x(t_i)|\} + \int_{E-H} (|\dot{x}_n(t)| + |\dot{x}(t)|)\,dt.$$

The first term can be made small by choosing n large, and the second term can be made small by choosing ϵ small. Thus for every measurable subset E of I we have

$$\lim_n \int_E \dot{x}_n(t)\,dt = \int_E \dot{x}(t)\,dt$$

so that $\dot{x}_n$ converges to $\dot{x}$ weakly in $L_1(I)$. From this we see that for every vector $z \in R^n$,

$$(2.1) \quad \overline{\lim_n} \langle z, \dot{x}_n(t)\rangle \ge \langle z, \dot{x}(t)\rangle \ge \underline{\lim_n} \langle z, \dot{x}_n(t)\rangle$$

a. e. in I. Since the set function $G(x, t)$ is upper semicontinuous in x for fixed t and since $\|x_n - x\| \to 0$, for every $z \in R^n$ and t in I we must have

$$(2.2)\qquad \begin{aligned} &\overline{\lim_{n}} \max_{y\in G(x_n(t),\, t)} \langle z, y\rangle \le \max_{y\in G(x(t),\, t)} \langle z, y\rangle , \\ &\underline{\lim_{n}} \min_{y\in G(x_n(t),\, t)} \langle z, y\rangle \ge \min_{y\in G(x(t),\, t)} \langle z, y\rangle . \end{aligned}$$

From (2.1) and (2.2) we deduce that for every $z \in R^n$ and almost all $t \in I$,

$$\max_{y\in G(x(t),\, t)} \langle z, y\rangle \ge \langle z, \dot{x}(t)\rangle \ge \min_{y\in G(x(t),\, t)} \langle z, y\rangle$$

so that since G is closed and convex,

$$\dot{x}(t) \in G(x(t), t) \qquad \text{a. e. in } I.$$

Q. E. D.

Remarks 2.1. a) In the proof of Theorem 2.1 we have not used either the assumption of continuity of f in t or the Lipschitzian condition (1.2). Therefore Theorem 2.1 is true and the same proof holds, if f is merely required to be measurable in t for fixed (x, u) and if condition (1.2) is eliminated. The proof of Theorem 2.2 however, makes critical use of (1.2).

b) If the initial set X_0 is merely required to be closed instead of compact, an immediate consequence of Theorem 2.2 is that $\mathcal{R}$ is closed in $\mathcal{C}$.

c) If, instead of the finite interval $I = [a, b]$, we consider the interval $\tilde{I} = [a, \infty)$, then the set $\tilde{\mathcal{R}}$ of relaxed trajectories defined on $\tilde{I}$ is a compact subset of $\tilde{\mathcal{C}}$ -- the Frechet space of all continuous function $\tilde{x}:\tilde{I}\to R^n$ with the topology of uniform convergence on finite intervals.

Definition 2.1. Let x be a fixed element of $\mathcal{R}$. Let $\mathcal{F}$ ($\mathcal{G}$) be the family of all measurable functions $f\,(g): I \to R^n$ such that $f(t) \in F(x(t),t)$ $\big(g(t) \in G(x(t),t)\big)$ a.e. in I.

We note that $\mathcal{F} \subset \mathcal{G}$ and $\mathcal{G}$ is a convex, closed, and bounded subset of $L_1(I)$.

Lemma 2.2. Let $\ell: I \to R^n$ be any bounded, measurable function. Then there exist functions $\bar{f}$ and $\underline{f}$ in $\mathcal{F}$, depending on ℓ, such that

$$(2.3) \qquad \int_I \langle \ell(t), \bar{f}(t)\rangle\, dt = \max_{g\in\mathcal{G}} \int_I \langle \ell(t), g(t)\rangle\, dt$$

$$(2.4) \qquad \int_I \langle \ell(t), \underline{f}(t)\rangle\, dt = \min_{g\in\mathcal{G}} \int_I \langle \ell(t), g(t)\rangle\, dt.$$

Proof. It is enough to prove (2.3). For each t in I let

$$M(t) = \max_{y\in G(x(t),t)} \langle \ell(t), y\rangle \quad .$$

Since $F(x(t),t)$ is compact and $G(x(t),t)$ is its convex hull we have

$$M(t) = \max_{y\in F(x(t),t)} \langle \ell(t), y\rangle \quad .$$

Clearly, M is measurable and $F(x(t),t)$ is upper semicontinuous in t. By mimicking the argument of Fillipov [1] it is easy to show that there is a function $\bar{f}$ in $\mathcal{F}$ which satisfies (2.3).

