CONTROLLABILITY AND OBSERVABILITY OF COMPOSITE SYSTEMS

by

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ABSTRACT

This paper solves completely the characterization of parallel tandem and feedback connection of linear time-invariant differential systems. It is also shown that even for nonlinear systems the controllability and observability of the feedback connection is equivalent to that of specific tandem connections.

1. INTRODUCTION

In this paper a complete characterization of the controllability and observability of some natural representations of composite systems is presented. These systems are obtained by connecting in parallel, in tandem, and in a feedback loop, linear time-invariant finite dimensional systems. Emphasis is placed on the distinction between a system and one of its representations. Some problems treated by Gilbert [1] for the case of distinct eigenvalues are completely solved here. In addition we establish the equivalence, in a very broad setting, of the controllability and the observability of a feedback connection to that of a tandem connection: it is interesting to note that for controllability this equivalence applies to nonlinear systems.

In Secs. 2 and 3 the notation and some basic facts are given, also a main theorem stated by Kalman [6] is proved. In Sec. 4 we consider the parallel connections. In Secs. 5 and 6 we cover the tandem and the feedback connections.

2. SYSTEMS AND THEIR REPRESENTATIONS

The basic notion associated with a system is the set of all its input-output pairs: this is the set of all inputs, $\underset{\sim}{u}$, and outputs $\underset{\sim}{y}$ where $\underset{\sim}{u}$ and $\underset{\sim}{y}$ are functions of time defined on $[t_0, \infty)$ where the initial time t_0 may vary from pair to pair and ranges over the whole real line. Given the set of all pairs $(\underset{\sim}{u}, \underset{\sim}{y})$, we assume that it is possible to assign a set of parameters, called the state. It is assumed that this parameterization satisfies all the consistency requirements [2] and is such that the output $\underset{\sim}{y}$ is a <u>function</u> of the input $\underset{\sim}{u}$ and the initial state. Given a system, there are many ways of assigning a state to it and, consequently, it is important to keep in mind the difference between a system and its representation. Indeed, the system--i.e., the set of all input-output pairs--consists of basic physical data whereas the state is an intellectual construct used for calculation. Given a system, there are many ways of assigning to it a state; some of these are more useful or more illuminating than others. Therefore, throughout this paper we make a sharp distinction between a system and one of its representations.

A representation of a system is a set of rules that allows the calculation of the output $\underset{\sim}{y}$ on the basis of the initial state and of the input $\underset{\sim}{u}$.

In this paper we consider almost exclusively differential systems that are linear and time-invariant. Typically, we consider a system $\mathcal{S}$ that has the following representation S:

$$S: \dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{B}\underset{\sim}{u}, \tag{2.1a}$$

$$\underset{\sim}{y} = \underset{\sim}{C}\underset{\sim}{x} + \underset{\sim}{D}\underset{\sim}{u}, \tag{2.1b}$$

where $\underset{\sim}{x}$, the state, is an n-dimensional vector; the input $\underset{\sim}{u}$, a $p \times 1$ vector; the output $\underset{\sim}{y}$, a $q \times 1$ vector. $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$, and $\underset{\sim}{D}$ are, respectively, $(n \times n)$, $(n \times p)$, $q \times n)$, and $q \times p)$ matrices. One of the simplest methods for obtaining <u>another</u> representation for the <u>same</u> system $\mathcal{S}$ is to perform a nonsingular constant transformation $\underset{\sim}{T}$ on the state space: $\underset{\sim}{x} = \underset{\sim}{T}\underset{\sim}{\bar{x}}$ where $\bar{x}$ is the new state. Both representations describe the same system $\mathcal{S}$ since they lead to the same set of input-output pairs; only the parametrization of the pairs is different.

The <u>zero-state</u> response of $\mathcal{S}$ is completely characterized by the $q \times n$ matrix (called the transfer function matrix)

$$\hat{\underset{\sim}{G}}(s) \triangleq \underset{\sim}{C}(s\underset{\sim}{I} - \underset{\sim}{A})^{-1}\underset{\sim}{B} + \underset{\sim}{D}. \tag{2.2}$$

Indeed

$$\hat{\underset{\sim}{y}}(s) = \underset{\sim}{G}(s)\,\underset{\sim}{u}(s), \tag{2.3}$$

where $\hat{\underset{\sim}{u}}(s)$ and $\hat{\underset{\sim}{y}}(s)$ are the Laplace transforms of the input $\underset{\sim}{u}$ and of the corresponding zero-state response $\underset{\sim}{y}$; the (i, j) component of $\hat{\underset{\sim}{G}}(s)$ is the Laplace transform of the zero-state response at the ith output when all inputs are identically zero except for the jth input which is a unit impulse, $\delta(t)$. From its definition, $\hat{\underset{\sim}{G}}(s)$ is independent of the chosen parametrization of the input-output pairs. It should be stressed that $\hat{\underset{\sim}{G}}(s)$ does not characterize $\mathcal{S}$, indeed it characterizes only the zero-state responses of $\mathcal{S}$ [2]. In terms of the Kalman decomposition [3] $\hat{\underset{\sim}{G}}(s)$ characterizes only the part that is both controllable and observable.

In order to test the representation of composite systems, we start by considering differential systems with representations S_1 and S_2, respectively:

$$S_i: \quad {}_i\dot{\underset{\sim}{x}} = {}_i\underset{\sim}{A}\,{}_i\underset{\sim}{x} + {}_i\underset{\sim}{B}\,{}_i\underset{\sim}{u} \tag{2.4a}$$

$${}_i\underset{\sim}{y} = {}_i\underset{\sim}{C}\,{}_i\underset{\sim}{x} + {}_i\underset{\sim}{D}\,{}_i\underset{\sim}{u} \tag{2.4b}$$

where ${}_i\underset{\sim}{A}$, ${}_i\underset{\sim}{B}$, ${}_i\underset{\sim}{C}$, ${}_i\underset{\sim}{D}$ are ${}_in \times {}_in$, ${}_in \times {}_ip$, ${}_iq \times {}_in$ and ${}_iq \times {}_ip$ matrices, respectively; ${}_i\underset{\sim}{u}$ is a ${}_ip \times 1$ input vector, ${}_i\underset{\sim}{y}$ is a ${}_iq \times 1$ output vector, and ${}_i\underset{\sim}{x}$ is an ${}_in \times 1$ state vector. A few words about notations: in general, the subscript on the left-hand side denotes the system (e.g., ${}_i\underset{\sim}{x}$, ${}_i\underset{\sim}{A}$, ${}_ip$). Subscripts on the right-hand side of these letters will be required in the next section. It is well known that Eq. (2.4a) has a unique solution for ${}_i\tilde{x}$ and that ${}_i\tilde{x}$ qualifies as the state of $\mathcal{S}_i$. Let Σ_i be the state space of the representation S_i, thus ${}_i\tilde{x} \in \Sigma_i$.

Consider now the three basic interconnections of S_1 and S_2: the parallel, the tandem, and the feedback connections. It is assumed that S_1 and S_2 are initially in the zero state and connected at $t = 0$. Let $\tilde{u}$ and $\tilde{y}$ be the input and output of the composite system.

a. To represent the parallel connection $\mathcal{S}_1$ and $\mathcal{S}_2$, we write Eq. (2.4) with ${}_1\tilde{u}(t) = {}_2\tilde{u}(t) = \tilde{u}(t)$ and $\tilde{y}(t) = {}_1\tilde{y}(t) + {}_2\tilde{y}(t)$ for all $t \geq 0$. It is clear that $\tilde{y}$ is a function of ${}_1\tilde{x}$, ${}_2\tilde{x}$, and $\tilde{u}$; hence the composite state $\begin{bmatrix} {}_1\tilde{x} \\ {}_2\tilde{x} \end{bmatrix}$, ranging over the direct sum of the state space of S_1 and S_2, denoted by $\Sigma_1 \oplus \Sigma_2$, qualifies as the state of the parallel connection of $\mathcal{S}_1$ and $\mathcal{S}_2$. The transfer function of the parallel connection is $\hat{\tilde{G}}_1(s) + \hat{\tilde{G}}_2(s)$.

b. For the representation of the tandem connection of $\mathcal{S}_1$ followed by $\mathcal{S}_2$, we choose Eq. (2.4) with ${}_1\tilde{u}(t) = \tilde{u}(t)$, ${}_2\tilde{u}(t) = {}_1\tilde{y}(t)$, $\tilde{y}(t) = {}_2\tilde{y}(t)$ for all $t \geq 0$; and the composite state $\begin{bmatrix} {}_1\tilde{x} \\ {}_2\tilde{x} \end{bmatrix}$ ranging over $\Sigma_1 \oplus \Sigma_2$, qualifies as the state. The transfer function of the tandem connection is $\hat{\tilde{G}}_2(s)\, \hat{\tilde{G}}_1(s)$.

c. The feedback connection with $\mathcal{S}_1$ in the forward path and $_2$ in the feedback path is shown in Fig. 2.1.

In order to make the connection possible, we need ${}_1p = {}_2q$, ${}_2p = {}_1q$. Its representation is Eq. (2.4) with ${}_1\underset{\sim}{u} = \underset{\sim}{u} - {}_2\underset{\sim}{y}$, $\underset{\sim}{y} = {}_1\underset{\sim}{y} = {}_2\underset{\sim}{u}$. However, in order to qualify $\begin{bmatrix} {}_1\underset{\sim}{x} \\ {}_2\underset{\sim}{x} \end{bmatrix}$ as the state of the feeback connection of $\mathcal{S}_1$ and $\mathcal{S}_2$, we need an additional condition. Taking the Laplace transform of Eq. (2.4), using ${}_1\hat{\underset{\sim}{u}}(s) = \hat{\underset{\sim}{u}}(s) - {}_2\hat{\underset{\sim}{y}}(s)$, $\hat{\underset{\sim}{y}}(s) = {}_1\hat{\underset{\sim}{y}}(s) = {}_2\hat{\underset{\sim}{u}}(s)$, and by making some simple manipulations, we obtain the following two sets of equations:

$$\begin{aligned} \left[\underset{\sim}{I} + \hat{\underset{\sim}{G}}_1(s)\, \hat{\underset{\sim}{G}}_2(s)\right] \hat{\underset{\sim}{y}}(s) = {} & \hat{\underset{\sim}{G}}_1(s)\, \hat{\underset{\sim}{u}}(s) \\ & + {}_1\underset{\sim}{C}(s\underset{\sim}{I} - {}_1\underset{\sim}{A})^{-1}\, {}_2\underset{\sim}{x}(0) \\ & - \hat{\underset{\sim}{G}}_1(s)\, {}_2\underset{\sim}{C}(s\underset{\sim}{I} - {}_2\underset{\sim}{A})^{-1}\, {}_2\underset{\sim}{x}(0)\,, \end{aligned} \qquad (2.5)$$

$$\left\{ \begin{aligned} \left[\underset{\sim}{I} + \hat{\underset{\sim}{G}}_2(s)\, \hat{\underset{\sim}{G}}_1(s)\right] {}_1\hat{\underset{\sim}{u}}(s) = {} & \hat{\underset{\sim}{u}}(s) - \hat{\underset{\sim}{G}}_2(s)\, {}_1\underset{\sim}{C}(s\underset{\sim}{I} - {}_1\underset{\sim}{A})^{-1}\, {}_1\underset{\sim}{x}(0) \\ & - {}_2\underset{\sim}{C}(s\underset{\sim}{I} - {}_2\underset{\sim}{A})^{-1}\, {}_2\underset{\sim}{x}(0)\,, \qquad (2.6a) \\ \hat{\underset{\sim}{y}}(s) = \hat{\underset{\sim}{G}}_1(s)\, {}_1\hat{\underset{\sim}{u}}(s) + {}_1\underset{\sim}{C}(s\underset{\sim}{I} - {}_1\underset{\sim}{A})^{-1}\, {}_1\underset{\sim}{x}(0)\,, & \qquad (2.6b) \end{aligned} \right.$$

where ${}_1\underset{\sim}{x}(0)$ and ${}_2\underset{\sim}{x}(0)$ are the initial states of S_1 and S_2, respectively. From either Eq. (2.5) or Eq. (2.6) we wish to solve for $\hat{\underset{\sim}{y}}(s)$. Before proceeding, we assert that:

$$\det\left(\underset{\sim}{I} + \hat{\underset{\sim}{G}}_2(s)\, \hat{\underset{\sim}{G}}_1(s)\right) = \det\left(\underset{\sim}{I} + \hat{\underset{\sim}{G}}_1(s)\, \hat{\underset{\sim}{G}}_2(s)\right) .$$

Observe that the order of the matrices are different: the one in the left-hand side is a ${}_2q \times {}_1p({}_2q = {}_1p$ by assumption) matrix; the one in the right-hand side is a ${}_1q \times {}_2p$ $({}_1q = {}_2p)$ matrix. The elements of these matrices are rational functions of s, and since the rational functions form a field [4], standard results in the theory of matrices can be applied here and the proof by Kalman [5] applies here. See also [9, p.46].

