BOUNDED-INPUT BOUNDED-OUTPUT STABILITY OF NONLINEAR TIME-VARYING DISCRETE CONTROL SYSTEMS

by

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ABSTRACT

It is shown that a discrete control system is bounded-input bounded-output (BIBO) stable if and only if there exist Liapunov functions possessing certain properties. As an illustration, a frequency criterion is developed for the BIBO stability of a class of nonlinear discrete systems.

I. INTRODUCTION

The direct or second method of Liapunov has greatly advanced the study of the stability of force-free differential system [1,2]. Many of these techniques have been extended to the study of force-free discrete systems [2, 3]. However, if one is interested in the qualitative behavior of the outputs of a differential or discrete system arising from a large class of inputs, then the situation is quite different because one is no longer dealing with a single equation and the usual Liapunov theory cannot be applied. One important problem of this nature is the study of the stability properties of a differential system subject to persistent disturbances [1].

Recently [4], the Liapunov direct method has been successfully extended to the study of the Bounded-Input Bounded-Output (BIBO) stability (see Def. 2.1) of arbitrary, nonlinear, time-varying, differential systems, and in a previous paper [5] these results have been applied to the Lur'e problem.

In this paper the results of an earlier paper [4] are extended to cover arbitrary, nonlinear, time-varying, discrete control systems. Specifically, it is shown (Theorems 3.1, 3.2) that a discrete control system is BIBO stable if and only if there exist Liapunov functions possessing certain properties. As an example (Theorem 4.1) we obtain a frequency criterion for the BIBO stability of a class of nonlinear sampled-data control systems.

II. DEFINITIONS AND NOTATIONS

Consider a discrete control system that can be represented as a vector difference equation of the form (2.1).

$$x(k+1) = f(x(k),\ u(k),\ k), \quad k \in I \tag{2.1}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control input, I is the set of nonnegative integers, and for each fixed $k \in I$, the next-state function f is a continuous mapping of $R^n \times R^m$ into R^n. If $x \in R^n$ $(u \in R^m)$, then $|x|$ $(|u|)$ denotes the usual Euclidean norm of x(u). By a control or input sequence $\underline{u} = (u(0),\ u(1),\ \ldots)$ we mean any sequence $\{u(k)\}$ of vectors in R^m, and $\|\underline{u}\| = \sup\{|u(k)| \mid k \in I\}$.

Let $(x(k), k)$ be any initial condition and let $\underline{u}$ be any control sequence. Then for $k' \geq k$ $(k' \in I)$ let $x_{\underline{u}}(k'; x(k), k)$ denote the state of the system (2.1) at time k' if it starts from state x(k) at time k and the input sequence $\underline{u}$ is applied. Thus,

$$x_{\underline{u}}(k; x(k), k) = x(k) \quad \text{and}$$

$$x_{\underline{u}}(k'+1; x(k), k) = f\Big(x_{\underline{u}}(k',\ x(k),\ k),\ u(k'),\ k'\Big)$$

for $k' \geq k$.

Definition 2.1. The system (2.1) is said to be Bounded-Input Bounded-Output (BIBO) stable if for every $a \geq 0$ and $\alpha \geq 0$ there is a finite number $\beta = \beta(a, \alpha)$ such that, for every initial condition $(x(k), k)$ with $|x(k)| \leq \alpha$ and every control sequence $\underline{u}$ with $\|\underline{u}\| \leq a$, we have

$$|x_{\underline{u}}(k'; x(k), k)| \leq \beta$$

for all $k' \geq k$.

In obtaining necessary and sufficient conditions for a system to be BIBO stable, a special class of Liapunov functions need to be considered.

Definition 2.2. a. For every number $\rho \geq 0$, let $\Delta_\rho = \{x \in R^n \mid |x| \geq \rho\}$.

b. A Liapunov function $V(x,k)$ defined on $\Delta_\rho \times I$ is said to possess property A if there exists a real-valued, continuous, nondecreasing function $\gamma(r) \geq 0$ defined for $r \geq 0$ such that

$$V(x, k) \leq \gamma(|x|)$$

for every $x \in \Delta_\rho$ and every $k \in I$. It is said to possess property B if there exists a real-valued, continuous, nondecreasing function $\delta(r) \geq 0$ defined for $r \geq 0$ with $\delta(r) \to \infty$ as $r \to \infty$ and such that

$$V(x, k) \geq \delta(|x|)$$

for every $x \in \Delta_\rho$ and every $k \in I$.

