PERTURBATIONAL SOLUTIONS TO THE DYADIC GREEN'S FUNCTIONS OF MAXWELL'S EQUATIONS IN ANISOTROPIC MEDIA

by

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ABSTRACT

The dyadic Green's functions of Maxwell's equations in anisotropic media are found by a perturbation technique. The solutions are in the form of multipole expansions that may be summed exactly in the case of uniaxial crystal media. When the medium is biaxial or gyrotropic, approximations by partial sum may be used. Second order solutions for biaxial and gyrotropic media are presented. The approximate solutions give the near fields in elementary functions and simple integrals, which may be used to solve integral equations arising from scattering and radiation problems.

1. INTRODUCTION

The dyadic Green's function of Maxwell's equations is defined as the solution of the vector wave equation

$$- \nabla \times \nabla \times G(\overline{r}/\overline{r}_0) + k_0^2 \underline{\underline{\epsilon}} \cdot G(\overline{r}/\overline{r}_0) = \delta(\overline{r}/\overline{r}_0) I \tag{1}$$

where $\underline{\underline{\epsilon}}$ is a tensor of rank 2 and I is a unit dyad. When $G(\overline{r}/\overline{r}_0)$ is known, the radiation from any current distribution may be obtained by integrating:

$$\vec{E}(\overline{r}) = j\omega\mu \int_V G(\overline{r}/\overline{r}_0) \cdot \vec{J}(\overline{r}_0)\, dv_0 . \tag{2}$$

The formal solution of (1) using Fourier transform is straightforward. Following the pioneering work of Bunkin [1957], many authors have contributed to the evaluation of the Fourier inversion of the solution using asymptotic methods that are valid for the far fields. In many important applications of $G(\overline{r}/\overline{r}_0)$, such as scattering and antenna problems, (2) is to be solved as an integral equation for the current distribution $\vec{J}(r)$ which will produce a prescribed tangential electric field over a known boundary. For example, in the problem of scattering by a conducting obstacle, the integral equation to be solved is

$$\hat{n} \times \vec{E}(\bar{r}_s) = j\omega\mu\, \hat{n} \times \int_s G(\bar{r}_s/\bar{r}_0) \cdot \vec{J}(\bar{r}_0)\, ds_0, \tag{3}$$

where $\hat{n}$ is the unit normal of the conducting surface, and r_s and r_0 are points on the surface of the obstacle. Numerical solution of such an integral equation [Mei and Van Bladel, 1963] is possible if the near fields of $G(\bar{r}/\bar{r}_0)$ are available.

The only known investigation of the near fields of $G(\bar{r}/\bar{r}_0)$ is by Mittra and Deschamps [1962], who found the singular terms of the dyadic, but the remaining terms were left in double integrals which are inconvenient for numerical applications. In this paper we shall investigate the problem in a different light. The solutions of Maxwell's equations for anisotropic media are considered as perturbations of the solutions for isotropic media. It is shown that the solutions so obtained result in power series of the perturbations. The series can be summed exactly for uniaxial crystals. A partial sum technique is used for the case of biaxial crystal and gyrotropic media. Second order solutions are presented for biaxial and gyrotropic media. They are in elementary functions and simple integrals feasible for numerical applications.

2. FORMAL SOLUTIONS

In this paper we shall assume that $\underline{\underline{\epsilon}}$ is in one of the following two forms

$$\underline{\underline{\epsilon}}_c = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \tag{4}$$

or

$$\underline{\underline{\epsilon}}_g = \begin{bmatrix} \epsilon_1 & -j\nu & 0 \\ j\nu & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \tag{5}$$

Let $G_0(\bar{r}/\bar{r}_0)$ be the dyadic Green's function of the vector wave equation in an infinite homogeneous isotropic medium, which satisfies the radiation condition at infinity. Thus,

$$-\nabla\times\nabla\times G_0(\bar{r}/\bar{r}_0) + k_1^2 G_0(\bar{r}/\bar{r}_0) = \delta(\bar{r}/\bar{r}_0)\, I, \tag{6}$$

where $k_1^2 = k_0^2 \epsilon_1$. The solution of (6) is known to be

$$G_0 = \left(I + \frac{\nabla\nabla}{k_1^2} \right) \phi_0(\bar{r}/\bar{r}_0), \tag{7}$$

where the function $\phi_0(\bar{r}/\bar{r}_0)$ is defined as

$$\phi_0(\bar{r}/\bar{r}_0) = - \frac{e^{-jk_1|\bar{r} - \bar{r}_0|}}{4\pi |\bar{r} - \bar{r}_0|}. \tag{8}$$

Equation (1) may be written as

$$- \nabla \times \nabla \times G(\bar{r}/\bar{r}_0) + k_1^2 G(\bar{r}/\bar{r}_0) = \delta(\bar{r}/\bar{r}_0) I - k_1^2 \underline{\underline{\delta}} \cdot G(\bar{r}/\bar{r}_0), \tag{9}$$

where

$$\underline{\underline{\delta}} = (\underline{\underline{\epsilon}} - \epsilon_1 I) \frac{1}{\epsilon_1}. \tag{10}$$

The solution of (9) is

$$G(\bar{r}/\bar{r}_0) = G_0(\bar{r}/\bar{r}_0) - k_1^2 \int_{v'} G_0(\bar{r}/\bar{r}') \cdot \underline{\underline{\delta}} \cdot G(\bar{r}'/\bar{r}_0) dv', \tag{11}$$

where the integration is a volume integral over the entire space. Iterating (11) once, we obtain

$$G(\bar{r}/\bar{r}_0) = G_0(\bar{r}/\bar{r}_0) - k_1^2 \int_{v'} G_0(\bar{r}/\bar{r}') \cdot \underline{\underline{\delta}} \cdot G_0(\bar{r}'/\bar{r}_0) dv'$$

$$+ k_1^4 \int_{v'} \int_{v''} G_0(\bar{r}/\bar{r}') \cdot \underline{\underline{\delta}} \cdot G_0(\bar{r}'/\bar{r}'') \cdot \underline{\underline{\delta}} \cdot G(\bar{r}''/\bar{r}_0) dv'' dv'. \tag{12}$$