We also observe that:

if $\det\left(I + \underset{\sim}{G}_2(s)\, \underset{\sim}{G}_1(s)\right) \neq 0$ for some s, then

$$\hat{\underset{\sim}{G}}_1(s)\left[\underset{\sim}{I} + \hat{\underset{\sim}{G}}_2(s)\, \hat{\underset{\sim}{G}}_1(s)\right]^{-1} = \left[\underset{\sim}{I} + \hat{\underset{\sim}{G}}_1(s)\, \hat{\underset{\sim}{G}}_2(s)\right]^{-1} \hat{\underset{\sim}{G}}_1(s) .$$

<u>Proof:</u> The assumption and the previous observation imply that $\left[\underset{\sim}{I} + \hat{\underset{\sim}{G}}_2(s)\, \hat{\underset{\sim}{G}}_1(s)\right]^{-1}$ and $\left[\underset{\sim}{I} + \hat{\underset{\sim}{G}}_1(s)\, \hat{\underset{\sim}{G}}_2(s)\right]^{-1}$ are well-defined rational functions. Consider the identity

$$\hat{\underset{\sim}{G}}_1(s)\left[\underset{\sim}{I} + \hat{\underset{\sim}{G}}_2(s)\, \hat{\underset{\sim}{G}}_1(s)\right]\left[\underset{\sim}{I} + \hat{\underset{\sim}{G}}_2(s)\, \hat{\underset{\sim}{G}}_1(s)\right]^{-1} = \hat{\underset{\sim}{G}}_1(s) ,$$

which can be rewritten as

$$\left[\underset{\sim}{I} + \hat{\underset{\sim}{G}}_1(s)\, \hat{\underset{\sim}{G}}_2(s)\right] \hat{\underset{\sim}{G}}_1(s)\left[\underset{\sim}{I} + \hat{\underset{\sim}{G}}_2(s)\, \hat{\underset{\sim}{G}}_1(s)\right]^{-1} = \hat{\underset{\sim}{G}}_1(s) ,$$

and from which the asserted equality follows.

With these observations at our disposal, we are ready to establish

<u>Theorem 2.1.</u> In the feedback connection shown in Fig. 2.1, for any input $\underset{\sim}{u}$ there exists a unique output $\underset{\sim}{y}$ if and only if

$$\det\left(\underset{\sim}{I} + \hat{\underset{\sim}{G}}_1(s)\,\hat{\underset{\sim}{G}}_2(s)\right) \neq 0 \text{ for some } s.$$

<u>Proof:</u> ($\Longleftarrow$) That $\det\left(\underset{\sim}{I} + \hat{\underset{\sim}{G}}_1(s)\,\hat{\underset{\sim}{G}}_2(s)\right) \neq 0$ for some s implies that $\left(\underset{\sim}{I} + \hat{\underset{\sim}{G}}_1(s)\,\hat{\underset{\sim}{G}}_2(s)\right)^{-1}$ is a well-defined rational function. Hence from Eq. (2.5) and the one-to-one correspondence between a time function and its Laplace transform, we conclude that for any $\underset{\sim}{u}$ there exists a unique $\underset{\sim}{y}$. Using our second observation, we can easily show that Eq. (2.6) gives the same $\underset{\sim}{y}$.

($\Longrightarrow$) Assume that $\det\left(\underset{\sim}{I} + \hat{\underset{\sim}{G}}_1(s)\,\hat{\underset{\sim}{G}}_2(s)\right) = 0$ for all s. In Eq. (2.6a) let $\underset{\sim}{x}_1(0) = \underset{\sim}{x}_2(0) = 0$ and pick $\hat{\underset{\sim}{u}}(s)$ outside the range of $\left(\underset{\sim}{I} + \hat{\underset{\sim}{G}}_2(s)\,\hat{\underset{\sim}{G}}_1(s)\right)$. Then there is no $\underset{\sim}{u}_1(s)$ satisfying Eq. (2.6a). Consequently, there is no $\underset{\sim}{y}(s)$ satisfying Eq. (2.6).

Q. E. D.

We given an example to illustrate this theorem. Let

$$S_1: \quad {}_1\dot{\underset{\sim}{x}} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} {}_1\underset{\sim}{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} {}_1\underset{\sim}{u}$$

$${}_1\underset{\sim}{y} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} {}_1\underset{\sim}{x} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} {}_1\underset{\sim}{u}$$

$$S_2: \quad {}_2\underset{\sim}{y} = {}_2\underset{\sim}{u},$$

then

$$\hat{\underset{\sim}{G}}_1(s) = \begin{bmatrix} \frac{-s}{s+1} & \frac{1}{s+2} \\ \frac{1}{s+1} & \frac{-s-1}{s+2} \end{bmatrix}, \quad \hat{\underset{\sim}{G}}_2(s) = \underset{\sim}{I} .$$

It is clear that $\det \underset{\sim}{I} + \hat{\underset{\sim}{G}}_1(s)\, \hat{\underset{\sim}{G}}_2(s) = 0$ for all s. Assuming that S_1 is in the zero state; i.e., ${}_1\underset{\sim}{x}(0) = \underset{\sim}{0}$, and choosing

$$\hat{\underset{\sim}{u}}(s) = \begin{bmatrix} \frac{1}{s+2} \\ \frac{1}{(s+1)^2} \end{bmatrix},$$

then Eq. (2.5) gives

$$\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+2} \end{bmatrix} \hat{\underset{\sim}{y}}(s) = \begin{bmatrix} \frac{1 - s(s+1)}{(s+1)^2 (s+2)} \\ 0 \end{bmatrix} \qquad (2.7)$$

It is obvious that there is no such $\hat{\underset{\sim}{y}}(s)$ to satisfy Eq. (2.7).

A direct consequence of theorem 2.1 is that if $\det\left(\underset{\sim}{I} + \hat{\underset{\sim}{G}}_1(s)\, \hat{\underset{\sim}{G}}_2(s)\right) \neq 0$ for some s, then the composite state $\begin{bmatrix} {}_1\underset{\sim}{x} \\ {}_2\underset{\sim}{x} \end{bmatrix}$, ranging over $\Sigma_1 \oplus \Sigma_2$, qualifies as the state of the representation of the feedback system. The transfer function matrix of the system is $\hat{\underset{\sim}{G}}_1(s)\left(\underset{\sim}{I} + \hat{\underset{\sim}{G}}_2(s)\, \hat{\underset{\sim}{G}}_1(s)\right)^{-1} = \left(\underset{\sim}{I} + \hat{\underset{\sim}{G}}_1(s)\, \hat{\underset{\sim}{G}}_2(s)\right)^{-1} \hat{\underset{\sim}{G}}_1(s)$.

The degree in s of the denominator of ${}_1\underset{\sim}{C}(s\underset{\sim}{I} - {}_1\underset{\sim}{A})^{-1}\, {}_1\underset{\sim}{B}$ is at least one degree higher than that of its numerator; hence, by choosing s arbitrary large, we can make ${}_1\underset{\sim}{C}(s\underset{\sim}{I} - {}_1\underset{\sim}{A})^{-1}\, {}_1\underset{\sim}{B}$ as small as we like. Consequently, if $\det(\underset{\sim}{I} + {}_1\underset{\sim}{D}) \neq 0$, then $\det\left(\underset{\sim}{I} + \hat{\underset{\sim}{G}}_1(s)\right) \neq 0$ for some s. Similarly, we find that if $\det(\underset{\sim}{I} + {}_1\underset{\sim}{D}\, {}_2\underset{\sim}{D}) \neq 0$ then $\det\left(\underset{\sim}{I} + \hat{\underset{\sim}{G}}_1(s)\, \hat{\underset{\sim}{G}}_2(s)\right) \neq 0$ for some s. Hence we have established

<u>Corollary 2.1.</u> In the feedback connection shown in Fig. 2.1, if $\det(\underset{\sim}{I} + {}_1\underset{\sim}{D}\, {}_2\underset{\sim}{D}) \neq 0$ then for any input $\underset{\sim}{u}$ there exists a unique output $\underset{\sim}{y}$ and $\begin{bmatrix} {}_1\underset{\sim}{x} \\ {}_2\underset{\sim}{x} \end{bmatrix}$ qualifies as the state of the feedback connection.

3. CONTROLLABILITY AND OBSERVABILITY

Consider the linear, time-invariant, finite dimensional differential system that has the representation S given in Eq. (2.1). The representation S is said to be <u>controllable on the interval $[t_0, t_1]$</u> if for any given state $x(t_0)$ in the state space Σ there exists an input

$\underset{\sim}{u}[t_0, t_1]$ that transfers $\underset{\sim}{x}(t_0)$ to the zero state at time t_1. The representation S is said to be observable on the interval $[t_0, t_1]$ if given any unknown state $\underset{\sim}{x}(t_0)$, the knowledge of the representation (2.1) and of the zero-input response over $[t_0, t_1]$ suffices to determine the state $\underset{\sim}{x}(t_0)$.

It is well known that the following characterizations of controllability are equivalent:

(i) S is controllable in any nonzero subinterval of $[0, \infty)$,

(ii) All the rows of $e^{\underset{\sim}{A}t}\underset{\sim}{B}$ are linearly independent over any open subinterval of $[0, \infty)$,

(iii) The matrix $[\underset{\sim}{B} \vdots \underset{\sim}{A}\underset{\sim}{B} \vdots \cdots \vdots \underset{\sim}{A}^{n-1}\underset{\sim}{B}]$ has rank n, and

(iv) $\underset{\sim}{W}_c(t) \triangleq \int_{t_0}^{t} (e^{\underset{\sim}{A}\tau}\underset{\sim}{B})(e^{\underset{\sim}{A}\tau}\underset{\sim}{B})^* d\tau$ is a positive definite matrix for all $t > t_0$ and any $t_0 \geq 0$. Here "*" denotes the complex conjugate transpose.

The following statements for the observability are equivalent:

(i)' S is observable in any nonzero subinterval of $[0, \infty)$,

(ii)' All the columns of $\underset{\sim}{C}e^{\underset{\sim}{A}t}$ are linearly independent over any open subinterval of $[0, \infty)$,

(iii)' The matrix $[\underset{\sim}{C}^* \vdots \underset{\sim}{A}^*\underset{\sim}{C}^* \vdots \cdots \vdots (\underset{\sim}{A}^*)^{n-1}\underset{\sim}{C}^*]$ has rank n, and (iv)' $\underset{\sim}{W}_0(t) \triangleq \int_{t_0}^{t} (\underset{\sim}{C}\, e^{\underset{\sim}{A}\tau})^* (\underset{\sim}{C}\, e^{\underset{\sim}{A}\tau})\, d\tau$ is a positive definite matrix for all $t > t_0$ and any $t_0 \geq 0$.

These equivalences are proved in the literature [2, 3, 6, 7].

One approach to the discussion of the controllability of, say, the tandem connection would be to observe that the state response starting from the zero state is related to the input ${}_1\underset{\sim}{u}$ of the tandem connection by

$$\begin{bmatrix} {}_1\hat{\underset{\sim}{x}}(s) \\ {}_2\hat{\underset{\sim}{x}}(s) \end{bmatrix} = \begin{bmatrix} (s\,{}_1\underset{\sim}{I} - {}_1\underset{\sim}{A})^{-1}\,{}_1\underset{\sim}{B} \\ (s\,{}_2\underset{\sim}{I} - {}_2\underset{\sim}{A})^{-1}\,{}_2\underset{\sim}{B}[\,{}_1\underset{\sim}{C}(s\,{}_1\underset{\sim}{I} - {}_1\underset{\sim}{A})^{-1}\,{}_1\underset{\sim}{B} + {}_1\underset{\sim}{D}] \end{bmatrix} {}_1\hat{\underset{\sim}{u}}(s)\,. \tag{3.1}$$

From condition (ii) above, it follows that the composite state $[{}_1\underset{\sim}{x}, {}_2\underset{\sim}{x}]'$ is controllable on any subinterval of $[0, \infty)$ if and only if the rows of the matrix in Eq. (3.1) are linearly independent (more precisely, if and only if there is no nonzero $({}_1n + {}_2n)$-tuple of real numbers $(\xi_1, \xi_2, \ldots, \xi_{{}_1n+{}_2n}) = \underset{\sim}{\xi}$ such that the product of $\underset{\sim}{\xi}$ by the matrix of Eq. (3.1) is not an identically zero row vector). This characterization of the controllability of the state $[{}_1\underset{\sim}{x}, {}_2\underset{\sim}{x}]'$ does not give any insight into the following phenomenon: it is possible that the representations S_1 and S_2 are controllable and observable, that the state ${}_1\underset{\sim}{x}$ of S_1 and the state ${}_2\underset{\sim}{x}$ of S_2 are controllable (separately) by the input ${}_1\underset{\sim}{u}$ of the tandem connection, but that the state $[{}_1\underset{\sim}{x}, {}_2\underset{\sim}{x}]'$ of the tandem connection is not controllable. Indeed consider

$$S_1: \; {}_1\dot{\underset{\sim}{x}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} {}_1\underset{\sim}{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} {}_1\underset{\sim}{u}$$

$${}_1\underset{\sim}{y} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -2 \end{bmatrix} {}_1\underset{\sim}{x} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} {}_1\underset{\sim}{u}$$

$$S_2: \; {}_2\dot{\underset{\sim}{x}} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} {}_2\underset{\sim}{x} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} {}_2\underset{\sim}{u}$$

$${}_2\underset{\sim}{y} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} {}_2\underset{\sim}{x}.$$

It is easy to verify that the representations S_1 and S_2 are controllable and observable. Observe that

$$\hat{\underset{\sim}{G}}_1(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{s+2}{s+3} \\ \frac{1}{s+1} & \frac{s+2}{s+4} \end{bmatrix}$$

has rank 2; hence (by remark (2) following theorem 5.1 below) after the tandem connection, ${}_2\tilde{x}$ in Σ_2 is controllable by ${}_1\tilde{u}$. The block diagram of the tandem connection of S_1 and S_2 is given in Fig. 3.1. This block diagram can be transformed by using partial fraction expansion or the transformation suggested in [1] to Fig. 3.2. Examining Fig. 3.2, we can readily see that the composite state $\begin{bmatrix} {}_1\tilde{x} \\ {}_2\tilde{x} \end{bmatrix}$ is not controllable in $\Sigma_1 \oplus \Sigma_2$. This is due to the fact that the states reachable from the origin must lie in the hyperplane defined by ${}_2x_1 - {}_1x_2 + {}_1x_3 = 0$. Also, the diagram shows that there is no connection between the inputs and ${}_2x_1$.

In order to gain some insight into these phenomena, we shall use the Jordan canonical form for the representation of each system.