Definition 2.3. Let $V(x,k)$ be a Liapunov function defined on $\Delta_\rho \times I$ and let $\underline{u}$ be any control sequence. Then for each $(x,k) \in \Delta_\rho \times I$, the forward difference at (x,k) corresponding to $\underline{u}$ is defined as

$$\Delta V_{\underline{u}}(x, k) = V(x_{\underline{u}}(k+1; x, k), k+1) - V(x, k).$$

III. CONDITIONS FOR BIBO STABILITY

Theorem 3.1. (Stability Theorem). If for each $a \geq 0$ there is a finite number $\rho = \rho(a) \geq 0$, and a Liapunov function $V = V_a$ defined on $\Delta_\rho \times I$, possessing properties A and B, and such that

$$\Delta V_{\underline{u}}(x, k) \leq 0$$

for each $(x,k) \in \Delta_\rho \times I$ and each control sequence $\underline{u}$ with $\|\underline{u}\| \leq a$, then the system (2.1) is BIBO stable.

Proof. Let $a \geq 0$ and $\alpha \geq 0$ be arbitrary. Let $\underline{u}$ be any input sequence with $\|\underline{u}\| \leq a$ and let (x,k) be any initial condition with $|x| \leq \alpha$. Without loss of generality it can be assumed that $|x| \geq \rho(a)$. Since V possesses property A,

$$V(x, k) \leq \gamma(|x|) \leq \gamma(\alpha).$$

Since V possesses property B, and since $\delta(r) \to \infty$ as $r \to \infty$, there is a real number $\beta = \beta(\alpha, a)$ such that $\delta(\beta) = \gamma(\alpha)$ and

$$V(x', k') \geq \delta(|x'|) \quad \text{for all} \quad (x', k') \in \Delta_\rho \times I.$$

Since V has nonpositive forward difference,

$$V(x, k) \geq V(x_{\underline{u}}(k'; x, k), k')$$

for $k' \geq k$. Therefore,

$$\delta(\beta) = \gamma(\alpha) \geq V(x, k) \geq V(x_{\underline{u}}(k'; x, k), k') \geq \delta(|x_{\underline{u}}(k'; x, k)|)$$

so that

$$\beta \geq |x_{\underline{u}}(k'; x, k)|$$

for all $k' \geq k$.

Q. E. D.

To prove the result converse to Theorem 3.1, we need a preliminary fact.

Definition 3.1. For each (x, k) in $R^n \times I$ and each control sequence $\underline{u}$, let

$$P_{\underline{u}}(x, k) = \left\{ y \in R^n \,\middle|\, x_{\underline{u}}(k; y, \tau) = x \text{ for some } \tau \text{ with } 0 \leq \tau \leq k \right\}.$$

Thus, $y \in P_{\underline{u}}(x, k)$ if and only if for some initial time τ, $0 \leq \tau \leq k$, the control sequence $\underline{u}$ takes the system from (y, τ) to (x, k).

The next lemma is an immediate consequence of the continuity of the next-state function f.

Lemma 3.1. For each $(x,k) \in R^n \times I$ and each control sequence $\underline{u}$, $P_{\underline{u}}(x,k)$ is a nonempty, closed subset of R^n.

Theorem 3.2. Suppose that the system (2.1) is BIBO stable. Then for each $a \geq 0$ there is a finite number $\rho = \rho(a) \geq 0$ and a Liapunov function $V(x,k) = V_a(x,k)$ defined on $\Delta_\rho \times I$ possessing properties A and B, and such that the forward difference

$$\Delta V_{\underline{u}}(x, k) \leq 0$$

for every $(x,k) \in \Delta_\rho \times I$, and every control sequence $\underline{u}$ with $\|\underline{u}\| \leq a$.