Substituting (7) in the second term on the right-hand side of (12), we get

$$- k_1^2 \int_{v'} \left(I + \frac{\nabla\nabla}{k_1^2}\right) \phi_0(\bar{r}/\bar{r}') \cdot \underline{\underline{\delta}} \cdot \left(I + \frac{\nabla_0 \nabla_0}{k_1^2}\right) \phi_0(\bar{r}'/\bar{r}_0)\, dv', \tag{13}$$

where the operators ∇ and ∇_0 operate on the variables $\bar{r}$ and $\bar{r}_0$, respectively. Since the integration in (13) is over the variable $\bar{r}'$, the dyadic operators may be taken outside of the integration. After changing the variable $\bar{r}'$ to $\bar{t} = \bar{r} - \bar{r}'$, we get

$$- k_1^2 \left(I + \frac{\nabla\nabla}{k_1^2}\right) \cdot \underline{\underline{\delta}} \cdot \left(I + \frac{\nabla_0 \nabla_0}{k_1^2}\right) \int_{v_t} \phi_0(\bar{t})\, \phi_0(\bar{t} - [\bar{r} - \bar{r}_0])\, dv_t . \tag{14}$$

The integral in (14) is recognized as a convolution integral of the functions $\phi_0(\bar{r})$ and $\phi_0(\bar{r} - \bar{r}_0)$. We define the Fourier transform pair of a function as

$$F(\vec{k}) = \int_v f(\vec{r})\, e^{j\vec{k}\cdot\vec{r}}\, dv , \tag{15}$$

$$f(\vec{r}) = \frac{1}{(2\pi)^3} \int_{v_k} F(\vec{k})\, e^{-j\vec{k}\cdot\vec{r}}\, dv_k . \tag{16}$$

By expressing (14) in the inverse Fourier transform we have

$$- k_1^2 \left(I + \frac{\nabla\nabla}{k_1^2}\right) \cdot \underline{\underline{\delta}} \cdot \left(I + \frac{\nabla_0 \nabla_0}{k_1^2}\right) \frac{1}{(2\pi)^3} \int_{v_k} \frac{e^{-j\vec{k}\cdot(\bar{r} - \bar{r}_0)}\, dv_k}{(k_1^2 - |\vec{k}|^2)^2} . \tag{17}$$

For convenience in the ensuing discussion, we define the function $\phi_n(p, \underline{\underline{a}}; \bar{r}/\bar{r}_0)$ as

$$\phi_n(p, \underline{\underline{a}}; \bar{r}/\bar{r}_0) = \frac{1}{(2\pi)^3} \int_{v_k} \frac{e^{-j\vec{k}\cdot(\bar{r} - \bar{r}_0)}\, dv_k}{(p - \vec{k}\cdot \underline{\underline{a}} \cdot \vec{k})^{n+1}} , \tag{18}$$

where the matrix $\underline{\underline{a}}$ is defined as

$$\underline{\underline{a}} = \begin{bmatrix} a_x & 0 & 0 \\ 0 & a_y & 0 \\ 0 & 0 & a_z \end{bmatrix} . \tag{19}$$

The integration of (18) is given in closed form in Appendix A. Equation (12) may then be written as

$$G(\bar{r}/\bar{r}_0) = (I + \frac{\nabla\nabla}{k_1^2}) \cdot \sum_{n=0}^{1} (-1)^n [k_1^2 \underline{\underline{\delta}} + \underline{\underline{\delta}} \cdot \nabla\nabla]^n \phi_n$$

$$+ k_1^4 \int_{v'} \int_{v''} G_0(\bar{r}/\bar{r}') \cdot \underline{\underline{\delta}} \cdot G_0(\bar{r}'/\bar{r}_0)\, dv''\, dv' .$$

Further iterations of (20) lead to the formal solution of $G(\bar{r}/\bar{r}_0)$ as

$$G(\bar{r}/\bar{r}_0) = (I + \frac{\nabla\nabla}{k_1^2}) \cdot \sum_{n=0}^{\infty} (-1)^n [k_1^2 \underline{\underline{\delta}} + \underline{\underline{\delta}} \cdot \nabla\nabla]^n \phi_n(k_1^2, I; \bar{r}/\bar{r}_0) . \tag{21}$$

This follows since $\phi_n(k_1^2, \underline{\underline{a}}; \bar{r}/\bar{r}_0)$ is a function of $|\bar{r} - \bar{r}_0|$, hence the operational relation $\nabla = -\nabla_0$ still holds. It is noticed that (21) is essentially a multipole expansion, i.e., the $\underline{n}$th order term in the summation contains a singularity of order $n + 1$ (for $n \neq 0$) at $\bar{r} = \bar{r}_0$. We shall discuss the summation of (21) for uniaxial, biaxial, and gyrotropic media in the following sections.

