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Using his concept of system function, Zadeh has shown that every self-dual one-port made of linear time-varying elements is a constant resistance one-port [1]. Recently we gave instances of constant resistance one-ports that have nonlinear, time-varying elements. Some of these one-ports are self-dual networks [2-4]. Here we give a precise condition for the truth of the statement "every self-dual one-port is constant resistance and conversely every constant resistance one-port is self-dual." This proposition has recently acquired more importance since wide classes of self-dual one-ports can easily be generated [4-5]. This paper is an extension of a previous paper [6] in that we adopt exclusively a black-box point of view, and it proves the equivalence completely.

We assume throughout that all one-ports under consideration have been created at $t = -\infty$ and that at the time of their creation they are in their zero-state. Similarly, any interconnection of one-ports is assumed to be done at $-\infty$. As a consequence, all waveforms under consideration are defined on $(-\infty, \infty)$.

By definition, a one-port $\mathcal{N}$ is specified as the set of all voltage current pairs $[v(\cdot), i(\cdot)]$ it allows. A one-port $\mathcal{N}^*$ is said to be the dual of $\mathcal{N}$ whenever the following condition holds: $[f, g] \in \mathcal{N}^*$ if and only if $[g, f] \in \mathcal{N}$. A one-port $\mathcal{N}$ is said to be self-dual whenever $[f, g] \in \mathcal{N}$ implies $[g, f] \in \mathcal{N}$. This point of view amounts to thinking of a one-port as a binary relation on some function space [7, p. 9]; the converse relation is the dual one-port; a one-port is self-dual if and only if its defining relation is symmetric. Given a one-port $\mathcal{N}$, we define the augmented one-port $\mathcal{N}_a$ by its ordered pairs: $[v + i, i] \in \mathcal{N}_a$ when and only when $[v, i] \in \mathcal{N}$; $\mathcal{N}_a$ has an obvious interpretation given in Fig. 1a. Any voltage $e(\cdot)$ such that $e = v + i$ for some $[v, i] \in \mathcal{N}$ is called an allowed voltage of $\mathcal{N}_a$. We now slightly extend the concept of solvability

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[8, p.113; 9, p.9] by considering only a restricted class of e(·)'s, namely those allowed by $\mathcal{N}_a$. $\mathcal{N}$ is said to be wide-sense solvable (abbreviated as w.s. solvable) if for all allowed e(·), the equation i(·) + v(·) = e(·) has a unique solution $[v(\cdot), i(\cdot)] \in \mathcal{N}$. Physically, w.s. solvability means that if a voltage source (whose voltage e(·) is an allowed voltage of $\mathcal{N}_a$) is connected to $\mathcal{N}_a$, then the port voltage and port current of $\mathcal{N}$ are uniquely determined. Note that the nullator is not solvable in the sense of Youla et al. [8] and Newcomb [9] but is w.s. solvable.

When we consider the one-port $\mathcal{N}$ as a "constant resistance one-port" we only allow $\mathcal{N}$ to be connected to one-ports $\mathcal{N}'$ such that the connection $\mathcal{N} - \mathcal{N}'$ is determinate, i.e., the port voltage v(·) and the port current i(·) of $\mathcal{N}$ are uniquely determined. Such one-ports $\mathcal{N}'$ are said to be compatible with $\mathcal{N}$. If $\mathcal{N}$ is solvable, then the series connection of a one-ohm resistor and a voltage source e where e is an allowed voltage of $\mathcal{N}_a$ is a one-port compatible with $\mathcal{N}$. If all connections $\mathcal{N} - \mathcal{N}'$ where $\mathcal{N}'$ is compatible with $\mathcal{N}$ have the property that the port voltage v(·) (of $\mathcal{N}$) is equal to the port current i(·) (of $\mathcal{N}$), we say that $\mathcal{N}$ is constant resistance. By including a scale factor, this definition can be extended to include the case where for all such connections, v(·) = ki(·), where k is a fixed non-zero real number independent of i(·), v(·), and t. We want now to prove the

Theorem. A one-port $\mathcal{N}$ is constant resistance if and only if $\mathcal{N}$ is w.s. solvable and self-dual.