Corollary 2.3. Let $g \in \mathcal{G}$, $t_1, t_2 \in I$ (with $t_1 \leq t_2$) be arbitrary. Then there exists $f \in \mathcal{F}$ (depending on g, t_1, t_2) such that

$$\int_{t_1}^{t_2} g(t)\,dt = \int_{t_1}^{t_2} f(t)\,dt.$$

Proof. Because of Lemma 2.2. it is enough to show that the set

$$L(t_1, t_2, \mathcal{F}) = \left\{ \int_{t_1}^{t_2} f(t)\,dt \;\middle|\; f \in \mathcal{F} \right\}$$

is a convex subset of R^n. Let $f_1, f_2 \in \mathcal{F}$ and let $\lambda \in [0,1]$ be arbitrary. Then we must show that

$$\int_{t_1}^{t_2} \left(\lambda f_1(t) + (1-\lambda) f_2(t)\right) dt \in L\left(t_1, t_2, \mathcal{F}\right) \tag{2.5}$$

For each Borel subset B of I let $f_B \in \mathcal{F}$ be defined by $f_B(t) = f_1(t)$ for $t \in B$ and $f_B(t) = f_2(t)$ for $t \notin B$. But then by Lyapunov's theorem [7] the set $L = \left\{ \int_{t_1}^{t_2} f_B(t)\,dt \mid B \text{ is a Borel subset of } I \right\}$ is a convex subset of R^n so that (2.5) is verified.

Q. E. D.

Theorem 2.2. $\mathcal{R}$ is the $\mathcal{C}$-closure of $\mathcal{T}$.

Proof. Let $x \in \mathcal{R}$ and let $\epsilon > 0$. We will first show that there exists f_ϵ in $\mathcal{F}$ (see Def. 2.1) such that for every t and t' in I

$$(2.6) \qquad \left| \int_t^{t'} \left(\dot{x}(\tau) - f_\epsilon(\tau)\right) d\tau \right| \leq \epsilon .$$

Indeed let $\delta > 0$ be so small that

$$(2.7) \qquad \int_t^{t'} \mu(\tau)\, d\tau < \epsilon/3$$

whenever $|t - t'| < \delta$ (μ is defined in Corollary 2.1). Choose a sequence $a = t_0 < t_1 < \ldots < t_m = b$ such that $t_{i+1} - t_i < \delta$ for each i. By Corollary 2.3 there exists a function f_i in $\mathcal{F}$ such that

$$(2.8) \qquad \int_{t_i}^{t_{i+1}} \left(\dot{x}(\tau) - f_i(\tau)\right) d\tau = 0.$$

Let $f_\epsilon \in \mathcal{F}$ be defined by $f_\epsilon(t) = f_i(t)$ for $t_i \leq t < t_{i+1}$. It is clear from (2.7) and (2.8) that f_ϵ satisfies (2.6). By Fillipov's lemma [1], there is a measurable function $u_\epsilon : I \to R^m$ with $u_\epsilon(t) \in U(x(t), t)$ such that $f_\epsilon(t) = f(x(t), t, u_\epsilon(t))$ a. e. in I. Let x_ϵ be the element in $\mathcal{T}$ defined by,

$$\dot{x}_\epsilon(t) = f(x_\epsilon(t), t, u_\epsilon(t)), \qquad x_\epsilon(a) = x(a)$$

Then, using (1.2) and (2.6), a simple application of Gronwall's lemma shows that $\| x - x_\epsilon \| \leq K\epsilon$ where K is a fixed number independent of ϵ.

Q. E. D.

Corollary 2.4. If $F(x,t)$ is convex for each (x,t), then $\mathcal{T}$ is closed in $\mathcal{C}$, and hence $\mathcal{T} = \mathcal{R}$.