3. UNIAXIAL CRYSTAL

The dielectric tensor for a uniaxial crystal is $\epsilon = \epsilon_1 \hat{x}\hat{x} + \epsilon_1 \hat{y}\hat{y} + \epsilon_3 \hat{z}\hat{z}$. It follows that

$$\underline{\underline{\delta}} = \frac{\epsilon_3 - \epsilon_1}{\epsilon_1} \hat{z}\hat{z} = \delta_3 \hat{z}\hat{z} . \tag{22}$$

Substituting (22) into (21) we have

$$G(\bar{r}/\bar{r}_0) = (I + \frac{\nabla\nabla}{k_1^2}) \sum_{n=0}^{\infty} (-1)^n \delta_3^n (k_1^2 \hat{z}\hat{z} + \hat{z}\hat{z} \cdot \nabla\nabla)^n \phi_n (k_1^2, I; \bar{r}/\bar{r}_0) . \tag{23}$$

Due to the orthogonality of the base vectors, the following relation holds for $n \neq 0$:

$$\begin{aligned}(k_1^2 \hat{z}\hat{z} + \hat{z} \nabla \frac{\partial}{\partial z})^n &= (k_1^2 \hat{z}\hat{z} + \hat{z}\hat{z} \frac{\partial^2}{\partial z^2})^{n-1} \cdot (k_1^2 \hat{z}\hat{z} + \hat{z} \nabla \frac{\partial}{\partial z}) \\ &= (k_1^2 + \frac{\partial^2}{\partial z^2})^{n-1} \hat{z}\hat{z} \cdot (k_1^2 \hat{z}\hat{z} + \hat{z} \nabla \frac{\partial}{\partial z}) \\ &= (k_1^2 \hat{z}\hat{z} + \hat{z} \nabla \frac{\partial}{\partial z})(k_1^2 + \frac{\partial^2}{\partial z^2})^{n-1} . \end{aligned} \tag{24}$$

Thus, (23) becomes

$$\begin{aligned} G(\bar{r}/\bar{r}_0) = (I + \frac{\nabla\nabla}{k_1^2}) \cdot \Big\{ & I \phi_0(k_1^2, I; \bar{r}/\bar{r}_0) \\ & + (k_1^2 \hat{z}\hat{z} + \hat{z} \nabla \frac{\partial}{\partial z}) \sum_{n=1}^{\infty} (-1)^n \delta_3^n (k_1^2 + \frac{\partial^2}{\partial z^2})^{n-1} \phi_n(k_1^2, I; \bar{r}/\bar{r}_0) \Big\} \end{aligned} \tag{25}$$

Making use of the results of Appendix B, we may write the summation in (25) in operational form as

$$\begin{aligned} \sum_{n=1}^{\infty} & (-1)^n \delta_3^n (k_1^2 + \frac{\partial^2}{\partial z^2})^{n-1} \phi_n(k_1^2, I; \bar{r}/\bar{r}_0) \\ &= \sum_{n=1}^{\infty} \frac{\delta_3^n}{n!} (k_1^2 + \frac{\partial^2}{\partial z^2})^{n-1} \frac{\partial^n}{\partial p^n} \phi_0 (k_1^2, I; \bar{r}/\bar{r}_0) \end{aligned}$$

(cont'd)

$$= \sum_{n=1}^{\infty} \frac{\delta_3^n}{n!} \left(k_1^2 \frac{\partial}{\partial p} + \frac{\partial}{\partial p} \frac{\partial^2}{\partial z^2}\right)^{n-1} \frac{\partial}{\partial p} \phi_0 (k_1^2, I; \bar{r}/\bar{r}_0)$$

$$= \sum_{n=1}^{\infty} \frac{\delta_3^n}{n!} \left(k_1^2 \frac{\partial}{\partial p} + \frac{\partial}{\partial a_z}\right)^{n-1} \frac{\partial}{\partial p} \phi_0 (k_1^2, I; \bar{r}/\bar{r}_0) . \tag{26}$$

And, introducing the parameter t in (26), we get

$$\sum_{n=1}^{\infty} (-1)^n \delta_3^n \left(k_1^2 + \frac{\partial^2}{\partial z^2}\right)^{n-1} \phi_n (k_1^2, I; \bar{r}/\bar{r}_0)$$

$$= \sum_{n=1}^{\infty} \int_0^{\delta_3} \frac{t^{n-1} dt}{(n-1)!} \left(k_1^2 \frac{\partial}{\partial p} + \frac{\partial}{\partial a_z}\right)^{n-1} \frac{\partial}{\partial p} \phi_0 (k_1^2, I; \bar{r}/\bar{r}_0)$$

$$= \int_0^{\delta_3} \sum_{n=0}^{\infty} \frac{\left(k_1^2\, t \frac{\partial}{\partial p} + t \frac{\partial}{\partial a_z}\right)^n}{n\,!} \frac{\partial}{\partial p} \phi_0 (k_1^2, I; \bar{r}/\bar{r}_0)\, dt . \tag{27}$$

The summation in (27) is recognized as Taylor's expansion of the function

$$\frac{\partial}{\partial p} \phi_0 [k_1^2 (1+t),\ I + t\hat{z}\hat{z}\,; \bar{r}/\bar{r}_0] = -\phi_1 [k_1^2 (1+t),\ I + t\hat{z}\hat{z}\,; \bar{r}/\bar{r}_0]$$

$$= \frac{j}{8\pi(1+t)k_1}\, e^{-jk_1(1+t)^{1/2} \left[(x-x_0)^2 + (y-y_0)^2 + \frac{(z-z_0)^2}{1+t}\right]^{1/2}} . \tag{28}$$

We designate the function (27) by $F(\bar{r}/\bar{r}_0)$, thus

$$F(\bar{r}/\bar{r}_0) = -\int_0^{\delta_3} \phi_1 [k_1^2 (1+t),\ I + t\hat{z}\hat{z}\,; \bar{r}/\bar{r}_0]\, dt . \tag{29}$$

It is readily shown that

$$(k_1^2 + \frac{\partial^2}{\partial z^2}) F(\bar{r}/\bar{r}_0) = \phi_0[k_1^2(1+\delta_3), I+\delta_3 \hat{z}\hat{z}; \bar{r}/\bar{r}_0] - \phi_0[k_1^2, I; \bar{r}/\bar{r}_0]. \tag{30}$$