Proof. Let $a \geq 0$ be fixed. By hypothesis there is a function $\beta(\alpha)$ such that $|x_{\underline{u}}(k'; x, k)| \leq \beta(\alpha)$ for all $k' \geq k$, for all initial conditions (x,k) with $|x| \leq \alpha$, and for all input sequences $\underline{u}$ with $\|\underline{u}\| \leq a$. Without loss of generality we can assume that β is a continuous, strictly increasing function, and that $\beta(\alpha) \to \infty$ as $\alpha \to \infty$. Hence the inverse function $\alpha(\beta)$ of β is defined for all $\beta \geq \beta(0)$ and has the same properties as β.

Let $\rho = \rho(a) = \beta(0)$. For each $(x,k) \in \Delta_\rho \times I$, and for each $\underline{u}$ with $\|\underline{u}\| \leq a$ let

$$K_{\underline{u}}(x, k) = \min\left\{ |y| \;\middle|\; y \in P_{\underline{u}}(x, k) \right\}$$

where $P_{\underline{u}}(x,k)$ is given by Def. 3.1. Now for each $(x,k) \in \Delta\rho \times I$ define $V(x,k)$ by

$$V(x, k) = \inf \left\{ K_{\underline{u}}(x, k) \;\middle|\; \|\underline{u}\| \leq a \right\}.$$

We now show that this Liapunov function has the desired properties. First of all, since $x \in P_{\underline{u}}(x,k)$ it follows that $K_{\underline{u}}(x,k) \leq |x|$, so that

$$V(x, k) \leq |x| ,$$

and hence V has property A. To show that V has property B it is enough to show that $V(x,k) \geq \alpha(|x|)$. Indeed, suppose that for some $(x,k) \in \Delta_\rho \times I$,

$$V(x, k) = \alpha(|x|) - \epsilon$$

for some $\epsilon > 0$. From the definition of V it follows that there is a control sequence $\underline{u}$ with $\|\underline{u}\| \leq a$ such that

$$K_{\underline{u}}(x, k) \leq \alpha(|x|) - \frac{\epsilon}{2}.$$

Thus $K_{\underline{u}}(x,k) + \frac{\epsilon}{2} \leq \alpha(|x|)$. Let $x^* \in P_{\underline{u}}(x,k)$ and $0 \leq \tau \leq k$ be such that $|x^*| = K_{\underline{u}}(x,k)$ and $x = x_{\underline{u}}(k; x^*, \tau)$. Then

$$|x^*| + \epsilon/2 \leq \alpha(|x|)$$

so that

$$\beta(|x^*|) < \beta(\alpha(|x|)) = |x|.$$

But since $x = x_{\underline{u}}(k; x^*, \tau)$ and System (2.1) is BIBO stable, it follows that $|x| \leq \beta(|x|)$ which gives a contradiction. Thus V has property B.

Next, let $(x, k) \in \Delta_\rho \times I$ and let $\underline{u}$ be any control sequence with $\|\underline{u}\| \leq a$. Let $x' = x_{\underline{u}}(k+1; x, k)$. It is then clear from the definition of V that

$$V(x', k+1) \leq V(x, k),$$

or

$$\Delta V_{\underline{u}}(x, k) \leq 0.$$

Q. E. D.

IV. AN EXAMPLE

In this section we apply the stability theorem to obtain a frequency criterion for a class of nonlinear discrete systems.

Consider the discrete control system (4.1).

$$\begin{aligned} x(k+1) &= Ax(k) - b\varphi(\sigma(k)), \\ \sigma(k) &= c'x(k) + u(k), \end{aligned} \tag{4.1}$$

where x, b, and c are real n-vectors, u is the scalar input, σ is the scalar error, x is the state, A is an $n \times n$ real matrix, and φ is a real-valued function of σ.

Definition 4.1. Let A be an $n \times n$ real matrix and b a real n-vector. We say that the pair (A, b) is controllable if the system

$$x(k+1) = Ax(k) - bu$$

is controllable [6].

The proof of the following lemma is similar to the proof of the Main Lemma by Lefschetz [7, pp. 114-118] and hence it is omitted.