Remark 2.2. This result shows that the "convexity assumption" on F is sufficient to insure compactness of $\mathcal{T}$. Theorem 2.3 states that if the mapping $(x,t) \to U(x,t)$ is continuous, then the convexity assumption is also necessary.

Henceforth we will assume that the function $(x,t) \to U(x,t)$ is continuous. The next lemma is immediate.

Lemma 2.3. The mapping $(x,t) \to F(x,t)$ and $(x,t) \to G(x,t)$ are continuous. The set of points (x,t) for which $F(x,t)$ is not convex is an open subset of $R^n \times R$.

Theorem 2.3. Let $x \in \mathcal{T}$ and suppose that for some $t^* \in I$. $F(x(t^*), t^*)$ is not convex. Then for every $\epsilon > 0$, there is an element $x_\epsilon \in \mathcal{R}$ such that $\| x - x_\epsilon \| \leq \epsilon$ and $x_\epsilon \notin \mathcal{T}$.

Proof. Because of Lemma 2.3 we can assume that $a < t^* < b$. Also there exist positive numbers δ and δ' such that if $|x - x(t^*)| \leq \delta$ and $|t - t^*| \leq \delta$, then $d_H(F(x,t), G(x,t)) \geq \delta'$ where $d_H(A,B)$ is the Hausdorff distance between A and B. Therefore there exists a measurable function $g: [t^*, t^* + \delta] \to R^n$ such that for each t, $g(t) \in G(x(t),t)$ and $|g(t) - y| \geq \delta$ for every $y \in F(x(t),t)$. By Fillipov's lemma there exist measurable functions α_i and u_i, $1 \leq i \leq n+1$, with $\alpha_i(t) \geqq 0$, $\sum_{i=1}^{n+1} \alpha_i(t) = 1$, $u_i(t) \in U(x(t),t)$ for each t, such that $g(t) = \sum_{i=1}^{n+1} \alpha_i(t) f(x(t), t, u_i(t))$ for $t \in [t^*, t^*+\delta]$. Also from the definition of g we see that there are positive numbers β and β' such that if

$$(2.9) \qquad t^* \leq t \leq t^*+\beta \quad \text{and} \quad |z - g(t)| \leq \beta , \quad \text{then}$$

$$(2.10) \qquad |z - y| \geq \beta' \quad \text{for every} \quad y \in F(z,t).$$

Now since $x \in \mathcal{T}$, there is a measurable function $u: I \to R^n$ with $u(t) \in U(x(t),t)$ such that $\dot{x}(t) = f(x(t),t,u(t))$ a. e. in I. For each positive integer m let $x_m \in \mathcal{R}$ be defined by

$$\dot{x}_m(t) = \begin{cases} f(x_m(t), t, u(t)) & t \in I, \quad t \notin [t^*, t^* + \frac{\delta}{m}] \\ \sum_{i=1}^{n+1} \alpha_i(t) \; f(x_m(t), t, u_i(t)) & t \in [t^*, t^* + \frac{\delta}{m}] \end{cases}$$

and $x_m(a) = x(a)$.

Clearly, $\| x_m - x \| \to 0$ as $m \to \infty$. Also for each m, since x_m and f are continuous functions, there is a number $t_m > 0$, such that for almost every $t \in [t^*, t^* + t_m]$, $| \dot{x}_m(t) - g(t) | \leq \beta$. From (2.9) and (2.10) we see that $x_m \notin \mathcal{T}$ and the theorem is proved.

Q. E. D.

Definition 2.2. A pair $(x', t') \in R^n \times R^1$ is said to be an <u>attainable phase</u> if there is a trajectory $x \in \mathcal{T}$ such that $x(t') = x'$.

Corollary 2.4. $\mathcal{T}$ is closed and a fortiori compact in $\mathcal{C}$ if and only if for every attainable phase (x', t') the set $F(x', t')$ is convex.

3. SOME CONSEQUENCES OF THEOREM 2.3

In this section we consider the class of control systems where Eq. (1.1) has the form:

$$(3.1) \qquad \dot{x}(t) = w(x(t), t) + \hat{v}(u(t), t).$$

The functions w and $\hat{v}$ are assumed continuous, the control set $U(t, x)$ is assumed to be independent of x i.e., $U(t, x) \equiv U(t)$, and the mapping $t \to U(t)$ is assumed to be continuous.