Criterion Based on the Jordan Canonical Form

The properties of controllability and observability are invariant under nonsingular transformations of the state; hence it is an additional justification for assuming that $\tilde{A}$ is in the Jordan canonical form. Let $\tilde{A}$ be of the form shown in Table 1: $\tilde{A}$ is an $n \times n$ matrix in Jordan canonical form with m ($m \leq n$) distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. Let $\tilde{A}_i$ denote all the Jordan blocks associated with the eigenvalue λ_i and $r(i)$ be the number of Jordan blocks in $\tilde{A}_i$. Let $\tilde{A}_{ij}$ be the jth Jordan block in $\tilde{A}_i$, then $\tilde{A}_i = \text{diag.}(\tilde{A}_{i1}, \tilde{A}_{i2}, \ldots, \tilde{A}_{ir(i)})$ and $\tilde{A} = \text{diag.}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_m)$. Let n_i and n_{ij} be the order of $\tilde{A}_i$ and $\tilde{A}_{ij}$, respectively; then

$$n = \sum_{i=1}^{m} n_i = \sum_{i=1}^{m} \sum_{j=1}^{r(i)} n_{ij} .$$

$\underset{\sim}{B}$ and $\underset{\sim}{C}$ are arbitrary $n \times p$, $q \times n$ matrices. Corresponding to $\underset{\sim}{A}_i$ and $\underset{\sim}{A}_{ij}$, $\underset{\sim}{B}$ and $\underset{\sim}{C}$ are partitioned as shown in Table 1. Call $\underset{\sim}{b}_{1ij}$, $\underset{\sim}{b}_{\ell ij}$ the first and the last row of $\underset{\sim}{B}_{ij}$; $\underset{\sim}{c}_{1ij}$, $\underset{\sim}{c}_{\ell ij}$ the first and the last column of $\underset{\sim}{C}_{ij}$.

Now we will give the necessary and sufficient condition for the representation S in the form of Table 1 to be controllable and observable. It turns out that the conditions depend only on $b_{\ell ij}$ and c_{1ij}, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, r(i)$. This result has been stated without proof by Kalman [5].

In the case of linear time-invariant systems, if the representation S is controllable in some interval of $[0, \infty)$, then it is controllable in any nonzero subinterval of $[0, \infty)$. Since in this paper we consider only the time-invariant representation, the qualification of the time interval will be dropped.

Theorem 3.1 [5]. The representation S is controllable if and only if for each $i = 1, 2, \ldots, m$, the set of $r(i)$ p-dimensional row vectors

$$\underset{\sim}{b}_{\ell i1}, \underset{\sim}{b}_{\ell i2}, \ldots, \underset{\sim}{b}_{\ell ir(i)}$$

is a linearly independent set.

The theorem is proved in the appendix.

Table 1

Notations for the Jordan Form Representation

$$\underset{(n \times n)}{\underset{\sim}{A}} = \begin{bmatrix} \underset{\sim}{A}_1 & & & & \\ & \underset{\sim}{A}_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \underset{\sim}{A}_m \end{bmatrix} \qquad \underset{(n \times p)}{\underset{\sim}{B}} = \begin{bmatrix} \underset{\sim}{B}_1 \\ \underset{\sim}{B}_2 \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{B}_m \end{bmatrix}$$

$$\underset{\sim}{C} = [\underset{\sim}{C}_1 \; \underset{\sim}{C}_2 \cdots \underset{\sim}{C}_m]$$

$$\underset{(n_i \times n_i)}{\underset{\sim}{A}_i} = \begin{bmatrix} \underset{\sim}{A}_{i1} & & & & \\ & \underset{\sim}{A}_{i2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \underset{\sim}{A}_{ir(i)} \end{bmatrix} \qquad \underset{(n_i \times p)}{\underset{\sim}{B}_i} = \begin{bmatrix} \underset{\sim}{B}_{i1} \\ \underset{\sim}{B}_{i2} \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{B}_{ir(i)} \end{bmatrix}$$

$$\underset{(q \times n_i)}{\underset{\sim}{C}_i} = [\underset{\sim}{C}_{i1} \; \underset{\sim}{C}_{i2} \cdots \underset{\sim}{C}_{ir(i)}]$$

$$\underset{(n_{ij} \times n_{ij})}{\underset{\sim}{A}_{ij}} = \begin{bmatrix} \lambda_i & 1 & & & & \\ & \lambda_i & 1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & \lambda_i & 1 \\ & & & & & \lambda_i \end{bmatrix} \qquad \underset{\sim}{B}_{ij} = \begin{bmatrix} \underset{\sim}{b}_{1ij} \\ \underset{\sim}{b}_{2ij} \\ \cdot \\ \cdot \\ \vdots \\ \underset{\sim}{b}_{\ell ij} \end{bmatrix}$$

$$\underset{\sim}{C}_{ij} = [\underset{\sim}{c}_{1ij} \; \underset{\sim}{c}_{2ij} \cdots \underset{\sim}{c}_{\ell ij}]$$

Remarks:

(1) The condition for controllability in the theorem requires that each of the m sets of vectors be individually tested for linear independence.

(2) The basic idea of the proof is essentially the following: write every state variable in the form of the convolution of its state impulse response (kernel) and the input function; if this kernel is linearly independent from all the other kernels corresponding to other state variables, then this state variable is controllable. And if this is the case for each state variable then the state is controllable.

(3) The necessary and sufficient conditions for controllability are immediately obvious if (a) one recalls that the linear independence of function of the form $p_i(t)\, e^{\lambda_i t}$ and $p_j(t)\, e^{\lambda_j t}$ (where p_i and p_j are polynomials and $\lambda_i \neq \lambda_j$) implies that the controllability of the state variables associated with λ_i can be considered independently from that of those associated with λ_j; and (b) one considers the block diagram representation of the Jordan form (Fig. 3.3). Clearly, if the vectors $\underset{\sim}{b}_{\ell i1}, \underset{\sim}{b}_{\ell i2}, \ldots, \underset{\sim}{b}_{\ell ir(i)}$ are linearly dependent, then whatever is the input waveform $\underset{\sim}{u}(\cdot)$, the r(i) inputs to the first column integrators are linearly dependent. Furthermore, since the functions $e^{\lambda_i t}$, $t e^{\lambda_i t}$, . . . are linearly independent over any open interval, any state variable associated with an integrator in any given row of Fig. 3.3 can be controlled by using only the input to the first integrator. Therefore, one

needs only consider the controllability of each set $x_{\ell ij}$, $j = 1, 2, \ldots, r(i)$ one at a time and for each different i.

(4) Observe that in order for the p-dimensional row vectors $\underset{\sim}{b}_{\ell i1}, \underset{\sim}{b}_{\ell i2}, \ldots, \underset{\sim}{b}_{\ell ir(i)}$ to be linearly independent, we must have $r(i) \leq p$. Hence, in the case of $p = 1$; i.e., $\underset{\sim}{B}$ is a column vector, we have $r(i) = 1$ and $b_{\ell i1} \neq 0$ for all i. Equivalently, the single-input representation S of a single input system $\mathcal{S}$ is controllable if and only if all the eigenvalues corresponding to each Jordan block are pairwise distinct and all the components of the column vector $\underset{\sim}{B}$ which correspond to the last row of each Jordan block are different from zero [2, p. 511].

Applying the duality theory of Kalman, we have

<u>Theorem 3.2.</u> The representation S is observable if and only if, for each $i = 1, 2, \ldots, m$, the set of r(i) q-dimensional column vectors

$$\underset{\sim}{c}_{1i1}, \underset{\sim}{c}_{1i2}, \ldots, \underset{\sim}{c}_{1ir(i)}$$

is a linearly independent set.

4. PARALLEL CONNECTION

In the remainder of this paper the controllability and observability of composite systems are considered. Given any two systems with representations S_1 and S_2 as given in Eq. (2.4), we have shown that the direct sum of the individual state space qualifies as the state space of the composite representation. Hence it is clear that if either S_1 or S_2 is not controllable (observable), then the composite representation is not controllable (observable); in other words, the controllability (observability) of S_1 and S_2 is a necessary condition for the composite representation to be controllable (observable).

In this section we consider only the parallel connection of S_1 and S_2. Recall that ${}_i\underset{\sim}{A}$, ${}_i\underset{\sim}{B}$, and ${}_i\underset{\sim}{C}$ are in the form of Table 1. Let ${}_i\lambda_j$, $j = 1, 2, \ldots, m_i$ be the distinct eigenvalues of ${}_i\underset{\sim}{A}$. Define

$$\Lambda_1 \triangleq \left\{ {}_1\lambda_j \mid j = 1, 2, \ldots, m_1 \right\}, \quad \Lambda_2 \triangleq \left\{ {}_2\lambda_j \mid j = 1, 2, \ldots, m_2 \right\}.$$

<u>Theorem 4.1</u>. Assume that the representations S_1 and S_2 are controllable and observable and that ${}_1p = {}_2p$, ${}_1q = {}_2q$. Then the parallel connection of S_1 and S_2 ($\underset{\sim}{u} = {}_1\underset{\sim}{u} = {}_2\underset{\sim}{u}$, $\underset{\sim}{y} = {}_1\underset{\sim}{y} + {}_2\underset{\sim}{y}$) is controllable (observable) if and only if <u>either</u> $\Lambda_1 \cap \Lambda_2 = \phi$ <u>or</u>, in case Λ_1 and Λ_2 are not disjoint, each pair of common eigenvalues, say ${}_1\lambda_\alpha = {}_2\lambda_\beta$, is such that the set of $(r_1(\alpha) + r_2(\beta))$ ${}_1p$-dimensional row vectors ${}_1\underset{\sim}{b}_{\ell\alpha 1}$, ${}_1\underset{\sim}{b}_{\ell\alpha 2}$, $\ldots$, ${}_1\underset{\sim}{b}_{\ell\alpha r_1(\alpha)}$, ${}_2\underset{\sim}{b}_{\ell\beta 1}$, ${}_2\underset{\sim}{b}_{\ell\beta 2}$, $\ldots$, ${}_2\underset{\sim}{b}_{\ell\beta r_2(\beta)}$ is a linearly independent set. (The set of $(r_1(\alpha) + r_2(\beta))$ ${}_1q$-dimensional column vectors.

$${}_1\underset{\sim}{c}_{1\alpha 1},\ {}_1\underset{\sim}{c}_{1\alpha 2},\ \ldots,\ {}_1\underset{\sim}{c}_{1\alpha r_1(\alpha)},\ {}_2\underset{\sim}{c}_{1\alpha r_1(\alpha)},\ {}_2\underset{\sim}{c}_{1\beta 1},\ {}_2\underset{\sim}{c}_{1\beta 2},\ \ldots,\ {}_2\underset{\sim}{c}_{1\beta r_2(\beta)}$$

is a linearly independent set.)

Proof. The representation of the parallel connection may be written as

$$\begin{bmatrix} {}_1\dot{\underset{\sim}{x}} \\ {}_2\dot{\underset{\sim}{x}} \end{bmatrix} = \begin{bmatrix} {}_1\underset{\sim}{A} & \underset{\sim}{0} \\ \underset{\sim}{0} & {}_2\underset{\sim}{A} \end{bmatrix} \begin{bmatrix} {}_1\underset{\sim}{x} \\ {}_2\underset{\sim}{x} \end{bmatrix} + \begin{bmatrix} {}_1\underset{\sim}{B} \\ {}_2\underset{\sim}{B} \end{bmatrix} \underset{\sim}{u} \tag{4.1}$$

$$\underset{\sim}{y} = \begin{bmatrix} {}_1\underset{\sim}{C} & {}_2\underset{\sim}{C} \end{bmatrix} \begin{bmatrix} {}_1\underset{\sim}{x} \\ {}_2\underset{\sim}{x} \end{bmatrix} + (\underset{\sim}{D}_1 + \underset{\sim}{D}_2)\,\underset{\sim}{u} \tag{4.2}$$

If $\Lambda_1 \cap \Lambda_2 = \phi$, Eq. (4.1) is already in the form of Table 1: for ${}_1\underset{\sim}{A}$ and ${}_2\underset{\sim}{A}$ are in the Jordan canonical form by assumption. If $\Lambda_1 \cap \Lambda_2 \neq \phi$, Eq. (4.1) can be transformed into the form of Table 1 by rearranging the order of Jordan blocks. Hence by Theorem 3.1, the conclusions follows.

Q. E. E.

If S_1 and S_2 are single-input single-output representations, then we have

Corollary 4.1. Assume that S_1 and S_2 are controllable and observable and that ${}_1p = {}_1q = {}_2p = {}_2q = 1$. Then the parallel connection of S_1 and S_2 is controllable and observable if and only if $\Lambda_1 \cap \Lambda_2 = \phi$.

This follows directly from the remark (4) following Theorem 3.1. An interesting interpretation of this corollary is that the sum of two irreducible transfer functions, with no common poles, $\hat{G}(s) = \hat{G}_1(s) + \hat{G}_2(s)$ is irreducible and the degree of the denominator of $\hat{G}(s)$ is equal to the sum of those of $\hat{G}_1(s)$ and $\hat{G}_2(s)$.

5. TANDEM CONNECTION

In this section we consider the tandem connection for S_1 and S_2. We consider first the case $\Lambda_1 \cap \Lambda_2 = \phi$, next some special cases where $\Lambda_1 \cap \Lambda_2 \neq \phi$ and finally the general case. In the first two cases we are able to obtain the necessary and sufficient conditions for the tandem connection to be controllable and observable. In the general case, only the sufficient conditions are obtained. All the results use the fact that ${}_i\underset{\sim}{A}$, ${}_i\underset{\sim}{B}$, and ${}_i\underset{\sim}{C}$ are in the form of Table 1 except Theorem 5.2 where the conditions are stated solely in terms of transfer function matrices; hence they are applicable for any state equation representations.

Compared to the parallel connection, the situation in the tandem connection is much more complicated. An important feature of our analysis is the use of the frequency domain concepts within the state equation representation.