Using the following abbreviations for the functions

$$\phi_0[k_1^2(1+\delta_3), I+\delta_3), I+\delta_3 \hat{z}\hat{z}; \bar{r}/\bar{r}_0] = \phi_0(\delta_3),$$

$$\phi_0(k_1^2, I, \bar{r}/\bar{r}_0) = \phi_0,$$

$$F(\bar{r}/\bar{r}_0) = F,$$

we get the final form of $G(\bar{r}/\bar{r}_0)$

$$G(\bar{r}/\bar{r}_0) = \begin{bmatrix} \phi_0 + \frac{\partial^2}{\partial x^2}\left[\frac{\phi_0(\delta_3)}{k_1^2} - F\right] & \frac{\partial^2}{\partial x \partial y}\left[\frac{\phi_0(\delta_3)}{k_1^2} - F\right] & \frac{\partial^2}{\partial x \partial y}\frac{\phi_0(\delta_3)}{k_1^2} \\ \frac{\partial^2}{\partial x \partial y}\left[\frac{\phi_0(\delta_3)}{k^2} - F\right] & \phi_0 + \frac{\partial^2}{\partial y^2}\left[\frac{\phi_0(\delta_3)}{k_1^2} - F\right] & \frac{\partial^2}{\partial y \partial z}\frac{\phi_0(\delta_3)}{k_1^2} \\ \frac{\partial^2}{\partial x \partial z}\frac{\phi_0(\delta_3)}{k_1^2} & \frac{\partial^2}{\partial y \partial z}\frac{\phi_0(\delta_3)}{k_1^2} & (1 + \frac{\partial^2}{k_1^2 \partial z^2})\phi_0(\delta_3) \end{bmatrix} \tag{31}$$

The third column in (31) which corresponds to the electric field produced by an infinitesimal dipole in the z-axis was previously solved by Clemmow [1962]. The above solution is exact and we notice that the effect of the perturbation is mainly on the propagation parameter. Therefore, if only a partial sum were used in (21), the solution would be valid only for near and intermediate fields.

4. BIAXIAL CRYSTAL

When the medium is biaxial, the tensor $\underline{\underline{\delta}}$ is of the form

$$\underline{\underline{\delta}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix} . \qquad (32)$$

In this case, the operator $[k_1^2 \underline{\underline{\delta}} + \underline{\underline{\delta}} \cdot \nabla\nabla]^n$ can no longer be simplified as it was for the uniaxial media. An exact summation of (21) cannot be obtained. If $||\delta||$ is small, it is possible to take finite terms. Let $G_N(\bar{r}/\bar{r}_0)$ be the $\underline{N}$ th order approximation of $G(\bar{r}/\bar{r}_0)$, i.e.,

$$G_N(\bar{r}/\bar{r}_0) = (I + \frac{\nabla\nabla}{k_1^2}) \cdot \sum_{n=0}^{N} (-1)^n [k_1^2 \underline{\underline{\delta}} + \underline{\underline{\delta}} \cdot \nabla\nabla]^n \phi_n(k_1^2, I; \bar{r}/\bar{r}_0) . \qquad (33)$$

There are, however, several disadvantages associated with the formula of (33) in that the multipole expansion requires $G_N(\bar{r}/\bar{r}_0)$ to be considered as a generalized function [Lighthill, 1958] near the singularities, which results in the measure of the high order derivatives of $\vec{J}(r)$ in (2). The integral equation (2) then becomes an integral differential equation involving high order derivatives of the unknown $\vec{J}(r)$. Furthermore, the high order derivative operators in $G_N(\bar{r}/\bar{r}_0)$ give rise to a multitude of terms which make the application cumbersome. In the following we shall remedy the situation by a "partial infinite sum" method which simplifies the approximate formula as well as reduces the order of singularity of $G_N(\bar{r}/\bar{r}_0)$. We shall only present the result of $G_2(\bar{r}/\bar{r}_0)$ in this paper. Higher order approximations may be obtained in essentially the same manner.

In the expansion of $G_N(\bar{r}/\bar{r}_0)$, we shall frequently come across the operator

$$\nabla\nabla \cdot \underline{\underline{\delta}} \cdot \nabla\nabla = \nabla(\delta_2 \frac{\partial^2}{\partial y^2} + \delta_3 \frac{\partial^2}{\partial y^2}) \nabla = \nabla\nabla(\delta_2 \frac{\partial^2}{\partial y^2} + \partial_3 \frac{\partial^2}{\partial z^2}) . \qquad (34)$$

Let the scalar operators ∇_1^2 and ∇_2^2 be defined as

$$\nabla_1^2 = \delta_2 \frac{\partial^2}{\partial y^2} + \delta_3 \frac{\partial^2}{\partial z^2} , \tag{35}$$

$$\nabla_2^2 = \delta_2^2 \frac{\partial^2}{\partial y^2} + \delta_3^2 \frac{\partial^2}{\partial z^2} . \tag{36}$$

We can then write $G_2(\bar{r}/\bar{r}_0)$ as

$$G_2(\bar{r}/\bar{r}_0) = (I\,\phi_0 - k_1^2 \underline{\underline{\delta}}^2 \phi_2) + \frac{\nabla\nabla}{k_1^2}(\phi_0 - \nabla_1^2 \phi_1 + \nabla_1^4 \phi_2)$$

$$- \nabla\nabla \cdot \underline{\underline{\delta}} (\phi_1 - \nabla_1^2 \phi_2) - \underline{\underline{\delta}} \cdot \nabla\nabla (\phi_1 - \nabla_1^2 \phi_2)$$

$$+ \nabla\nabla\nabla_2^2 \phi_2 + (k_1^2 \underline{\underline{\delta}}^2 \cdot \nabla\nabla + k_1^2 \underline{\underline{\delta}} \cdot \nabla\nabla \cdot \underline{\underline{\delta}} + k_1^2 \nabla\nabla \cdot \underline{\underline{\delta}}^2)\phi_2 .$$

(37)