Lemma 4.1. Given

(i) real $n \times n$ matrix A whose eigenvalues λ_i satisfy $|\lambda_i| < 1$,

(ii) real n-vectors γ, g, and b such that (A, b) is controllable,

and

(iii) real positive number τ_1,

such that $T_0(z) > 0$ for all values of the complex variable z with $|z| \geq 1$, where

$$T_0(z) = \tau_1 + m^*(z)\gamma + \gamma' m(z) - m^*(z) gg' m(z)$$

and

$$m(z) = (zI - A)^{-1} b.$$

Then there exists a real n-vector q and symmetric positive definite real matrices Q and P such that

(i) $A'QA - Q + gg' = -(P + qq')$,

(ii) $\tau_1 > b'Qb > 0$,

and

(iii) $\sqrt{\tau_1 - b'Qb}\, q + \gamma = A'Qb$.

Theorem 4.1. Suppose that the system (4.1) has the following properties:

(i) The eigenvalues λ_i of A satisfy $|\lambda_i| < 1$,

(ii) (A, b) is controllable,

(iii) $\varphi(0) = 0$; $0 \leq \sigma\varphi(\sigma) \leq K\sigma^2$ for some $K < \infty$ and for all σ,

and

$$0 \leq \left|\frac{d\varphi(\sigma)}{d\sigma}\right| \leq K'$$

for some $K' < \infty$ and for all σ, and

(iv) There exist real numbers $\alpha \geq 0$ and β such that $\operatorname{Re} T(z) > 0$ for all $|z| \geq 1$, where

$$T(z) = \frac{\alpha}{K} + [\alpha + \beta(z-1)]\,W(z) - \frac{K'}{2}\,|\beta|\,|(z-1)W(z)|^2$$

and

$$W(z) = c'(zI - A)^{-1}b = c'm(z)$$

Then

(a) System (4.1) is BIBO stable,

and (b) System (4.1) is asymptotically stable in the large under zero-input ($u \equiv 0$) conditions.

Proof. Let

$$\tau_1 = \frac{\alpha}{K} - \frac{1}{2}K'|\beta|\,(b'c)^2 + \beta b'c,$$

$$2\gamma' = \alpha c' + \beta c'(A-I) - K'|\beta|\,(b'c)\,c'(A-I),$$

and

$$g = \sqrt{(1/2)K'|\beta|}\,(A' - I)\,c.$$

Then a slight amount of algebraic manipulation yields

$$\operatorname{Re} T(z) = \tau_1 + 2 \operatorname{Re} \gamma' m(z) - |g' m(z)|^2 .$$

Let

$$T_0(z) = \tau_1 + \gamma' m(z) + m^*(z)\gamma - m^*(z) g g' m(z) .$$

Then item (iv) of the hypothesis implies that $T_0(z) > 0$ for $|z| \geq 1$ so that we can apply Lemma 4.1. Let $a \geq 0$ be fixed and let $\underline{u}$ be any input sequence with $\|\underline{u}\| \leq a$. Define

$$V(x,k) = x'Qx + \beta \int_0^{\sigma} \varphi(\sigma')\, d\sigma' .$$

Clearly since Q is positive definite, V possesses properties A and B (see Def. 2.2). Also

$$\Delta V_{\underline{u}}(x,k) = x'(A'QA - Q)x - 2\varphi(\sigma) b'QAx + \varphi^2(\sigma) b'Qb$$
$$+ \beta \int_{\sigma}^{\sigma_1} \varphi(\sigma')\, d\sigma' . \qquad (4.2)$$

where $\qquad \sigma_1 = c'x(k+1) + u(k+1) .$

By (iii) of the hypothesis,

$$\beta \int_{\sigma}^{\sigma_1} \varphi(\sigma')\,d\sigma' = \beta \int_{\sigma}^{\sigma_1} \frac{\varphi(\sigma') - \varphi(\sigma)}{\sigma' - \sigma} (\sigma' - \sigma)\,d\sigma' + \beta\varphi(\sigma) \int_{\sigma}^{\sigma_1} d\sigma' .$$

Hence

$$\beta \int_{\sigma}^{\sigma_1} \varphi(\sigma') \leq \frac{1}{2} K'|\beta| (\sigma_1 - \sigma)^2 + \beta\varphi(\sigma)(\sigma_1 - \sigma) , \tag{4.3}$$

where

$$\sigma_1 - \sigma = c'(A - I)x - c'b\varphi(\sigma) + \Delta u(k) , \tag{4.4}$$

where $\Delta u(k) = u(k+1) - u(k)$.