Proof. 1. Wide-sense solvability and self-duality imply constant resistance. Let $\mathcal{K}(v)$ denote any member of $\{\tilde{i} : [v, \tilde{i}] \in \mathcal{N}\}$; $\mathcal{K}$ is not necessarily a function but describes the relation defining $\mathcal{N}$. From Fig. 1b, and the w.s. solvability assumption, the equation

$$e = v + \mathcal{K}(v) \tag{1}$$

has a unique solution for all allowed e. Figure 1c shows the dual of Fig. 1b; then, with the notations shown in Fig. 1c, $\hat{i} = v$ and $\hat{v} = i$, by duality. By self-duality, $i = \mathcal{K}(v)$ implies $v = \mathcal{K}(i)$, or what is the same $\hat{i} = \mathcal{K}(\hat{v})$. From Fig. 1c, KCL gives $\hat{j} = e = \hat{v} + \hat{i}$,

hence
$$e = \hat{v} + \mathcal{K}(\hat{v}). \tag{2}$$

Since for all allowed e, this equation has a unique solution, Eqs. (1) and (2) imply that $v = \hat{v}$. Hence, v = i and the one-port $\mathcal{N}$ is equivalent to a one-ohm resistor when it is driven by any allowed voltage source in series with a one-ohm resistor. That it is equivalent to a one-ohm resistor under all compatible connections is obvious by contradiction: suppose it were not true, then there would exist a compatible one-port $\mathcal{N}'$ such that the connection $\mathcal{N} - \mathcal{N}'$ has a solution $[\tilde{v}, \tilde{i}]$ with $\tilde{v} \neq \tilde{i}$. Now consider $\mathcal{N}_a$ driven by the allowed voltage source $\tilde{e} \triangleq \tilde{v} + \tilde{i}$: by the w.s. solvability assumption and the definition of $\tilde{v}$, $\tilde{i}$ there is only one possible port voltage and port current, namely, $\tilde{v}$ and $\tilde{i}$. But the previous proof requires $\tilde{i} = \tilde{v}$. This is a contradiction, hence $\mathcal{N}$ is equivalent to a one-ohm resistor under all compatible connections, i.e., $\mathcal{N}$ is constant resistance.

2. Constant resistance implies self-duality and w.s. solvability. Let $[v_0, i_0]$ be an arbitrary pair of $\mathcal{N}$. Consider the one-port $\mathcal{N}'_0$ shown in Fig. 2: the current source i_0 and the voltage source v_0 of $\mathcal{N}'_0$ are independent sources; the nullator admits only the pair $[0, 0]$. By KCL, KVL and the defining relations of the elements of $\mathcal{N}'_0$, the one-port $\mathcal{N}'_0$ admits only one pair $[v_0, -i_0]$. The connection $\mathcal{N} - \mathcal{N}'_0$ has a unique solution: $[v_0, i_0]$, i.e., $\mathcal{N}'_0$ is compatible with $\mathcal{N}$. By the constant resistance assumption, $v_0 = i_0$. Thus we have shown that, for all $[v, i] \in \mathcal{N}$, $v = i$. This implies that $\mathcal{N}$ is self-dual. Given any allowed voltage e, the only solution of $e = v + i$, with $[v, i] \in \mathcal{N}$, is $v = i = (e/2)$, i.e., $\mathcal{N}$ is w.s. solvable.

Q.E.D.

It follows from the proof of the theorem that $\mathcal{N}$ is constant resistance if and only if $v = i$ for all $[v, i] \in \mathcal{N}$.

Remarks.

a. By interpreting all voltages and all currents as n-vectors one sees that all definitions and derivations are still valid, consequently the theorem holds for n-ports.

b. It should be stressed that the point of view adopted in this paper is strictly black box: only the port voltage and the port current are observable and the set of all pairs $[v, i]$ constitute the complete description of the one-port. An immediate consequence is that the theorem applies to any one-port: its elements may be lumped or distributed, active or passive, linear or nonlinear, time-varying or time-invariant. On the other hand one should keep in mind that the black box self-duality defined here does not imply, for example, that the graph of the network inside the box is a self-dual graph. For example, the linear time-invariant network of Fig. 3 of a previous paper [3] is self dual in the present (black box) sense but its graph is not a self-dual graph.

c. Given an arbitrary one-port $\mathcal{N}$ and its dual $\mathcal{N}^*$ (as defined in this paper), it is possible to use $\mathcal{N}$ and $\mathcal{N}^*$ as elements to obtain constant resistance one-ports. (See Examples 1 and 2 of Sec. III in Ref. [4].)