Let $V(t) = \{\hat{v}(u, t) \mid u \in U(t)\}$ and let $G(t)$ be the convex closure of $V(t)$. Let $\mathcal{T}$ and $\mathcal{R}$, respectively, denote the trajectories and relaxed trajectories of (3.1), defined on $I = [a, b]$ with $a < b$, and starting at time a, in some fixed closed subset X_0 of R^n. We note that $\mathcal{T}$ consists of those absolutely continuous functions $x : I \to R^n$ for which $x(a) \in X_0$ and $\dot{x}(t) \in w(x(t), t) + V(t)^{\dagger}$ a.e., whereas $\mathcal{R}$ consists of those absolutely continuous functions x for which $x(a) \in X_0$ and $\dot{x}(t) \in w(x(t), t) + G(t)$ a.e. in I.

Definition 3.1. a) Let $\mathcal{V}$ denote the set of all measurable functions $v : I \to R^n$ such that $v(t) \in V(t)$ a.e. in I.

b) Let $\mathcal{G}$ denote the set of all measurable functions $g : I \to R^n$ such that $g(t) \in G(t)$ a.e. in I.

c) Let $\mathcal{Q} = \mathcal{G} - \mathcal{V}$ denote the complement of $\mathcal{V}$ in $\mathcal{G}$.

Remark 3.1. We consider $\mathcal{V}$, $\mathcal{Q}$ and $\mathcal{G}$ as subset of the real Banach space L_1 consisting of all integrable functions $\ell : I \to R^n$ with the norm of ℓ given by $\|\ell\|_1 = \int_I |\ell(t)|\, dt$.

Definition 3.2. For $x_0 \in X_0$ and $g \in \mathcal{G}$, let $\tau(x_0, g)$ denote the element x of $\mathcal{C}$ given by

$$\dot{x}(t) = w(x(t), t) + g(t) \quad \text{a.e. in } I$$

$$x(a) = x_0$$

Lemma 3.1. The mapping $\tau : X_0 \times \mathcal{G} \to \mathcal{C}$ is continuous and one-to-one.

Proof. Let x_0, x_0' belong to X_0 and g, g' belong to $\mathcal{G}$. Let $x = \tau(x_0, g)$ and $x' = \tau(x_0', g')$. Then, for $a \le t \le b$,

$$|x(t) - x'(t)| \le |x_0 - x_0'| + \int_a^t |\dot{x}(s) - \dot{x}'(s)|\, ds$$

$$\le |x_0 - x_0'| + \int_a^t |w(x(s), s) - w(x'(s), s)|\, ds + \int_a^t |g(s) - g'(s)|\, ds$$

$$\le \int_a^t k(s)\, |x(s) - x'(s)|\, ds + \|g - g'\|_1 + |x_0 - x_0'|$$

by (1.2). By Gronwall's lemma we obtain

$$|x(t) - x'(t)| \leq \exp\left(\int_a^t k(s)\,ds\right) \left[|x_0 - x_0'| + \|g - g'\|_1 \right].$$

so that τ is continuous.

Now suppose that $x = \tau(x_0, g) = \tau(x_0', g') = x'$. Then certainly $x_0 = x_0'$. Furthermore, $\dot{x}(t) = \dot{x}'(t)$ a.e., so that

$$w(x(t), t) + g(t) = w(x'(t), t) + g'(t)$$

$$= w(x(t), t) + g'(t) \quad \text{a.e.}$$

and hence $g = g'$. Therefore τ is one-to-one.

Lemma 3.2. The sets $\mathcal{V}$ and $\mathcal{Q}$ ($= \mathcal{G} - \mathcal{V}$) are pathwise-connected†† in L_1.

Proof. Let v_0 and v_1 be in $\mathcal{V}$. For $t \in [0,1]$ let h_t in $\mathcal{V}$ be given by

$$h_t(\tau) = v_0(\tau) \quad \text{for} \quad \tau \geq t,$$

and $$h_t(\tau) = v_1(\tau) \quad \text{for} \quad \tau < t.$$

Clearly, the map $t \to h_t$ is continuous; $h_0 = v_0$ and $h_1 = v_1$ so that $\mathcal{V}$ is pathwise connected.