We also have the identity

$$0 = \alpha\varphi(\sigma)c'x - \frac{\alpha}{K}\varphi^2(\sigma) + \alpha\varphi(\sigma)u(k) - \alpha\varphi(\sigma)\Big(\sigma - \frac{1}{K}\varphi(\sigma)\Big) . \tag{4.5}$$

Adding (4.5) to (4.2) and using (4.3) and rearranging terms, we obtain (4.6).

$$\begin{aligned} \Delta V_{\underline{u}}(x,k) \leq\ & x'(A'QA - Q + gg')x + 2\varphi(\sigma)(\gamma' - b'QA)x \\ & - \varphi^2(\sigma)(\tau_1 - b'Qb) + \sqrt{2K'|\beta|}\ \Delta u(k)\,g'x \\ & + \big[(\beta - K'|\beta|\, c'b)\Delta u(k) + \alpha u(k)\big]\varphi(\sigma) \\ & - \alpha\varphi(\sigma)\Big(\sigma - \frac{1}{K}\varphi(\sigma)\Big) + \frac{1}{2}K'|\beta|\big(\Delta u(k)\big)^2 . \end{aligned} \tag{4.6}$$

By Lemma 4.1 therefore,

$$\Delta V_{\underline{u}}(x,k) \leq -x'Px - \left[g'x + \sqrt{\tau_1 - b'Qb}\;\varphi(\sigma)\right]^2$$
$$+ \sqrt{2K'|\beta|}\,\Delta u(k)\,g'x + \left[(\beta - K'|\beta|\,c'b)\,\Delta u(k) + \alpha u(k)\right]\varphi(\sigma)$$
$$- \alpha\left(\sigma - \frac{1}{K}\varphi(\sigma)\right)\varphi(\sigma) + \frac{1}{2}K'|\beta|\left(\Delta u(k)\right)^2. \tag{4.7}$$

Since $\|\underline{u}\| \leq a$, it follows that $|u(k)| \leq a$ and $|\Delta u(k)| \leq 2a$. Let $\xi_1(a) = 2a\sqrt{2K'|\beta|}$, $\xi_2(a) = |(\beta - K'|\beta|\,c'b)|\,2a + |\alpha|\,a$, $\xi_3(a) = 2K'|\beta|\,a^2$ and $\zeta = \inf_{\sigma} \alpha(\sigma - \frac{1}{K}\varphi(\sigma))\varphi(\sigma)$ Then $\zeta \geq 0$ and

$$\Delta V_{\underline{u}}(x,k) \leq -x'Px - \left[g'x + \sqrt{\tau_1 - b'Qb}\;\varphi(\sigma)\right]^2$$
$$+ \xi_1(a)\,g'(x) + \xi_2(a)\varphi(\sigma) - \zeta + \xi_3(a).$$

By Lemma 4.1 P is positive definite and $(\tau_1 - b'Qb) > 0$ so that there exists a finite number $\rho(a)$ such that $|x| \geq \rho(a)$ implies that

$$\Delta V_{\underline{u}}(x,k) \leq 0,$$

and hence (4.1) is BIBO stable. Also, if $\underline{u} \equiv 0$, we see from (4.7) that

$$\Delta V(x,k) \leqq -x'Px$$

so that (4.1) is asymptotically stable in the large.

Q.E.D.

Remarks.

1. Szegö [8] has presented Lemma 4.1 except that in his result P is equal to the zero matrix. For our purposes, however, it is crucial that P be positive definite.

2. The conclusion (b) of Theorem 4.1 has also been proved by a different argument of Jury and Lee [9].

3. A slightly weaker version of Theorem 4.1 has been presented with a different proof by Iwens and Bergen [10].

V. CONCLUSIONS

It has been shown that the BIBO stability of any discrete control system is equivalent to the existence of certain Liapunov functions and the results have been applied to an interesting class of discrete systems. An interesting area of further study is the investigation of the relationship between zero-input stability and BIBO stability. A relatively weak result in this direction has been presented in [4].

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