We now construct a new function, $G_{II}(\bar{r}/\bar{r}_0)$, by adding to (37) those terms selected from the higher order parts of (21) to get

$$G_{II}(\bar{r}/\bar{r}_0) = \sum_{n=0}^{\infty} (-1)^n k_1^{2n} \underline{\underline{\delta}}^n \phi_n + \frac{\nabla\nabla}{k_1^2} \sum_{n=0}^{\infty} (-1)^n \nabla_1^{2n} \phi_n$$

$$- \nabla\nabla \cdot \underline{\underline{\delta}} \sum_{n=1}^{\infty} (-1)^{n-1} \nabla_1^{2(n-1)} \phi_n - \underline{\underline{\delta}} \cdot \nabla\nabla \sum_{n=1}^{\infty} (-1)^{n-1} \nabla_1^{2(n-1)} \phi_n$$

$$+ \nabla\nabla \sum_{n=2}^{\infty} (-1)^n [\delta_2^n \frac{\partial^{2n-2}}{\partial y^{2n-2}} + \partial_3^n \frac{\partial^{2n-2}}{\partial z^{2n-2}}] \phi_n$$

$$+ (k_1^2 \underline{\underline{\delta}}^2 \cdot \nabla\nabla + k_1^2 \underline{\underline{\delta}} \cdot \nabla\nabla \cdot \underline{\underline{\delta}} + k_1^2 \nabla\nabla \cdot \underline{\underline{\delta}}^2) \phi_2 . \tag{38}$$

Making use of the results in Appendices A and B, we obtain the following identities:

$$\sum_{n=0}^{\infty} (-1)k_1^{2n} \underline{\underline{\delta}}^n \phi_n = \sum_{n=0}^{\infty} [\frac{k_1^{2n} \underline{\underline{\delta}}^n}{n!}] \phi_0 (k_1^2, I; \bar{r}/\bar{r}_0)$$

$$= \sum_{n=0}^{\infty} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{k_1^{2n} \delta_2^n}{n!} & 0 \\ 0 & 0 & \frac{k_1^{2n} \delta_3^n}{n!} \end{pmatrix} \frac{\partial^n}{\partial pn} \phi_0 (k_1^2, I; \bar{r}/\bar{r}_0) + \hat{x}\hat{x} \phi_0 (k_1^2, I, \bar{r}/\bar{r}_0)$$

$$= \begin{pmatrix} \phi_0(k_1^2, I; \bar{r}/\bar{r}_0) & 0 & 0 \\ 0 & \phi_0[k_1^2(1+\delta_2), I; \bar{r}/\bar{r}_0] & 0 \\ 0 & 0 & \phi_0[k_1^2(1+\delta_3), I; \bar{r}/\bar{r}_0] \end{pmatrix} \quad (39)$$

$$\sum_{n=0}^{\infty} (-1)^n \nabla^{2n} \phi_n = \sum_{n=0}^{\infty} \frac{(\delta_2 \frac{\partial}{\partial a_y} + \delta_3 \frac{\partial}{\partial a_z})^n}{n!} \phi_0(k_1^2, I; \bar{r}/\bar{r}_0)$$

$$= \phi_0[k_1^2, I + \delta_2 \hat{y}\hat{y} + \delta_3 \hat{z}\hat{z}; \bar{r}/\bar{r}_0], \quad (40)$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \nabla_1^{2(n-1)} \phi_n = \sum_{n=1}^{\infty} \frac{(\delta_2 \frac{\partial}{\partial a_y} + \delta_3 \frac{\partial}{\partial a_z})^{n-1}}{n!} \phi_1(k_1^2, I; \bar{r}/\bar{r}_0)$$

$$= \sum_{n=1}^{\infty} \frac{1}{\delta_2} \int_0^{\delta_2} \frac{(\frac{\partial}{\partial a_y} + \frac{\delta^3}{\delta_2} \frac{\partial}{\partial a_z})^{n-1} t^{n-1}}{(n-1)!} \phi_1 (k_1^2, I; \bar{r}/\bar{r}_0)\, dt$$

(cont'd)

$$= \frac{1}{\delta_2} \sum_{n=0}^{\infty} \int_0^{\delta_2} \frac{\left[t \frac{\partial}{\partial a_y} + \frac{\delta^3}{\delta_2} t \frac{\partial}{\partial a_z}\right]^n}{n!} \phi_1(k_1^2, I; \bar{r}/\bar{r}_0)\, dt$$

$$= \frac{1}{\delta_2} \int_0^{\delta_2} \phi_1(k_1^2, I + t\,\hat{y}\hat{y} + \frac{\delta_3}{\delta_2} t\,\hat{z}\hat{z}; \bar{r}/\bar{r}_0)\, dt, \tag{41}$$

$$\sum_{n=2}^{\infty} (-1)^n \delta_2^n \frac{\partial^{2n-2}}{\partial y^{2n-2}} \phi_n = \sum_{n=2}^{\infty} (-1)^{2n-1} \frac{\delta_2^n}{n!} \frac{\partial^{n-1}}{\partial a_y^{n-1}} \phi_1(k_1^2, I; \bar{r}/\bar{r}_0)$$

$$= - \sum_{n=2}^{\infty} \int_0^{\delta_2} \frac{t^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial a_y^{n-1}} \phi_1(k_1^2, I; \bar{r}/\bar{r}_0)\, dt$$

$$= - \sum_{n=1}^{\infty} \int_0^{\delta_2} \frac{t^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial a_y^{n-1}} \phi_1(k_1^2, I; \bar{r}/\bar{r}_0)\, dt + \delta_2 \phi_1(k_1^2, I; \bar{r}/\bar{r}_0)$$

$$= - \sum_{n=0}^{\infty} \int_0^{\delta_2} \frac{t^n}{n!} \frac{\partial^n}{\partial a_y^n} \phi_1(k_1^2, I; \bar{r}/\bar{r}_0)\, dt + \delta_2 \phi_1(k_1^2, I; \bar{r}/\bar{r}_0)$$

$$= - \int_0^{\delta_2} \phi_1(k_1^2, I + t\,\hat{y}\hat{y}; \bar{r}/\bar{r}_0)\, dt + \delta_2 \phi_1(k_1^2, I; \bar{r}/\bar{r}_0). \tag{42}$$