5.1 LET $\Lambda_1 \cap \Lambda_2 = \phi$. We consider the case where the set of eigenvalues of ${}_1\underset{\sim}{A}$ is disjoint from that of ${}_2\underset{\sim}{A}$. Two theorems are given. Recall that $r_2(i)$ is the number of Jordan blocks in ${}_2\underset{\sim}{A}$ associated with the eigenvalue ${}_2\lambda_i$; ${}_2\underset{\sim}{b}_{\ell ij}$ is the row in ${}_2\underset{\sim}{B}$ corresponding the last row of ${}_2\underset{\sim}{A}_{ij}$; ${}_1\underset{\sim}{c}_{1ij}$ is the column in ${}_1\underset{\sim}{C}$ corresponding the the first column of ${}_1\underset{\sim}{A}_{ij}$. The condition ${}_1q = {}_2p$ means that the number of outputs of S_1 is equal to the number of inputs of S_2.

Theorem 5.1. Assume that the representations S_1 and S_2 are controllable and observable, and that ${}_1q = {}_2p$. If $\Lambda_1 \cap \Lambda_2 = \phi$, then S_{12}, the state representation of the tandem connection of S_1 followed by S_2, is controllable (observable) if and only if, for each $i = 1, 2, \ldots, m_2$, the set of $r_2(i)$ ${}_1p$-dimensional row vectors

$${}_2\underset{\sim}{b}_{\ell i1}\, \hat{\underset{\sim}{G}}_1({}_2\lambda_i),\ {}_2\underset{\sim}{b}_{\ell i2}\, \hat{\underset{\sim}{G}}_1({}_2\lambda_i),\ \ldots,\ {}_2\underset{\sim}{b}_{\ell i r_2(i)}\, \hat{\underset{\sim}{G}}_1({}_2\lambda_i)$$

is a linearly independent set. (For each $i = 1, 2, \ldots, m_1$, the set of $r_1(i)$ ${}_2q$-dimensional column vectors

$$\hat{\underset{\sim}{G}}_2({}_1\lambda_i)\, {}_1\underset{\sim}{c}_{1i1},\ \hat{\underset{\sim}{G}}_2({}_1\lambda_i)\, {}_1\underset{\sim}{c}_{1i2},\ \ldots,\ \hat{\underset{\sim}{G}}_2({}_1\lambda_i)\, {}_1\underset{\sim}{c}_{1ir_1(i)}$$

is a linearly independent set.)

The proof of Theorem 5.1 is given at the end of this subsection.

Remarks:

(1) Since we use $\Sigma_1 \oplus \Sigma_2$ as the state space of representation S_{12} of the tandem connection, a necessary condition for the representation S_{12} to be controllable is that S_1 and S_2 be controllable.

In addition, it is necessary that, after the connection, S_2 be controllable by ${}_1\underset{\sim}{u}$; i.e., for any ${}_2\underset{\sim}{x}$ in Σ_2, there exists an ${}_1\underset{\sim}{u}$ which, passing through S_1, transfers ${}_2\underset{\sim}{x}$ to the zero state in a finite time. These two conditions are only necessary, because even though we can control ${}_1\underset{\sim}{x}$ in Σ_1 and ${}_2\underset{\sim}{x}$ in Σ_2 by ${}_1\underset{\sim}{u}$ separately, as shown by our previous example we may not be able to control ${}_1\underset{\sim}{x}$ in Σ_1 and ${}_2\underset{\sim}{x}$ in Σ_2 simultaneously.

(2) It is obvious that the tandem connection of S_1 followed by S_2 does not change the controllability of ${}_1\underset{\sim}{x}$; in most cases, the tandem connection does not change the controllability of ${}_2\underset{\sim}{x}$ in Σ_2 by ${}_1\underset{\sim}{u}$ (passing through S_1), either. For example, if $\rho\hat{\underset{\sim}{G}}_1(s) \triangleq$ rank of $\hat{\underset{\sim}{G}}_1(s) = {}_1q$ (in the field of rational functions), then for any output ${}_1\underset{\sim}{y}$ (hence the input ${}_2\underset{\sim}{u}$ of S_2), as long as its Laplace transform is a rational function of s, there exists an input $\underset{\sim}{u}$ such that $\hat{\underset{\sim}{G}}_1(s)\,\underset{\sim}{u}(s) = {}_2\hat{\underset{\sim}{u}}(s)$. Now this class of inputs ${}_2\underset{\sim}{u}$ suffices to control S_2[8] hence, if $\rho\hat{\underset{\sim}{G}}_1(s) = {}_1q$, then S_2 is controllable by $\underset{\sim}{u} = {}_1\underset{\sim}{u}$. If $\rho\hat{\underset{\sim}{G}}_1(s) < {}_1q$, there always exists a ${}_1\underset{\sim}{y}$ such that ${}_1\hat{\underset{\sim}{y}}(s) = \hat{\underset{\sim}{G}}_1(s)\,{}_1\underset{\sim}{u}(s)$ does not have a solution ${}_1\underset{\sim}{u}$. In other words, if $\rho\hat{\underset{\sim}{G}}_1(s) < {}_1q$, we do not have complete freedom in controlling the input of S_2. In this case S_2 may or may not be controllable by $\underset{\sim}{u}$.

<u>Theorem 5.2.</u> Assume that S_1 and S_2 are controllable and observable, and that ${}_1q = {}_2p$. If $\Lambda_1 \cap \Lambda_2 = \phi$ and if

$\hat{\underset{\sim}{G}}_1({}_2\lambda_i)$ has rank ${}_1q$ for all ${}_2\lambda_i \in \Lambda_2$ and

$(\hat{\underset{\sim}{G}}_2({}_1\lambda_i)$ has rank ${}_2p$ for all ${}_1\lambda_i \in \Lambda_1)$,

then S_{12} is controllable (observable).

Proof. Let

$$
{}_2\underset{\sim}{B}_i^{\ell} = \begin{bmatrix} {}_2\underset{\sim}{b}_{\ell i1} \\ {}_2\underset{\sim}{b}_{\ell i2} \\ \cdot \\ \cdot \\ \cdot \\ {}_2\underset{\sim}{b}_{\ell i r_2(i)} \end{bmatrix} \tag{5.1}
$$

then Theorem 5.1 states that S_{12} is controllable if and only if for each i, $\rho\left[{}_2\underset{\sim}{B}_i^{\ell}\ \hat{\underset{\sim}{G}}_1({}_2\lambda_i)\right] = r_2(i)$. From Sylvester's inequality [9, p. 66], we have

$$
\begin{aligned}
\rho_2\underset{\sim}{B}_i^{\ell} + \rho\hat{\underset{\sim}{G}}_1({}_2\lambda_i) - {}_1q \le \rho\left[{}_2B_i^{\ell}\ \hat{\underset{\sim}{G}}_1({}_2\lambda_i)\right] \\
\le \min\left\{\rho_2 B_i^{\ell},\ \rho\hat{\underset{\sim}{G}}_1({}_2\lambda_i)\right\} .
\end{aligned} \tag{5.2}
$$

Now $\rho_2\underset{\sim}{B}_i^{\ell} = r_2(i)$ by the assumption that S_2 is controllable, hence if $\rho\hat{\underset{\sim}{G}}_1({}_2\lambda_i) = {}_1q$, then (5.2) implies that

$$r_2(i) \leq \rho\left[{}_2\underset{\sim}{B}_i^{\ell}\ \hat{\underset{\sim}{G}}_1({}_2\lambda_i)\right]$$

and S_{12} is controllable.

The observability part can be similarly proved.

Q.E.D.

Using the first inequality of (5.2), we have obtained a sufficient condition for S_{12} to be controllable. Now we use the second inequality of (5.2) to obtain a necessary condition:

Assume that S_1, S_2 are controllable and observable, ${}_2p = {}_1q$ and $\Lambda_1 \cap \Lambda_2 = \phi$. If

$$\rho\hat{\underset{\sim}{G}}_1({}_2\lambda_i) < r_2(i) \left(\rho\hat{\underset{\sim}{G}}_2({}_1\lambda_i) < r_1(i)\right)$$

for some i, then S_{12} is not controllable (observable).

The significance of Theorem 5.2 is that it is independent of the specific representation of $\mathcal{S}_i$. However in applying Theorem 5.1, we must first transform the representation into the Jordan canonical form.

Proof of Theorem 5.1.

(a) Recall that in considering the controllability of a representation, it is legitimate to consider the set of state variables associated with the same eigenvalue independently from all the other state variables associated with different eigenvalues. Furthermore, among the state variables associated with the same eigenvalue, say ${}_2\lambda_i$, we need to consider only those state variables corresponding to the last row of each Jordan block in ${}_2\underset{\sim}{A}$ associated with ${}_2\lambda_i$; namely, ${}_2x_{\ell i1}$, ${}_2x_{\ell i2}$, . . ., ${}_2x_{\ell i r_2(i)}$. Because we know from Theorem 3.1 that if the state variables ${}_2x_{\ell i1}$, ${}_2x_{\ell i2}$, . . ., ${}_2x_{\ell i r_2(i)}$ are controllable, then all the state variables of ${}_2\underset{\sim}{x}$ associated with ${}_2\lambda_i$ will be controllable.

The preliminary step in the proof is to write the state variable as a convolution of state impulse response (kernel) and the input. If all the rows of the kernel matrix are linearly independent over any open interval, then the representation is controllable. Assume that S_1 and S_2 are in the zero state, then for the state vector in S_1, we have

$$ {}_1\hat{\underset{\sim}{x}}(s) = (s\underset{\sim}{I} - {}_1\underset{\sim}{A})^{-1}\, {}_1\underset{\sim}{B}\,\hat{\underset{\sim}{u}}(s) , \tag{5.3} $$

or

$$ {}_1\hat{\underset{\sim}{x}}(t) = \underset{\sim}{H}_1(t) * \underset{\sim}{u}(t) . \tag{5.4} $$

where * denotes the convolution, the ${}_1n \times {}_1p$ matrix $\underset{\sim}{H}_1(t)$ is the inverse Laplace transform of $(s\underset{\sim}{I} - {}_1\underset{\sim}{A})^{-1}\, {}_1\underset{\sim}{B}$. The assumed controllability of S_1 implies that all the ${}_1n$ rows of $\underset{\sim}{H}_1(t)$ are linearly independent over (0, T) for any $T > 0$. The output of S_1, which is also the input of S_2, is ${}_1\hat{\underset{\sim}{y}}(s) = \hat{\underset{\sim}{G}}_1(s)\,\hat{\underset{\sim}{u}}(s)$. Hence, for any state variable ${}_2x_{\ell ij}$, $j = 1, 2, \ldots, r_2(i)$, $i = 1, 2, \ldots, m_2$, we have

$$
{}_2\hat{x}_{\ell ij}(s) = \frac{1}{s - {}_2\lambda_i} \, {}_2 b_{\ell ij} \, \hat{G}_1(s) \, \hat{u}(s)
$$

$$
= {}_2 b_{\ell ij} \, \frac{1}{s - {}_2\lambda_i} \, {}_1 D \, \hat{u}(s) + {}_2 b_{\ell ij} \, {}_1 C \, \frac{1}{s - {}_2\lambda_i} \, (s I - {}_1 A)^{-1} \, {}_1 B \, \hat{u}(s) \, . \tag{5.5}
$$

Substituting the identity,

$$
\frac{1}{s - {}_2\lambda_i} (s I - {}_1 A)^{-1} = \frac{1}{s - {}_2\lambda_i} ({}_2\lambda_i I - {}_1 A)^{-1} - ({}_2\lambda_i I - {}_1 A)^{-1} (s I - {}_1 A)^{-1}
$$

--to prove it, multiply it on both sides by $(s I - {}_1 A)({}_2\lambda_i I - {}_1 A)$--into Eq. (5.5), we obtain

$$
{}_2\hat{x}_{\ell ij}(s) = {}_2 b_{\ell ij} \left[{}_1 D + {}_1 C ({}_2\lambda_i I - {}_1 A)^{-1} \, {}_1 B \right] \frac{1}{s - {}_2\lambda_i} \, \hat{u}(s)
$$

$$
- \, {}_2 b_{\ell ij} \, {}_1 C ({}_2\lambda_i I - {}_1 A)^{-1} (s I - {}_1 A)^{-1} \, {}_1 B \, \hat{u}(s) \, . \tag{5.6}
$$

The existence of $({}_2\lambda_i I - {}_1 A)^{-1}$ follows from the assumption that $\Lambda_1 \cap \Lambda_2 = \phi$. Substituting $\hat{G}_1({}_2\lambda_i) = {}_1 C ({}_2\lambda_i I - {}_1 A)^{-1} \, {}_1 B + {}_1 D$ and Eq. (5.3) into Eq. (5.6) and taking their inverse Laplace transform, we obtain

$$
{}_2 x_{\ell ij}(t) + {}_2 b_{\ell ij} \, {}_1 C ({}_2\lambda_i I - {}_1 A)^{-1} \, {}_1 x(t) = {}_2 b_{\ell ij} \, \hat{G}_1({}_2\lambda_i) \, e^{{}_2\lambda_i t} * u(t) \tag{5.7}
$$

for $i = 1, 2, \ldots, m_2$; $j = 1, 2, \ldots, r_2(i)$. With these preliminaries, we now proceed to prove the necessity and sufficiency of the controllability part of Theorem 5.1.