Let q_0 and q_1 be in $\mathcal{Q}$. Since $q_0 \notin \mathcal{V}$, the set E of all points t for which $q_0(t) \notin V(t)$ has positive measure. Let χ_E be the indicator function of E, i.e., $\chi_E(t) = 1$ if $t \in E$, and $\chi_E(t) = 0$

if $t \notin E$. Let

$$\xi(t) = \int_a^t \chi_E(s)\,ds \qquad \text{for} \quad a \leq t \leq b\,.$$

Then ξ is a continuous, non-decreasing function of t and $\xi(t) = 0$. Let $t^* \in I$ be such that $\xi(t) = 0$ for $t \leq t^*$ and $\xi(t) > 0$ for $t > t^*$. Since measure of E is positive, $t^* < b$. This implies that

$$\text{(3.2)} \qquad \text{measure}\ (E \cap (t^*, t^* + \delta)) > 0 \qquad \text{for} \quad \delta > 0\,.$$

Now for each $t \in [a, b]$ define the function h_t as follows:

(i) Let $a \leq t \leq t^*$. Then $h_t(\tau) = q_0(\tau)$ for $\tau \geq t$ and $h_t(\tau) = q_1(\tau)$ for $\tau < t$.

(ii) Let $t^* < t \leq b$. Then $h_t(\tau) = q_1(\tau)$ for $\tau \leq t^*$ and for $\tau \geq b - t + t^*$; whereas $h_t(\tau) = q_0(\tau)$ for $t^* < \tau < b - t + t^*$.

It is easy to check that $t \to h_t$ is continuous and $h_0 = q_0$, $h_1 = q_1$; also (3.2) implies that $h_t \in \mathcal{Q}$ for each t in I.

Definition 3.3. Let $\mathcal{S} = \bar{\mathcal{T}} - \mathcal{T} = \mathcal{R} - \mathcal{T}$. Thus $\mathcal{S}$ is the set of limit points of $\mathcal{T}$ which are not themselves members of $\mathcal{T}$.

Theorem 3.1. Suppose that for some $t^* \in I$ the set $V(t^*)$ is not convex. Then

(i) $\mathcal{S} \neq \phi$ (ϕ denotes the empty set)

(ii) $\bar{\mathcal{S}} = \bar{\mathcal{T}} = \mathcal{R}$

(iii) $(\mathcal{S} \cap \mathcal{T}) = \phi$ and

(iv) $\mathcal{S}$ and $\mathcal{T}$ are pathwise connected subsets of $\mathcal{C}$, if and only if the initial set X_0 is pathwise-connected.

Proof. Since $V(t^*)$ is not convex, the set $\{w(z,t^*) + V(t^*)\}$ is not convex for every z in R^n. By Theorem 2.3, for every x in $\mathcal{T}$ and every $\epsilon > 0$, there is an element $x_\epsilon \in \mathcal{S}$ such that $\| x - x_\epsilon \| < \epsilon$. This implies (i) and (ii). (iii) follows from the definition of $\mathcal{S}$.

By Lemma 3.1, since τ is one-to-one,

$$\mathcal{T} = \{\tau(x_0, v) \mid x_0 \in X_0,\ v \in \mathcal{V}\}$$

and
$$\mathcal{S} = \{\tau(x_0, q) \mid x_0 \in X_0,\ q \in \mathcal{Q}\}$$

By Lemma 3.2 the sets $\mathcal{V}$ and $\mathcal{Q}$ are pathwise connected; and by Lemma 3.1 τ is continuous so that (iv) follows.

FOOTNOTES

† For $q \in R^n$ and $Q \subset R^n$, $q + Q$ denotes the set $\{q + q' \mid q' \in Q\}$.

†† A subset K of a topological space T is pathwise-connected if given k_0, k_1 in K there is a continuous mapping $h: [0,1] \rightarrow T$ such that $h(0) = k_0$, $h(1) = k_1$ and $h(t) \in K$ for each t.

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