Similarly,

$$\sum_{n=2}^{\infty} (-1)^n \delta_3^n \frac{\partial^{2n-2}}{\partial z^{2n-2}} \phi_n = - \int_0^{\delta_3} \phi_1(k_1^2, I + t\,\hat{z}\hat{z}; \bar{r}/\bar{r}_0)\, dt + \delta_3 \phi_1(k_1^2, I; \bar{r}/\bar{r}_0). \tag{43}$$

Using the formulas (39) to (43) in (38), we get the final form for the second order solution as

$$G_{II}(\bar{r}/\bar{r}_0) = \phi_0(k_1^2, I; \bar{r}/\bar{r}_0)\hat{x}\hat{x} + \phi_0[k_1^2(1+\delta_2), I; \bar{r}/\bar{r}_0]\hat{y}\hat{y}$$

$$+ \phi_0[k_1^2(1+\delta_3), I; \bar{r}/\bar{r}_0]\hat{z}\hat{z} + \nabla\nabla\left\{\frac{\phi_0}{k_1^2}[k_1^2, I+\delta_2\hat{y}\hat{y}; \bar{r}/\bar{r}_0]\right.$$

$$- \int_0^{\delta_2} \phi_1(k_1^2, I+t\hat{y}\hat{y}; \bar{r}/\bar{r}_0)\,dt - \int_0^{\delta_3} \phi_1(k_1^2, I+t\hat{z}\hat{z}; \bar{r}/\bar{r}_0)\,dt$$

$$\left. + (\delta_2+\delta_3)\phi_1(k_1^2, I; \bar{r}/\bar{r}_0)\right\} - [\nabla\nabla \cdot \underline{\underline{\delta}} \cdot \nabla\nabla]\frac{1}{\delta_2}$$

$$\cdot \int_0^{\delta_2} \phi_1(k_1^2, I+t\hat{y}\hat{y} + \frac{\delta_3}{\delta_2}t\hat{z}\hat{z}; \bar{r}/\bar{r}_0)\,dt$$

$$+\left\{k_1^2\underline{\underline{\delta}}^2 \cdot \nabla\nabla + k_1^2\underline{\underline{\delta}} \cdot \nabla\nabla \cdot \underline{\underline{\delta}} + k_1^2\nabla\nabla \cdot \underline{\underline{\delta}}^2\right\}\phi_2(k_1^2, I; \bar{r}/\bar{r}_0). \tag{44}$$

We notice that the order of singularity of $G_{II}(\bar{r}/\bar{r}_0)$ is the same as that of $G_0(\bar{r}/\bar{r}_0)$.

5. GYROTROPIC MEDIA

In the case of gyrotropic medium, $\underline{\underline{\delta}}$ is of the form

$$\underline{\underline{\delta}} = \begin{bmatrix} 0 & -j\nu & 0 \\ j\nu & 0 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix}. \tag{45}$$

The second order solution for this case is identical to (37). The matrix $\underline{\underline{\delta}}^n$ takes different forms for even and odd n. They are

$$\underline{\underline{\delta}}^{2m} = \begin{bmatrix} \nu^{2m} & 0 & 0 \\ 0 & \nu^{2m} & 0 \\ 0 & & \delta_3^{2m} \end{bmatrix}, \tag{46}$$

$$\underline{\underline{\delta}}^{2m+1} = \begin{bmatrix} 0 & -j\nu^{2m+1} & 0 \\ j\nu^{2m+1} & 0 & 0 \\ 0 & 0 & \delta_3^{2m+1} \end{bmatrix}. \tag{47}$$

And, the operational products $\nabla\nabla \cdot \underline{\underline{\delta}}^n \cdot \nabla\nabla$ become

$$\nabla\nabla \cdot \underline{\underline{\delta}}^{2m} \cdot \nabla\nabla = \nabla\nabla \left(\nu^{2m} \frac{\partial^{2m}}{\partial x^{2m}} + \nu^{2m} \frac{\partial^{2m}}{\partial y^{2m}} + \delta_3^{2m} \frac{\partial^{2m}}{\partial_z^{\;2m}}\right), \tag{48}$$

$$\nabla\nabla \cdot \delta^{2m+1} \cdot \nabla\nabla = \nabla\nabla\, \delta_3^{2m} \frac{\partial^{2m}}{\partial z^{2m}}. \tag{49}$$

Using the "partial infinite sum," we have

$$G_{II}(\bar{r}/\bar{r}_0) = \sum_{n=0}^{\infty} (-1)^n k_1^{2n} \underline{\underline{\delta}}^n \phi_n + \frac{\nabla\nabla}{k_1^2} \sum_{n=0}^{\infty} (-1)^n \delta_3^n \frac{\partial^{2n}}{\partial z^{2n}} \phi_n$$

$$- [\nabla\nabla \cdot \delta + \delta \cdot \nabla\nabla] \sum_{n=1}^{\infty} (-1)^{n-1} \delta_3^{n-1} \frac{\partial^{2(n-1)}}{\partial z^{2(n-1)}} \phi_n$$

$$+ \frac{\nabla\nabla}{k_1^2} \sum_{n=1}^{\infty} k_1^{2n} \nu^{2n} \left\{ \frac{\partial^{2n}}{\partial x^{2n}} + \frac{\partial^{2n}}{2y^{2n}} \right\} \phi_{2n} + \nabla\nabla \sum_{n=2}^{\infty} (-1)^n$$

$$\cdot \delta_3^n \frac{\partial^{2(n-1)}}{\partial z^{2(n-1)}} \phi_n + k_1^2 \left\{ \underline{\underline{\delta}}^2 \cdot \nabla\nabla + \underline{\underline{\delta}} \cdot \nabla\nabla \cdot \underline{\underline{\delta}} + \nabla\nabla \cdot \underline{\underline{\delta}}^2 \right\} \phi_2 . \tag{50}$$