($\Longrightarrow$) Use contradiction. Suppose ${}_2\underset{\sim}{b}_{\ell i1}\,\hat{\underset{\sim}{G}}_1({}_2\lambda_i)$, ${}_2\underset{\sim}{b}_{\ell i2}\,\hat{\underset{\sim}{G}}_1({}_2\lambda_i)$, $\ldots$, ${}_2\underset{\sim}{b}_{\ell i r_2(i)}\,\hat{\underset{\sim}{G}}_1({}_2\lambda_i)$ are linearly dependent for some i, then there exists a nonzero $r_2(t)$-tuple of complex numbers, say $\underset{\sim}{q}$, such that

$$\underset{\sim}{q}\;{}_2\underset{\sim}{B}_i^{\ell}\;\hat{\underset{\sim}{G}}_1({}_2\lambda_i) = \underset{\sim}{0}, \tag{5.8}$$

where ${}_2\underset{\sim}{B}_i^{\ell}$ is defined in Eq. (5.1). Equation (5.8) with Eq. (5.7) imply that

$$\underset{\sim}{q}\begin{bmatrix} {}_2x_{\ell i1}(t) + {}_2\underset{\sim}{b}_{\ell i1}\,{}_1\underset{\sim}{C}({}_2\lambda_i\underset{\sim}{I} - {}_1\underset{\sim}{A})^{-1}\,{}_1\underset{\sim}{x}(t) \\ {}_2x_{\ell i2}(t) + {}_2\underset{\sim}{b}_{\ell i2}\,{}_1\underset{\sim}{C}({}_2\lambda_i\underset{\sim}{I} - {}_1\underset{\sim}{A})^{-1}\,{}_1\underset{\sim}{x}(t) \\ \vdots \\ {}_2x_{\ell i r_2(i)}(t) + {}_2\underset{\sim}{b}_{\ell i r_2(i)}\,{}_1\underset{\sim}{C}({}_2\lambda_i\underset{\sim}{I} - {}_1\underset{\sim}{A})^{-1}\,{}_1\underset{\sim}{x}(t) \end{bmatrix} = 0\,. \tag{5.9}$$

It implies that for any $\underset{\sim}{u}$, the states (in $\Sigma_1 \oplus \Sigma_2$) reachable from the origin lie in the hyperplane (5.9). Hence the composite state $\begin{bmatrix} {}_1\underset{\sim}{x} \\ {}_2\underset{\sim}{x} \end{bmatrix}$ is not controllable in $\Sigma_1 \oplus \Sigma_2$. This contradicts the hypothesis that S_{12} is controllable.

($\Leftarrow$) Suppose that for each $i = 1, 2, \ldots, m_2$, the set of $r_2(i)$ ${}_1p$-dimensional row vectors ${}_2\underset{\sim}{b}_{\ell i1}\, \hat{\underset{\sim}{G}}_1({}_2\lambda_i)$, ${}_2\underset{\sim}{b}_{\ell i2}\, \hat{\underset{\sim}{G}}_2({}_2\lambda_i)$, $\ldots$, ${}_2b_{\ell i r_2(i)}\, \hat{\underset{\sim}{G}}_2({}_2\lambda_i)$ is a linearly independent set. Then, since $\Lambda_1 \cap \Lambda_2 = \phi$, all the rows of $\underset{\sim}{H}_1(t)$ and ${}_2\underset{\sim}{b}_{\ell ij}\, \hat{\underset{\sim}{G}}_1({}_2\lambda_i)\, e^{{}_2\lambda_i t}$, $i = 1, 2, \ldots, m_2$: $j = 1, 2, \ldots, r_2(i)$ are linearly independent. Hence ${}_1\underset{\sim}{x}$ and ${}_2x_{\ell ij}$, $i = 1, 2, \ldots, m_2$; $j = 1, 2, \ldots, r_2(i)$ are controllable. However, we know that if ${}_2x_{\ell ij}$, $j = 1, 2, \ldots, r_2(i)$ are controllable, then all the state variables associated with ${}_2\lambda_i$ are controllable. Hence we conclude that $\begin{bmatrix} {}_1\underset{\sim}{x} \\ {}_2\underset{\sim}{x} \end{bmatrix}$ is controllable in $\Sigma_1 \oplus \Sigma_2$.

(b) In discussing the observability part, we assume that $\underset{\sim}{u}(t) \equiv 0$. However the input of S_2 is in general not equal to zero. Recall that a representation is said to be observable if and only if the knowledge of the representation and that of the input and output over a common interval suffice to determine the initial state. Thus, if S_1 is not observable from the output of S_2 (hence the input of S_2 is unknown), although S_2 is observable by itself, S_2 as a part of S_{12} is not observable because of the lack of information of the input of S_2. If S_1 is observable from the output of S_2, then the input of S_2 is known, hence S_2 is observable. Consequently S_{12} is observable. Similar to the controllability part, we know that in order for S_1 to be observable from the output of S_2, it is sufficient to know that ${}_1x_{1ij}$, $i = 1, 2, \ldots, m_1$; $j = 1, 2, \ldots, r_1(i)$ are observable. Now assume that at $t = 0$, all the components of ${}_1\underset{\sim}{x}$ except ${}_1x_{1ij}$, $i = 1, 2, \ldots, m_1$; $j = 1, 2, \ldots, r_1(i)$ are zero, then the output due to the initial states ${}_1\underset{\sim}{x}(0)$ and ${}_2\underset{\sim}{x}(0)$ is given by

$$\hat{\underset{\sim}{y}}(s) \triangleq {}_2\hat{\underset{\sim}{y}}(s) = \sum_{i=0}^{m_1} \sum_{j=1}^{r_1(i)} \hat{\underset{\sim}{G}}_2(s)\, {}_1\underset{\sim}{c}_{1ij} \frac{1}{s - {}_1\lambda_i}\, {}_1x_{1ij}(0)$$

$$+ {}_2\underset{\sim}{C}(s\underset{\sim}{I} - {}_2\underset{\sim}{A})^{-1}\, {}_2\underset{\sim}{x}(0) . \tag{5.10}$$

By the same manipulations used in Eq. (5.5), Eq. (5.10) can be written as

$$\hat{\underset{\sim}{y}}(s) = \sum_{i=1}^{m_1} \sum_{j=1}^{r_1(i)} \hat{\underset{\sim}{G}}_2({}_1\lambda_i)\, {}_1\underset{\sim}{c}_{1ij} \frac{1}{s - {}_1\lambda_i}\, {}_1x_{1ij}(0) - {}_2\underset{\sim}{C}(s\underset{\sim}{I} - {}_2\underset{\sim}{A})^{-1}$$

$$\cdot \left(\sum_{i=1}^{m_1} \sum_{j=1}^{r_1(i)} ({}_1\lambda_i \underset{\sim}{I} - {}_2\underset{\sim}{A})^{-1}\, {}_2\underset{\sim}{B}\, {}_1\underset{\sim}{c}_{1ij}\, {}_1x_{1ij}(0) - {}_2\underset{\sim}{x}(0) \right) . \tag{5.11}$$

Taking its inverse Laplace transform, we obtain

$$
\underset{\sim}{y}(t) = \sum_{i=1}^{m_1} \sum_{j=1}^{r_1(i)} \hat{\underset{\sim}{G}}_2({}_1\lambda_i)\, {}_1\underset{\sim}{c}_{1ij}\, e^{{}_1\lambda_i t}\, {}_1x_{1ij}(0)
$$

$$
-H_2(t)\left(\sum_{i=1}^{m_1} \sum_{j=1}^{r_1(i)} ({}_1\lambda_i \underset{\sim}{I} - {}_2\underset{\sim}{A})^{-1}\, {}_2\underset{\sim}{B}\, {}_1\underset{\sim}{c}_{1ij}\, {}_1x_{1ij}(0) - {}_2\underset{\sim}{x}(0) \right) , \qquad (5.12)
$$

where $\underset{\sim}{H}_2(t)$ is the inverse Laplace transform of ${}_2\underset{\sim}{C}(s\underset{\sim}{I} - {}_2\underset{\sim}{A})^{-1}$, and all its ${}_2n$ columns are linearly independent by the assumption that S_2 is observable. Now we will prove the observability part of Theorem 5.1.

($\Longrightarrow$) Use contradiction. Assume that $\hat{\underset{\sim}{G}}_2({}_1\lambda_i)\, {}_1\underset{\sim}{c}_{1ij}$, $j = 1, 2, \ldots, r_1(i)$ are linearly dependent for $i = k$. Let all the components of ${}_1\underset{\sim}{x}$ except ${}_1x_{1kj}$, $j = 1, 2, \ldots, r_1(k)$ be zero, and choose

$$
{}_2\underset{\sim}{x}(0) = \sum_{j=1}^{r_1(k)} ({}_1\lambda_k \underset{\sim}{I} - {}_2\underset{\sim}{A})^{-1}\, {}_2\underset{\sim}{B}\, {}_1\underset{\sim}{c}_{1kj}\, {}_1x_{1kj}(0) .
$$

Then Eq. (5.12) reduces to

$$
\underset{\sim}{y}(t) = \sum_{j=1}^{r_1(k)} \hat{\underset{\sim}{G}}_2({}_1\lambda_k)\, {}_1\underset{\sim}{c}_{1kj}\, e^{{}_1\lambda_k t}\, {}_1x_{1kj}(0) . \qquad (5.13)
$$

Since $\hat{G}_2({}_1\lambda_k)\ {}_1c_{1kj}$, $j = 1, 2, \ldots, m_1(k)$ are linearly dependent, we are not able to determine ${}_1x_{1jk}(0)$ from Eq. (5.13) with the knowledge of $y(t)$. Hence, S_{12} is not observable. This contradicts the hypothesis. Hence for each $i = 1, 2, \ldots, m_1$, the set $\hat{G}_2({}_1\lambda_i)\ {}_1c_{1ij}$, $j = 1, 2, \ldots, r_1(i)$ must be a linearly independent set.

($\Leftarrow$) Similar to the controllability part except that we have linearly independent columns instead of rows.

Q. E. D.

5.2 SPECIAL CASES WHERE $\Lambda_1 \cap \Lambda_2 = \phi$. In this section, we consider the case $\Lambda_1 \cap \Lambda_2 = \phi$. But for each pair of common eigenvalues, some conditions are imposed. We give four theorems here, two of them are independent of the specific representation chosen for the systems.

Theorem 5.3. Assume that S_1 and S_2 are controllable and observable, and that ${}_1q = {}_2p$. If Λ_1 and Λ_2 are not disjoint, then for each pair of common eigenvalues, say ${}_1\lambda_\alpha = {}_2\lambda_\beta$, we assume that (i) $r_1(\alpha) = 1$, $r_2(\beta) = 1$; i.e., corresponding to ${}_1\lambda_\alpha = {}_2\lambda_\beta$, there is only one Jordan block in ${}_1A$ and ${}_2A$; and (ii) ${}_2b_{\ell\beta 1}\ {}_1c_{1\alpha 1} \neq 0$. Under these assumptions, S_{12} is controllable (observable) if and only if, for each $i = 1, 2, \ldots, \beta-1, \beta+1, \ldots, m_2$, the set of $r_2(i)$ ${}_1p$-dimensional row vectors

$$ {}_2b_{\ell i1}\ \hat{G}_1({}_2\lambda_i),\ {}_2b_{\ell i2}\ \hat{G}_1({}_2\lambda_i),\ \ldots,\ {}_2b_{\ell i r_2(i)}\ \hat{G}_1({}_2\lambda_i) $$

is a linearly independent set. (For each $i = 1, 2, \ldots, \alpha-1, \alpha+1, \ldots, m_1$, the set of $r_1(i)$ ${}_2q$-dimensional column vectors

$$\hat{\underset{\sim}{G}}_2({}_1\lambda_i)\ {}_1\underset{\sim}{c}_{1i1},\ \hat{\underset{\sim}{G}}_2({}_1\lambda_i)\ {}_1\underset{\sim}{c}_{1i2},\ \ldots,\ \hat{\underset{\sim}{G}}_2({}_1\lambda_i)\ {}_1\underset{\sim}{c}_{1ir_1(i)}$$

is a linearly independent set.)

<u>Proof.</u> Consider first the controllability of the state variables in ${}_2\underset{\sim}{x}$ associated with the eigenvalue ${}_2\lambda_\beta$. As in theorem 5.1 it is sufficient to consider that state variable corresponding to the last row of the Jordan block associated with ${}_2\lambda_\beta$ in ${}_2\underset{\sim}{A}$; namely ${}_2x_{\ell\beta 1}$. Corresponding to any input ${}_1\underset{\sim}{u} = \underset{\sim}{u}$, ${}_2x_{\ell\beta 1}$ contains a term of the form

$${}_2\underset{\sim}{b}_{\ell\beta 1}\ {}_1\underset{\sim}{c}_{1\alpha 1}\ {}_1\underset{\sim}{b}_{\ell\alpha 1}\, t^{{}_1n_\alpha}\, e^{{}_2\lambda_\beta t} * \underset{\sim}{u}\ .$$

(Where ${}_1n_\alpha$ is the order of the Jordan block associated with ${}_1\lambda_\alpha$ in ${}_1\underset{\sim}{A}$.) Since ${}_1\underset{\sim}{b}_{\ell\alpha 1} \neq \underset{\sim}{0}$ (by the controllability of S_1) and since ${}_2\underset{\sim}{b}_{\ell\beta 1}\ {}_1\underset{\sim}{c}_{1\alpha 1} \neq 0$ (by assumption), the coefficient of $t^{{}_1n_\alpha} e^{{}_2\lambda_\beta t} * \underset{\sim}{u}$ is different from zero. Now associated with ${}_2\lambda_\beta$, there is only one Jordan block, hence the state variable ${}_2x_{\ell\beta 1}$ is the only term having $t^{{}_1n_\alpha} e^{{}_2\lambda_\beta t}$ in its kernel. Thus, ${}_2x_{\ell\beta 1}$ is controllable independently from all other state variables. Consequently, all the state variables associated with ${}_2\lambda_\beta$ are controllable. Consider next the observability of the state variables in ${}_1\underset{\sim}{x}$ associated with the eigenvalue ${}_1\lambda_\alpha$. As in theorem 5.1, we need to consider only ${}_1x_{1\alpha 1}$. If ${}_1x_{1\alpha 1}(0) \neq 0$, since ${}_2\underset{\sim}{b}_{\ell\beta 1}\,{}_1\underset{\sim}{c}_{1\alpha 1} \neq 0$ and ${}_2\underset{\sim}{c}_{1\alpha 1} \neq \underset{\sim}{0}$, the output $\underset{\sim}{y}$ contains a term $t^{{}_2n_\beta} e^{{}_1\lambda_\alpha t}\,{}_1x_{1\alpha 1}(0)$ (${}_2n_\beta$ is the order of the Jordan block associated with ${}_2\lambda_\beta$ in ${}_2\underset{\sim}{A}$) which is independent of all other kernels in S_{12}; hence ${}_1x_{1\alpha 1}$ is observable. We conclude up to here that the assumptions (i) and (ii) in the theorem imply that all the components of the composite state $\begin{bmatrix} {}_1\underset{\sim}{x} \\ {}_2\underset{\sim}{x} \end{bmatrix}$ associated with ${}_1\lambda_\alpha = {}_2\lambda_\beta$ are controllable and observable. Furthermore, the controllability and observability of these state variables are independent from those of all other state variables in S_{12}. Hence, in the remainder of proof we may disregard those state variables associated with common eigenvalues, and the method of proving theorem 5.1 applies.