d. Let a be a fixed real number. If in the one-port shown in Fig. 2 we set $v_0(t) = -i_0(t) = a$ for all t, we then obtain a constant resistance one-port: indeed, its only pair is $[a, a]$. With $a = 0$, we see that the nullator is a constant resistance one-port.

e. The following one-port $\mathcal{N}_1$ shows that self-duality implies neither constant resistance nor w.s. solvability. Let $\mathcal{N}_1$ admit only constant voltages and currents and let its admissible pairs be $[V, I]$ where either $V = 2I$ or $V = 2^{-1}I$. $\mathcal{N}_1$ is clearly self-dual but neither constant resistance nor w.s. solvable.

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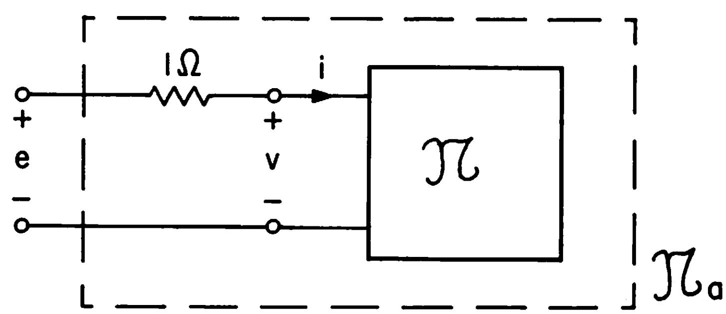

Fig. 1a. Physical relation between e, i, and v.

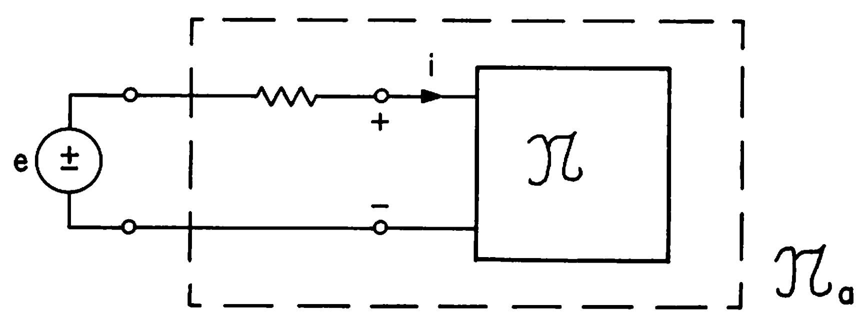

Fig. 1b. Circuit required for testing solvability.

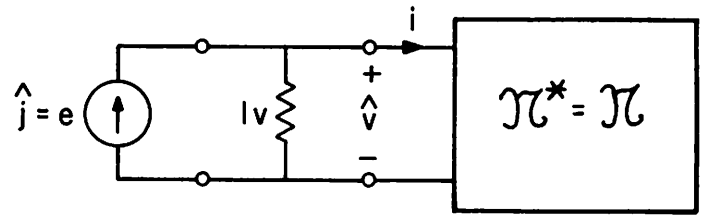

Fig. 1c. The dual of (b). $\mathcal{N}^*$, the dual of $\mathcal{N}$, is identical to $\mathcal{N}$ since $\mathcal{N}$ is self dual.

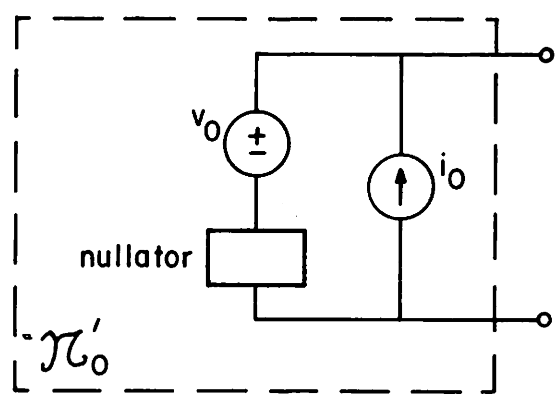

Fig. 2. $\mathcal{N}_0'$ is compatible with $\mathcal{N}$, and $\mathcal{N}_0'$ has only one admissible voltage current pair, namely $[v_0, -i_0]$.