The summations in (50) can be calculated as in the previous section except for the following two terms:

$$\sum_{n=0}^{\infty} (-1)^n k_1^{2n} \underline{\underline{\delta}}^n \phi_n =$$

$$\sum_{n=0}^{\infty} \begin{pmatrix} (k_1^2 \nu)^{2n} \dfrac{\partial^{2n}}{2n!\,\partial p^{2n}} & -j(k_1^2 \nu)^{2n+1} \dfrac{\partial^{2n+1}}{(2n+1)!\,\partial p^{2n+1}} & 0 \\ j(k_1^2 \nu)^{2n+1} \dfrac{\partial^{2n+1}}{(2n+1)!\,\partial p^{2n+1}} & (k_1^2 \nu)^{2n} \dfrac{\partial^{2n}}{2n!\,\partial p^{2n}} & 0 \\ 0 & 0 & k_1^{2n} \delta_3^n \dfrac{\partial^n}{n!\,\partial p^n} \end{pmatrix} \phi_0(k_1^2, I, \bar{r}/\bar{r}_0)$$

$$= \frac{1}{2} \underline{\underline{S}}\, \phi_0[k_1^2(1+\nu), I; \bar{r}/\bar{r}_0] + \frac{1}{2} \underline{\underline{S}}^* \phi_0[k_1^2(1-\nu), I; \bar{r}/\bar{r}_0] + \phi_0[k_1^2(1+\delta_3), I; \bar{r}/\bar{r}_0]\, \hat{z}\hat{z}, \tag{51}$$

where $\underline{\underline{S}}$ is defined as

$$\underline{\underline{S}} = \begin{bmatrix} 1 & j & 0 \\ -j & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \qquad (52)$$

And, using the result of Appendix C, we obtain

$$\sum_{n=1}^{\infty} k_1^{2n} \nu^{2n} \frac{\partial^{2n}}{\partial x^{2n}} \phi_{2n} = \sum_{n=1}^{\infty} (-1)^n \frac{k_1^{2n} \nu^{2n}}{(2\pi)^3} \int_{\nu_k} \frac{e^{-j\vec{k}\cdot(\bar{r}-\bar{r}_0)} k_x^2 dV_k}{[k_1^2 - |\bar{k}|^2]^{2n+1}}$$

$$= \sum_{n=1}^{\infty} (-j)^{2n} \frac{k_1^{2n} \nu^{2n}}{(2\pi)^3 \, 2n!} \frac{\partial^{2n}}{\partial b^{2n}} \int_{V_k} \frac{e^{-j\vec{k}\cdot(\bar{r}-\bar{r}_0)} dV_k}{[k_1^2 - |\bar{k}|^2 + bk_x]_{b=0}}$$

$$= \phi_0[k_1^2(1 - \frac{\nu^2}{4}), I; \bar{r}/\bar{r}_0] \cosh\left[\frac{k_1 \nu}{2}(x - x_0)\right] - \phi_0(k_1^2, I; \bar{r}/\bar{r}_0) . \qquad (53)$$

The final form for $G_{II}(\bar{r}/\bar{r}_0)$ in gyrotropic media is

$$G_{II}(\bar{r}/\bar{r}_0) = \frac{1}{2} \underline{\underline{S}} \, \phi_0[k_1^2(1+\nu), I; \bar{r}/\bar{r}_0] + \frac{1}{2} \underline{\underline{S}}^* \phi_0[k_1^2(1-\nu), I; \bar{r}/\bar{r}_0]$$

$$+ \phi_0[k_1^2(1+\delta_3), I; \bar{r}/\bar{r}_0] \hat{z}\hat{z} - [\nabla\nabla \cdot \underline{\underline{\delta}} + \underline{\underline{\delta}} \cdot \nabla\nabla] \frac{1}{\delta_3} \int_0^{\delta_3} \phi_1[k_1^2, I + t\,\hat{z}\hat{z}; \bar{r}/\bar{r}_0] \, dt$$

$$+ \frac{\nabla\nabla}{k_1^2} \phi_0[k_1^2, I + \delta_3 \hat{z}\hat{z}; \bar{r}/\bar{r}_0] + \phi_0[k_1^2(1 - \frac{\nu^2}{4}), I; \bar{r}/\bar{r}_0][\cosh \frac{k_1 \nu}{2}(x - x_0)$$

$$+ \cosh \frac{k_1 \nu}{2}(y - y_0)] - k_1^2 \int_0^{\delta_3} \phi_0(k_1^2, I + t\,\hat{z}\hat{z} : \bar{r}/\bar{r}_0) \, dt$$

(cont'd)

$$- 2\phi_0(k_1, I; \bar{r}/\bar{r}_0) - k_1^2 \delta_3 \phi_1(k_1^2, I; \bar{r}/\bar{r}_0)\}$$

$$+ k_1^2 \; \underline{\underline{\delta}}^2 \cdot \nabla\nabla + \underline{\underline{\delta}} \cdot \nabla\nabla \cdot \underline{\underline{\delta}} + \nabla\nabla \cdot \underline{\underline{\delta}}^2 \quad \phi_2(k_1^2, I; \bar{r}/\bar{r}_0) . \tag{54}$$

6. DISCUSSION

The perturbation method may be started from different zeroth order solutions other than those presented in this paper. In the biaxial problem, for example, using $k_a \left(= k_0 \dfrac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3}\right)$ in place of k_1, we can obviously obtain better second order solution. We chose k_1 because it gives a simpler result.