Q.E.D.

<u>Theorem 5.4</u>. Under the same assumptions as in theorem 5.3, if

$$\hat{\underset{\sim}{G}}_1({}_2\lambda_i) \text{ is of rank } {}_1q \text{ for all } {}_2\lambda_i \in \Lambda_2 - (\Lambda_1 \cap \Lambda_2),$$

and

$$(\hat{\underset{\sim}{G}}_2({}_1\lambda_i) \text{ is of rank } {}_2p \text{ for all } {}_1\lambda_i \in \Lambda_1 - (\Lambda_1 \cap \Lambda_2)),$$

then S_{12} is controllable (observable).

The proof is the same as that in theorem 5.2.

5.3 GENERAL CASES.

In this section we consider the general case, i.e., without assuming that the number of Jordan blocks associated with common eigenvalues is one. For the single-input single-output representation, the results are seen to be a special case of theorem 5.3. For the multivariable case, we give only some sufficient conditions.

5.3.1. Single-input single-output case.

As shown in the previous sections, in the multivariable case, in order to obtain the necessary and sufficient conditions for controllability and observability, we must have the knowledge of the internal connections of the system. However, in the single-input single-output case, we need only the external relations: the transfer function.

Corollary 5.1. Assume that S_1 and S_2 are controllable and observable and that ${}_1p = {}_1q = {}_2p = {}_2q = 1$. Then S_{12}, the representation of the tandem connection of $\mathcal{S}_1$ followed by $\mathcal{S}_2$, is controllable (observable) if and only if

$$\hat{G}_1({}_2\lambda_i) \neq 0 \text{ for all } {}_2\lambda_i \in \Lambda_2 - (\Lambda_1 \cap \Lambda_2)$$

$$(\hat{G}_2({}_1\lambda_i) \neq 0 \text{ for all } {}_1\lambda_i \in \Lambda_1 - (\Lambda_1 \cap \Lambda_2)).$$

Proof For a single-input single-output representation, the controllability and observability conditions imply that $r_k(i) = 1$, ${}_kb_{\ell i1} \neq 0$ and

${}_{k}c_{1i1} \neq 0$ for all i and k = 1, 2, ; hence the assumptions (i) and (ii) in theorem 5.3 are satisfied. By applying theorem 5.3, we find that S_{12} is controllable if and only if ${}_{2}b_{\ell i1}\ \hat{G}_1({}_{2}\lambda_i) \neq 0$ for all ${}_{2}\lambda_i \in \Lambda_2 - (\Lambda_1 \cap \Lambda_2)$. However, ${}_{2}b_{\ell i1} \neq 0$ by assumption; hence, S_{12} is controllable if and only if $\hat{G}_1({}_{2}\lambda_i) \neq 0$ for all ${}_{2}\lambda_i \in \Lambda_2 - (\Lambda_1 \cap \Lambda_2)$.

Q.E.D.

Corollary 5.1 says that the tandem connection of two controllable and observable single-input single-output representations is controllable if and only if there is no cancellation of one or more poles of the second transfer function by some zeros of the first transfer function. The tandem connection is observable if and only if there is no cancellation of one or more poles of the first transfer function by some zeros of the second transfer function. An immediate consequence of these observations is that under the assumptions of the corollary, S_{12}($\mathcal{S}_1$ followed by $\mathcal{S}_2$) is observable if and only if S_{21}($\mathcal{S}_2$ followed by $\mathcal{S}_1$) is controllable, and S_{12} is controllable and observable if and only if S_{21} is controllable and observable. This is obvious from the transfer function point of view because these conditions are satisfied if and only if there is no cancellations in the product of $\hat{G}_1(s)\ \hat{G}_2(s)$.

5.3.2 Multivariable case.

Next we consider the tandem connection of multivariable systems. Recall that all the representations are still in the Jordan canonical form but now the number of Jordan blocks associated with common eigenvalues is arbitrary. Here we give only the sufficient conditions. Before stating the theorems, we define

$$\hat{\underset{\sim}{G}}_{1\alpha}(s) = \hat{\underset{\sim}{G}}_1(s)\Big|_{{}_1\underset{\sim}{B}_\alpha=\underset{\sim}{0},\ {}_1\underset{\sim}{C}_\alpha=\underset{\sim}{0}}, \tag{5.14}$$

$$\hat{\underset{\sim}{G}}_{2\beta}(s) = \hat{\underset{\sim}{G}}_2(s)\Big|_{{}_2\underset{\sim}{B}_\beta=\underset{\sim}{0},\ {}_2\underset{\sim}{C}_\beta=\underset{\sim}{0}}, \tag{5.15}$$

where ${}_1\underset{\sim}{B}_\alpha$, ${}_1\underset{\sim}{C}_\alpha$ are defined as in Table 1, the left subscript 1 says that they below to S_1. $\hat{\underset{\sim}{G}}_{1\alpha}(s)$ denotes part of the transfer function $\hat{\underset{\sim}{G}}_1(s)$ by disconnecting inputs and outputs to all subsystems associated with the eigenvalue ${}_1\lambda_\alpha$.

<u>Theorem 5.5a</u>. Assume that S_1 and S_2 are observable and controllable and that ${}_1q = {}_2p$. If, for each ${}_2\lambda_i \in \Lambda_2 - (\Lambda_1 \cap \Lambda_2)$, the set of $r_2(i)$ ${}_1p$-dimensional row vectors

$${}_2\underset{\sim}{b}_{\ell i1}\,\hat{\underset{\sim}{G}}_1({}_2\lambda_i),\ {}_2\underset{\sim}{b}_{\ell i2}\,\hat{\underset{\sim}{G}}_1({}_2\lambda_i),\ \ldots,\ {}_2\underset{\sim}{b}_{\ell i r_2(i)}\,\hat{\underset{\sim}{G}}_1({}_2\lambda_i)$$

is a linearly independent set; and if, for each ${}_2\lambda_j \in \Lambda_1 \cap \Lambda_2$, say ${}_2\lambda_\beta = {}_1\lambda_\alpha$, the set of $r_1(\alpha) + r_2(\beta)$ ${}_1p$-dimensional row vectors

$$ {}_1\underset{\sim}{b}_{\ell\alpha 1},\ {}_1\underset{\sim}{b}_{\ell\alpha 2},\ \ldots,\ {}_1\underset{\sim}{b}_{\ell\alpha r_1(\alpha)},\ {}_2\underset{\sim}{b}_{\ell\beta 1}\ \hat{\underset{\sim}{G}}_{1\alpha}({}_2\lambda_\beta), $$

$$ {}_2\underset{\sim}{b}_{\ell\beta 2}\ \hat{\underset{\sim}{G}}_{1\alpha}({}_2\lambda_\beta),\ \ldots,\ {}_2\underset{\sim}{b}_{\ell\beta r_2(\beta)}\ \hat{\underset{\sim}{G}}_{1\alpha}({}_2\lambda_\beta) $$

is a linearly independent set, then S_{12} is controllable.

<u>Proof.</u> We need to consider only those state variables associated with common eigenvalues. For any $\underset{\sim}{u}$, and for each $j = 1, 2, \ldots, r_2(\beta)$, we have

$$ {}_2x_{\ell\beta j}(t) = {}_2\underset{\sim}{b}_{\ell\beta j}\ \hat{\underset{\sim}{G}}_{1\alpha}({}_2\lambda_\beta)\ e^{{}_2\lambda_\beta t} * \underset{\sim}{u}(t) + \underset{\sim}{F}_j(t) * \underset{\sim}{u}(t), \tag{5.16} $$

where in $F_j(t)$, we have terms associated $t^k e^{{}_2\lambda_\beta t}$, $k \geq 1$ and all other modes due to ${}_1\lambda_i \in \Lambda_1 - (\Lambda_1 \cap \Lambda_2)$. It is important to observe that the only coefficient associated with $e^{{}_2\lambda_\beta t}$ in Eq. (5.16) is ${}_2\underset{\sim}{b}_{\ell\beta j}\ \underset{\sim}{G}_{1\alpha}({}_2\lambda_\beta)$, hence it cannot be cancelled by the terms in $\underset{\sim}{F}_j$. For ${}_1x_{\ell\alpha j}$, $j = 1, 2, \ldots, r_1(\alpha)$, we have

$$ {}_1x_{\ell\alpha j} = {}_1\underset{\sim}{b}_{\ell\alpha j}\ e^{{}_1\lambda_\alpha t} * \underset{\sim}{u}(t). \tag{5.17} $$

Now ${}_1\lambda_\alpha = {}_2\lambda_\beta$ by assumption, hence if the set of ${}_1p$-dimensional row vectors.

$$ {}_1\underset{\sim}{b}_{\ell\alpha 1},\ {}_1\underset{\sim}{b}_{\ell\alpha 2},\ \ldots,\ {}_1\underset{\sim}{b}_{\ell\alpha r_1(\alpha)},\ {}_2\underset{\sim}{b}_{\ell\beta_1}\ \hat{\underset{\sim}{G}}_{1\alpha}({}_2\lambda_\beta), $$

$$ {}_2\underset{\sim}{b}_{\ell\beta 2}\ \hat{\underset{\sim}{G}}_{1\alpha}({}_2\lambda_\beta),\ \ldots,\ {}_2\underset{\sim}{b}_{\ell\beta r_2(\beta)}\ \hat{\underset{\sim}{G}}_{1\alpha}({}_2\lambda_\beta) $$

is a linearly independent set, from Eqs. (5.16-17) we can conclude that ${}_1x_{\ell\alpha j}$, $j = 1, 2, \ldots, r_1(\alpha)$ and ${}_2x_{\ell\beta j}$, $j = 1, 2, \ldots, r_2(\beta)$ are controllable.

Q.E.D.

Theorem 5.5b. Assume that S_1 and S_2 are controllable and observable and that ${}_1q = {}_2p$. If, for each ${}_1\lambda_i \in \Lambda_1 - (\Lambda_1 \cap \Lambda_2)$, the set of $r_1(i)$ ${}_2q$-dimensional column vectors

$$ \hat{\underset{\sim}{G}}_2({}_1\lambda_i)\ {}_1\underset{\sim}{c}_{1i1},\ \hat{\underset{\sim}{G}}_2({}_1\lambda_i)\ {}_1\underset{\sim}{c}_{1i2},\ \ldots,\ \hat{\underset{\sim}{G}}_2({}_1\lambda_i)\ {}_1\underset{\sim}{c}_{1ir_1(i)} $$

is a linearly independent set, and if, for each ${}_1\lambda_j \in (\Lambda_1 \cap \Lambda_2)$, say ${}_1\lambda_\alpha = {}_2\lambda_\beta$, the set of $r_2(\beta) + r_1(\alpha)$ ${}_2q$-dimensional column vectors

$$ {}_2\underset{\sim}{c}_{1\beta 1},\ {}_2\underset{\sim}{c}_{1\beta 2},\ \ldots,\ {}_2\underset{\sim}{c}_{1\beta r_2(\beta)},\ \hat{\underset{\sim}{G}}_{2\beta}({}_1\lambda_\alpha)\ {}_1\underset{\sim}{c}_{1\alpha 1}, $$

$$ \hat{\underset{\sim}{G}}_{2\beta}({}_1\lambda_\alpha)\ {}_1\underset{\sim}{c}_{1\alpha 2},\ \ldots,\ \hat{\underset{\sim}{G}}_{2\beta}({}_1\lambda_\alpha)\ {}_1\underset{\sim}{c}_{1\alpha r_1(\alpha)} $$

is a linearly independent set, then S_{12} is observable.

6. FEEDBACK CONNECTIONS

We propose to show that for a very general class of systems the problem of controllability and observability of a feedback connection reduces to that of the tandem connection. As an example of the class of system to which these results apply we consider nonlinear time-varying systems with finite-dimensional differential equation representations:

$$\overline{S}_i: \quad {}_i\dot{x} = {}_if({}_ix, {}_iu, t) \tag{6.1a}$$

$$ {}_iy = {}_ig({}_ix, {}_iu, t) \tag{6.1b}$$

where ${}_ix$, ${}_iu$ and ${}_iy$ are, respectively, ${}_in \times 1$, ${}_ip \times 1$ and ${}_iq \times 1$ vectors; ${}_if$ is an ${}_in$ vector-valued function, ${}_ig$ is a ${}_iq$ vector-valued function. The state space of $\overline{S}_i$ is denoted by Σ_i and the input space is assumed to be a linear vector space. (We write $\overline{S}_i$ instead of S_i to indicate that we are considering nonlinear systems.) $\overline{S}_i$ is said to be <u>determinate</u> [2, p. 96] if for any ${}_iu$, any t_0 and any initial state ${}_ix(t_0)$, there exist unique ${}_ix(t)$ and $y_i(t)$ for all $t \geq t_0$. It is well known [10] that $\overline{S}_i$ is determinate if ${}_if$ is Lipschitz in ${}_ix$ and continuous in u_i and t and ${}_ig$ is continuous in ${}_ix$, ${}_iu$, and t.

For a determinate representation $\overline{S}_i$, since ${}_ix(t)$ and ${}_iy(t)$ are functions of the input and the initial state, all the concepts of controllability and observability can be applied with appropriate modifications. As in the linear case, the nonlinear representation $\overline{S}_i$ is said to be controllable on $[t_0, t_1]$ if for any given state ${}_i\underset{\sim}{x}(t_0) \in \Sigma_i$, there exists an input $\underset{\sim}{u}_i$ which transfers ${}_i\underset{\sim}{x}(t_0)$ to the zero state at time t_1. $\overline{S}_i$ is said to be observable on $[t_0, t_1]$ if for any given input $\underset{\sim}{u}$ over $[t_0, t_1]$, the response $\underset{\sim}{y}$ (due to this input $\underset{\sim}{u}$ and an arbitrary initial state $x(t_0)$) and the knowledge of the representation (6.1) suffice to determine that initial state $\underset{\sim}{x}(t_0)$. Similarly, $\overline{S}_i$ is said to be zero-input observable on $[t_0, t_1]$ if, with $\underset{\sim}{u}(t) \equiv \underset{\sim}{0}$, the response $\underset{\sim}{y}$ over $[t_0, t_1]$ (due to an arbitrary intial state $\underset{\sim}{x}(t_0)$) and the knowledge of the representation (6.1) suffice to determine the initial state $\underset{\sim}{x}(t_0)$. No explicit criterion for the controllability and observability of such determinate representation $\overline{S}_i$ is known. However it turns out that if a feedback connection is determinate, the representation of the feedback connection is controllable (observable) if and only if some appropriate open-loop connection is controllable (observable).

Theorem 6.1. We are given two determinate systems with representations $\overline{S}_1$ and $\overline{S}_2$ of the form (6.1) with ${}_1p = {}_2q$, ${}_1q = {}_2p$. Let $\overline{S}_f$, the representation of the feedback connection shown in Fig. 6.1a, be determinate and $\Sigma_1 \oplus \Sigma_2$ be the state space of $\overline{S}_f$. Under these assumptions, $\overline{S}_f$ is controllable on $[t_0, t_1]$ if and only if $\overline{S}_{12}$ is controllable on $[t_0, t_1]$. Furthermore, $\overline{S}_f$ is zero-input observable on $[t_0, t_1]$ if $\overline{S}'_{21}$ is observable on $[t_0, t_1]$, where $\overline{S}'_{21}$ is the tandem connection of $\overline{S}_2$, a sign inverter, and $\overline{S}_1$.

Let us prove the first statement. Refer to Fig. 6.1a and b. Suppose that $\overline{S}_{12}$ is controllable on $[t_0, t_1]$, then for any $\left[{}_1\underset{\sim}{x}(t_0), {}_2\underset{\sim}{x}(t_0)\right]' \in \Sigma_1 \oplus \Sigma_2$ there is an input to $\overline{S}_{12}$, say ${}_1\underset{\sim}{u}$, which transfers this initial state to $[\underset{\sim}{0}, \underset{\sim}{0}]'$ at time t_1. Since $\overline{S}_f$ is determinate, this input ${}_1\underset{\sim}{u}$ (to $\overline{S}_1$) produces in $\overline{S}_f$ a unique output ${}_2\underset{\sim}{y}$. Clearly then, in order to have ${}_1\underset{\sim}{u}$ as input to $\overline{S}_1$ we must apply to $\overline{S}_f$ the input $\underset{\sim}{u} = {}_1\underset{\sim}{u} + {}_2\underset{\sim}{y}$: $\underset{\sim}{u}$ will transfer $\left[{}_1\underset{\sim}{x}(t_0), {}_2\underset{\sim}{x}(t_0)\right]'$ of $\overline{S}_f$ to $[\underset{\sim}{0}, \underset{\sim}{0}]$. Therefore, $\overline{S}_f$ is controllable on $[t_0, t_1]$. Conversely, if $\overline{S}_f$ is controllable, there is an input $\underset{\sim}{u}$ (to $\overline{S}_f$) which causes the required state transfer and a corresponding output ${}_2\underset{\sim}{y}$. Clearly, the input ${}_1\underset{\sim}{u} = \underset{\sim}{u} - {}_2\underset{\sim}{y}$, applied to $\overline{S}_{12}$, will cause the required transfer.

Let us now prove the second statement. Let $\overline{S}'_{21}$ be observable, then the knowledge of the output ${}_1\underset{\sim}{y}$ of the feedback connection $\overline{S}_f$ (with $\underset{\sim}{u} \equiv \underset{\sim}{0}$) gives an input output pair of $\overline{S}'_{21}$ and thus determines $\left[{}_1\underset{\sim}{x}(t_0), {}_2\underset{\sim}{x}(t_0)\right]'$. Thus the observability of $\overline{S}'_{21}$ implies the zero-input observability of $\overline{S}_f$.

<u>Corollary 6.2</u>. Let S'_{21} be the tandem connection of S_2, a sign inverter and S_1. In the linear case (i.e., S_1 and S_2 are described by (2.4) where the matrices are possibly time-varying), S_f is observable on $[t_0, t_1]$ if and only if S'_{21} is observable on $[t_0, t_1]$.

Proof. Suppose that S'_{21} is observable, let us show that S_f is observable. Call $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$ the linear operators mapping the inputs of S_1 and S_2, into their zero-state responses, and call z_1 and z_2 their zero-input responses; then, on $[t_0, t_1]$, with the notations of Fig. 6.1a

$$ {}_1\underset{\sim}{y} = \underset{\sim}{K}_1 \underset{\sim}{u} - \underset{\sim}{K}_1 \underset{\sim}{K}_2 \, {}_1\underset{\sim}{y} - \underset{\sim}{K}_1 \underset{\sim}{z}_2 + \underset{\sim}{z}_1 \, . \tag{6.2} $$

Therefore, the knowledge of the description of the systems S_1 and S_2 and of $\underset{\sim}{u}$ and ${}_1\underset{\sim}{y}$ gives $-\underset{\sim}{K}_1 \underset{\sim}{z}_2 + \underset{\sim}{z}_1$ which is the zero-input response of S'_{21}. Since S'_{21} is observable, this zero-input response determines $\left[{}_1\underset{\sim}{x}(t_0), \; {}_2\underset{\sim}{x}(t_0)\right]'$. Therefore, the observability of S'_{21}, in the linear case, implies that of S_f.

Conversely, suppose that S_f is observable and let us show that S'_{21} is observable. Given any input-output pair of S'_{21} we can extract from it (as above) the zero-input response of S'_{21}: $-\underset{\sim}{K}_1 \underset{\sim}{z}_2 + \underset{\sim}{z}_1$. This data together with any $\underset{\sim}{u}$, ${}_1\underset{\sim}{y}$ which satisfy (6.2) reduces the problem to: given an input output pair of S_f find the initial state. Since S_f is observable such determination is possible. Hence the observability of S_f implies that of S'_{21}.

We can specialize to the linear time-invariant case quite easily and the results of Sec. 5 give necessary and sufficient conditions for the feedback connection S_f to be controllable and observable.

7. CONCLUSION

The problem of characterizing the controllability and observability of the representations of linear time-invariant finite-dimensional composite systems has been completely solved. In this sense it is the generalization of the classic paper by Gilbert, who considered exclusively the case of distinct eigenvalues and where $\Lambda_1 \cap \Lambda_2 = \phi$. The very general conditions under which the controllability and observability of a feedback connection is equivalent to that of a tandem connection has also been exhibited.

APPENDIX

PROOF OF THEOREM 3.1

We use the following equivalent definition to prove the theorem: S is controllable if and only if every state in Σ is reachable from the origin in a finite time. Assume that S is in the zero state, then

$$\underset{\sim}{x}(t) = \int_0^t e^{\underset{\sim}{A}(t-\tau)} \underset{\sim}{B}\,\underset{\sim}{u}(\tau)\, d\tau \triangleq e^{\underset{\sim}{A}t} \underset{\sim}{B} * u(t) \ . \tag{1}$$

Since $\underset{\sim}{A}$ is in the Jordan canonical form, Eq. (1) can be decomposed as

$$\underset{\sim}{x}(t) = \begin{bmatrix} e^{\underset{\sim}{A}_{11}t} \underset{\sim}{B}_{11} \\ e^{\underset{\sim}{A}_{12}t} \underset{\sim}{B}_{12} \\ \cdot \\ \cdot \\ \cdot \\ e^{\underset{\sim}{A}_{mr(m)}} \underset{\sim}{B}_{mr(m)} \end{bmatrix} * \underset{\sim}{u} \tag{2}$$

Writing our explicity for the state variables ($\triangleq$ components of the state) corresponding to the i jth Jordan block, we obtain

$$x_{1ij}(t) = \left\{ \frac{1}{(n_{ij}-1)!} t^{n_{ij}-1} e^{\lambda_i t} \underset{\sim}{b}_{\ell ij} + \frac{1}{(n_{ij}-2)!} t^{n_{ij}-2} e^{\lambda_i t} \underset{\sim}{b}_{(\ell-1)ij} \right.$$

$$\left. + \cdots + e^{\lambda_i t} \underset{\sim}{b}_{1ij} \right\} * \underset{\sim}{u} ,$$

$$x_{2ij}(t) = \left\{ \frac{1}{(n_{ij}-2)!} t^{n_{ij}-2} e^{\lambda_i t} \underset{\sim}{b}_{\ell ij} + \cdots + e^{\lambda_i t} \underset{\sim}{b}_{2ij} \right\} * \underset{\sim}{u} ,$$

$$\vdots$$

$$x_{\ell ij}(t) = \underset{\sim}{b}_{\ell ij} \, e^{\lambda_i t} * \underset{\sim}{u} . \tag{3}$$

Observe that all of the first terms in the right-hand side of Eq. (3) are associated with the vector $\underset{\sim}{b}_{\ell ij}$ and are linearly independent. With these preliminaries, we are ready to prove the necessity and the sufficiency of theorem 3.1.

($\Longrightarrow$) Use contradiction. Suppose, for some i, the set $\underset{\sim}{b}_{\ell i1}$, $\underset{\sim}{b}_{\ell i2}$, . . ., $\underset{\sim}{b}_{\ell ir(i)}$ is not a linearly independent set. Then there exists a nonzero r(i)-tuple of complex numbers $\underset{\sim}{q}$ such that

$$\underset{\sim}{q}\begin{bmatrix} \underset{\sim}{b}_{\ell i1} \\ \underset{\sim}{b}_{\ell i2} \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{b}_{\ell ir(i)} \end{bmatrix} = 0 . \tag{4}$$

It implies that

$$\underset{\sim}{q}\begin{bmatrix} x_{\ell i1} \\ x_{\ell i2} \\ \cdot \\ \cdot \\ \cdot \\ x_{\ell ir(i)} \end{bmatrix} = \underset{\sim}{q}\begin{bmatrix} \underset{\sim}{b}_{\ell i1} \\ \underset{\sim}{b}_{\ell i2} \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{b}_{\ell ir(i)} \end{bmatrix} e^{\lambda_i t} * \underset{\sim}{u} = 0 \tag{5}$$

for any $\underset{\sim}{u}$. Hence the states reachable from the origin are in the hyperplane defined by Eq. (5). This contradicts the hypothesis that S is controllable.

($\Longleftarrow$) From Eq. (3) it is clear that if $\underset{\sim}{b}_{\ell ij} \neq 0$, then all the n_{ij} rows in $e^{\underset{\sim}{A}_{ij}t} \underset{\sim}{B}_{ij}$ are linearly independent. By hypothesis that $\underset{\sim}{b}_{\ell i1}$, $\underset{\sim}{b}_{\ell i2}, \ldots, \underset{\sim}{b}_{\ell ir(i)}$ is a linearly independent set; hence all the n_i rows in $e^{\underset{\sim}{A}_i t} \underset{\sim}{B}_i$ are linearly independent. Recall that if $\lambda_i \neq \lambda_j$, then the two functions $p_i(t)\, e^{\lambda_i t}$ and $p_j(t)\, e^{\lambda_j t}$ (where $p_i(\cdot)$ and $p_j(\cdot)$ are polynomials) are linearly independent over any nonempty interval. Consequently, any row (or any linear combination of rows) of $e^{\underset{\sim}{A}_i t} \underset{\sim}{B}_i$ is linearly independent of any row (or any linear combination of rows) of $e^{\underset{\sim}{A}_j t} \underset{\sim}{B}_j$ with $i \neq j$. Thus if the rows of $e^{\underset{\sim}{A}t} \underset{\sim}{B}$ are linearly dependent it is because there is a linear dependence relation that applies to rows lying exclusively in Jordan blocks associated with <u>one</u> eigenvalue. Hence we conclude that the hypotheses imply that all the n rows in $e^{\underset{\sim}{A}t} \underset{\sim}{B}$ are linearly independent over any interval (0, T) with $T > 0$. Now for any T, let

$$\underset{\sim}{u}(t) = \left(e^{\underset{\sim}{A}(T-t)} \underset{\sim}{B}\right)^* \underline{\alpha}, \qquad 0 \le t \le T, \tag{6}$$

where * denotes the complex conjugate transpose and $\underset{\sim}{\alpha}$ is an n-dimensional constant vector to be determined from the desired state at time T. Then from Eq. (1), we have

$$\underset{\sim}{x}(T) = \left\{ \int_0^T (e^{\underset{\sim}{A}\tau} \underset{\sim}{B})(e^{\underset{\sim}{A}\tau} \underset{\sim}{B})^* \, d\tau \right\} \underset{\sim}{\alpha} \triangleq \underset{\sim}{W}(T)\, \underset{\sim}{\alpha} . \tag{7}$$

Since all the n rows of $e^{\underset{\sim}{A}\tau} \underset{\sim}{B}$ are linearly independent over any non-zero interval, using the extension of lemma 11. 2.4 in [2], we conclude that $\underset{\sim}{W}(T)$ is nonsingular for any $T \in (0, \infty)$. Hence for any $\underset{\sim}{x}(T)$, the input $\underset{\sim}{u}(t) = \underset{\sim}{B}^* e^{\underset{\sim}{A}^*(T-t)} W^{-1}(T)\, \underset{\sim}{x}(T)$ transfers the zero state to $\underset{\sim}{x}(T)$ in finite time T.

Q. E. D.

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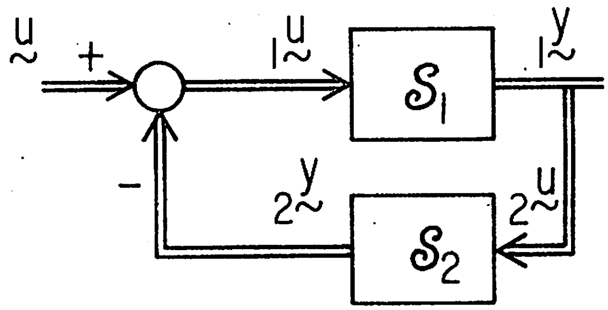
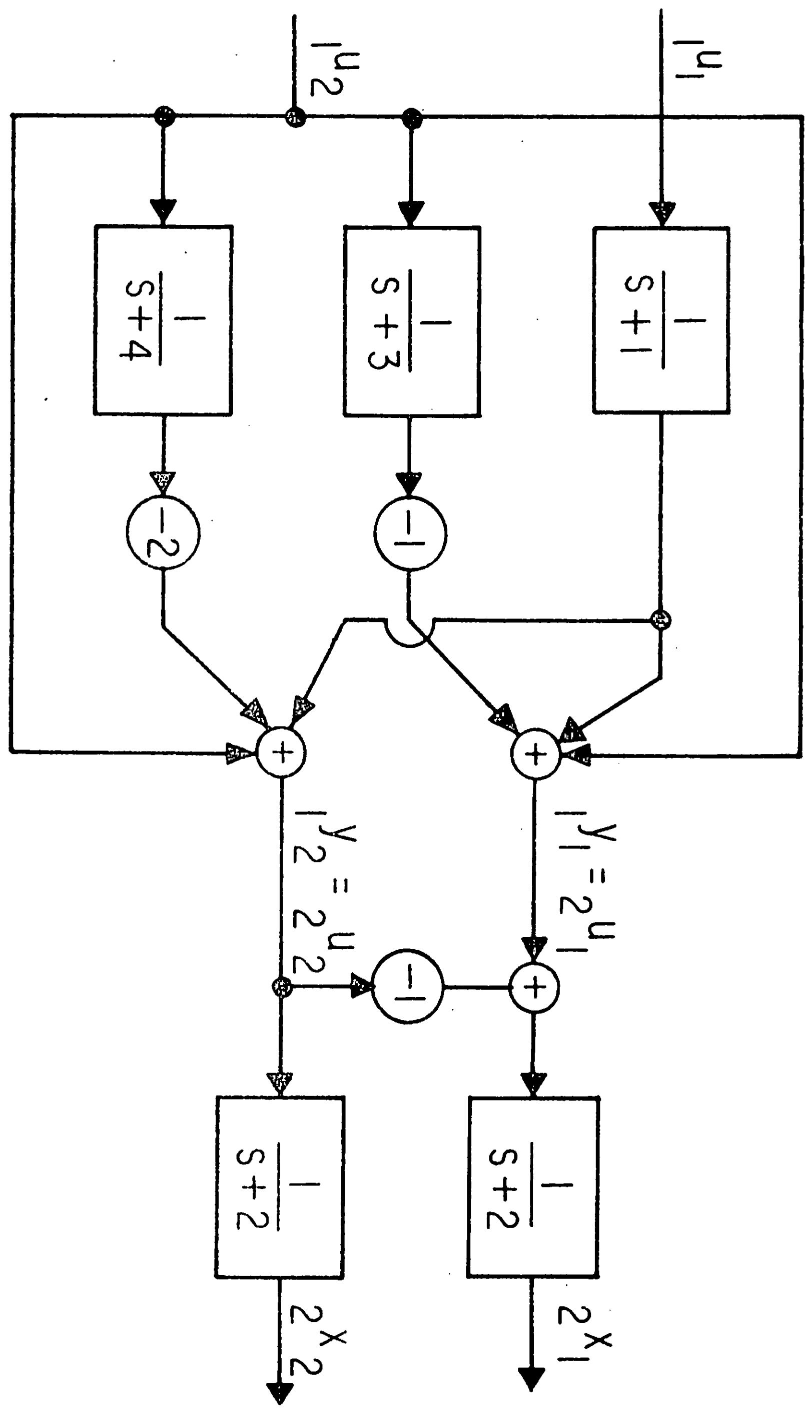
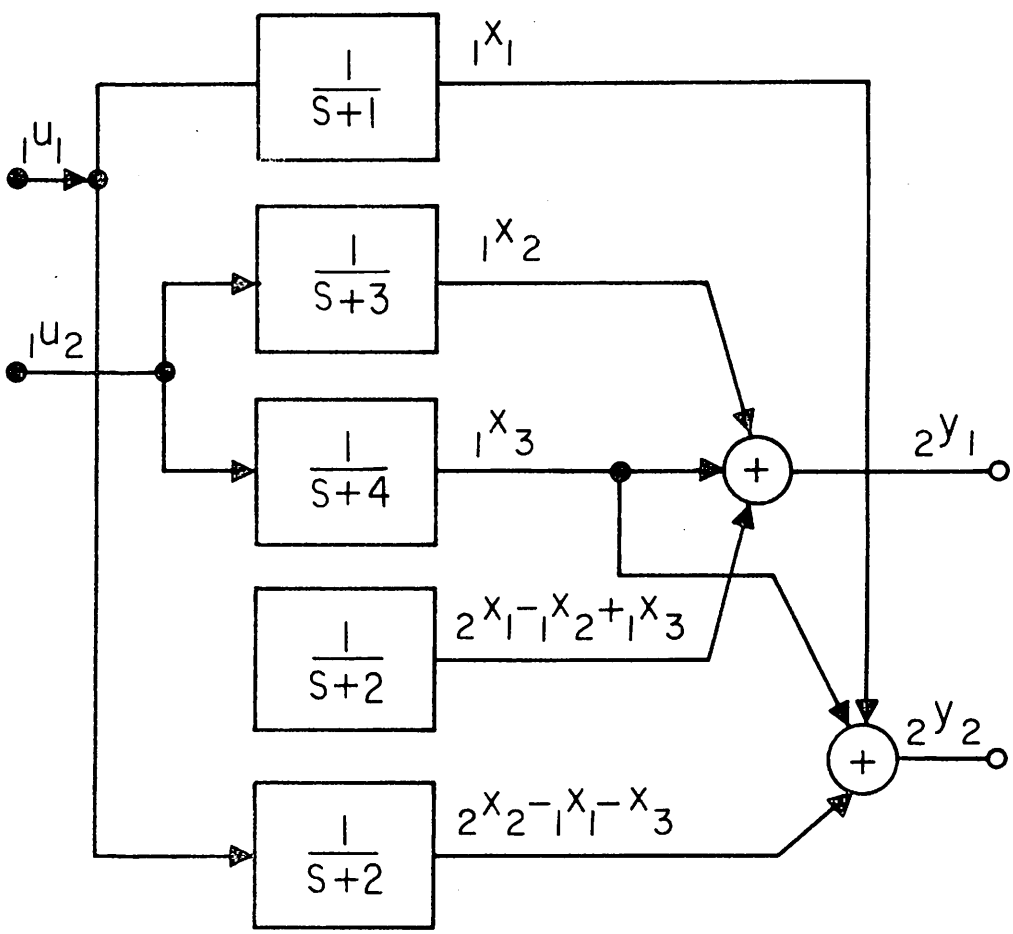
LIST OF FIGURES

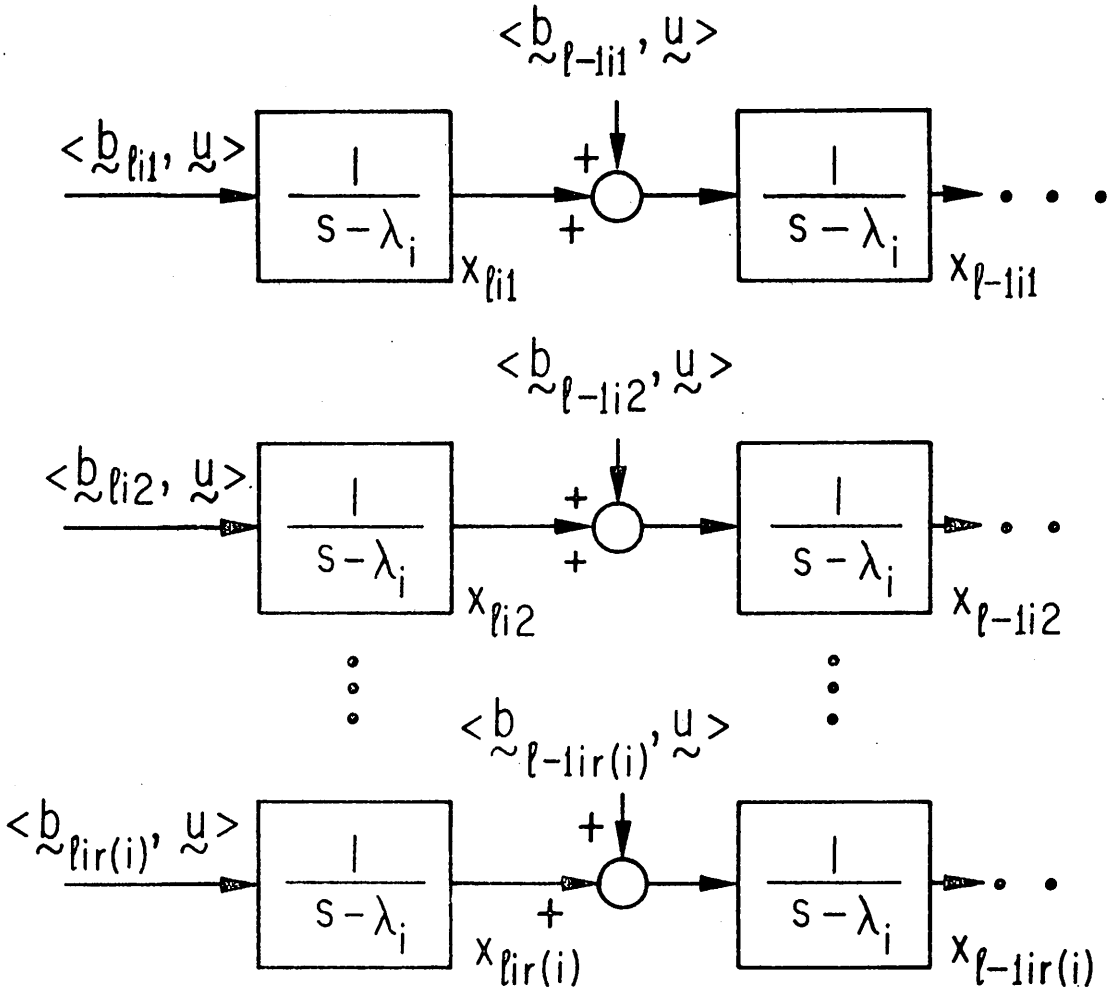

u
+
−
$_1u$
$_1y$
$_2y$
$_2u$
$\mathcal{S}_1$
$\mathcal{S}_2$



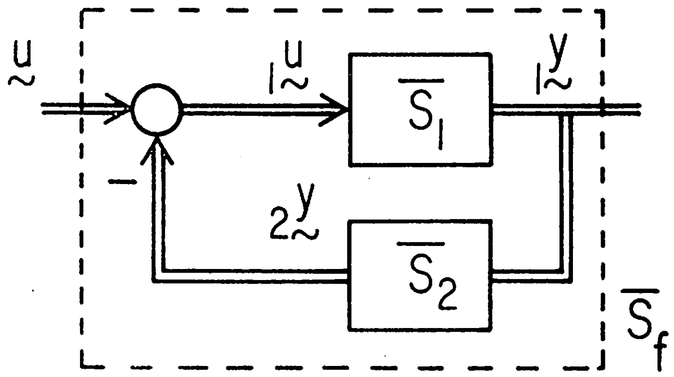
1u1
1u2
1/(s+1)
1/(s+3)
1/(s+4)
-1
-2
+
1y1 = 2u1
1y2 = 2u2
1/(s+2)
2x1
2x2



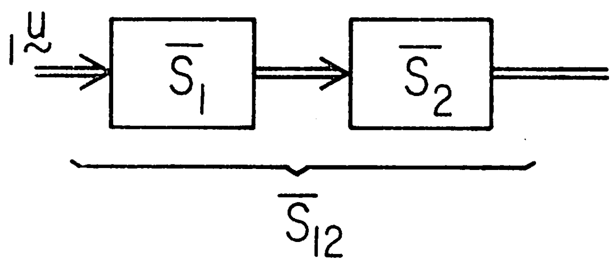
$_1u_1$
$_1u_2$
$\frac{1}{S+1}$
$\frac{1}{S+3}$
$\frac{1}{S+4}$
$\frac{1}{S+2}$
$\frac{1}{S+2}$
$_1x_1$
$_1x_2$
$_1x_3$
$_2x_1 - _1x_2 + _1x_3$
$_2x_2 - _1x_1 - x_3$
+
+
$_2y_1$
$_2y_2$



$\langle \underset{\sim}{b}_{\ell-1i1}, \underset{\sim}{u} \rangle$
$\langle \underset{\sim}{b}_{\ell i1}, \underset{\sim}{u} \rangle$
$\frac{1}{s-\lambda_i}$
$x_{\ell i1}$
$x_{\ell-1i1}$
$\langle \underset{\sim}{b}_{\ell-1i2}, \underset{\sim}{u} \rangle$
$\langle \underset{\sim}{b}_{\ell i2}, \underset{\sim}{u} \rangle$
$x_{\ell i2}$
$x_{\ell-1i2}$
$\langle \underset{\sim}{b}_{\ell-1ir(i)}, \underset{\sim}{u} \rangle$
$\langle \underset{\sim}{b}_{\ell ir(i)}, \underset{\sim}{u} \rangle$
$x_{\ell ir(i)}$
$x_{\ell-1ir(i)}$



(a)



(b)