The solution of the uniaxial problem is exact. We may use it to obtain certain qualitative information about the perturbation method, which is difficult to obtain for the solutions concerning biaxial and gyrotropic media. The accuracy of the second order solutions, for example, is easily obtained for the uniaxial solution using Taylor's theorem. We may use that information to estimate the accuracy of the approximate solutions for biaxial and gyrotropic problems. Another use for the uniaxial solution is to determine the branch for the fractional power terms. In general, the branch is determined by the radiation condition at infinity. The second order solutions are valid at short or intermediate distances from the sources, hence the radiation condition can not be applied to it. However, in the limit of infinite dc magnetic field, the $\underline{\underline{\epsilon}}$ of a magnetoionic medium approaches that of a uniaxial medium. Thus the rules which govern the choice of the branch of fractional powers in the uniaxial problems should apply to the problem in gyrotropic media. Consider the following term of the uniaxial solution:

$$e^{-jk_0 \epsilon_1^{1/2} (1+\delta_3)^{1/2} \left[(x - x_0)^2 + (y - y_0)^2 + \frac{(z - z_0)^2}{1+\delta_3} \right]^{1/2}}$$

$$= e^{-jk_0 |\bar{r} - \bar{r}_0| \{\epsilon_1 [1 + \delta_3 \sin^2\theta]\}^{1/2}} , \tag{55}$$

which is valid for all real θ and $|\bar{r} - \bar{r}_0|$. When ϵ_1 and δ_3, or both are negative, the term inside the braces may be negative. If so, the radiation condition requires that

$$\{\epsilon_1 [1 + \delta_3 \sin^2\theta]\}^{1/2} = -j |\epsilon_1 [1 + \delta_3 \sin^2\theta]|^{1/2} . \tag{56}$$

This rule should be applied to the second order solution of the gyrotropic problem near the plasma resonance.

APPENDIX A

The function $\phi_n(p, \underline{\underline{a}}; \bar{r}/\bar{r}_0)$

$$\phi_n(p, \underline{\underline{a}}; \bar{r}/\bar{r}_0) = \frac{1}{(2\pi)^3} \int_{V_k} \frac{e^{-j\vec{k}\cdot(\bar{r}-\bar{r}_0)}\, dV_k}{(p - \vec{k}\cdot \underline{\underline{a}} \cdot \vec{k})^{n+1}}$$

$$= \frac{(-1)^n}{(2\pi)^3 n!} \frac{\partial^n}{\partial p^n} \int_{V_k} \frac{e^{-j\vec{k}\cdot(\bar{r}-\bar{r}_0)}\, dV_k}{(p - \vec{k}\cdot \underline{\underline{a}} \cdot \vec{k})}$$

$$= \frac{(-1)^n}{(2\pi)^3 n!} \frac{\partial^n}{\partial p^n} \iiint_{-\infty}^{\infty} \frac{e^{-j[k_x(x-x_0)+k_y(y-y_0)+k_z(z-z_0)]}\, dk_x dk_y dk_z}{[p - a_x k_x^2 - a_y k_y^2 - a_z k_z^2]}$$

Let $k_x^1 = a_x^{1/2} k_x$, $k_y^1 = a_y^{1/2} k_y$, $k_y^1 = a_z^{1/2} k_z$, we get

$$\phi_n(p, \underline{\underline{a}}; \bar{r}/\bar{r}_0) = \frac{(-1)^n}{(2\pi)^3 n!\, (a_x a_y a_z)^{1/2}} \frac{\partial^n}{\partial p^n}$$

$$\iiint_{-\infty}^{\infty} \frac{e^{-j\left[k_x' \frac{(x-x_0)}{a_x^{1/2}} + k_y' \frac{(y-y_0)}{a_y^{1/2}} + k_z' \frac{(z-z_0)}{a_z^{1/2}}\right]} \frac{dk_z'}{dk_x' dk_y'}}{\left[p - k_x'^2 - k_y'^2 - k_z'^2\right]}$$

(cont'd)

$$= \frac{(-1)^{n+1}}{n!\,(a_n a_y a_z)^{1/2}\,4\pi} \frac{\partial^n}{\partial p^n} \frac{e^{-jp^{1/2}\left[\frac{(x-x_0)^2}{a_x} + \frac{(y-y_0)^2}{a_y} + \frac{(z-z_0)^2}{a_z}\right]^{1/2}}}{\left[\frac{(x-x_0)^2}{a_x} + \frac{(y-y_0)^2}{a_y} + \frac{(z-z_0)^2}{a_z}\right]^{1/2}}$$

$$= (-1)^n \frac{1}{n!} \frac{\partial^n}{\partial p_n} \phi_0(p, \underline{\underline{a}}; \bar{r}/\bar{r}_0) \,.$$

APPENDIX B

The function $\nabla_1^{2n} \phi_n(p, I; \bar{r}/\bar{r}_0)$

$$\nabla_1^{2n}\phi_n(p, I; \bar{r}/\bar{r}_0) = \left(\delta_2 \frac{\partial^2}{\partial y^2} + \delta_3 \frac{\partial^2}{\partial z^2}\right)^n \frac{1}{(2\pi)^3} \int_{V_k} \frac{e^{-j\vec{k}\cdot(\bar{r}-\bar{r}_0)}\,dV_k}{(p - \vec{k}\cdot I\cdot \vec{k})^{n+1}}$$

$$= \frac{1}{(2\pi)^3} \int_{V_k} \frac{(-1)^n\,(\delta_2 k_y^2 + \delta_3 k_z^2)^n\, e^{-j\vec{k}\cdot(\bar{r}-\bar{r}_0)}\,dV_k}{(p - \vec{k}\cdot I\cdot \vec{k})^{n+1}}$$

$$= \frac{(-1)^n}{(2\pi)^3\, n!} \left[\delta_2 \frac{\partial}{\partial a_y} + \delta_3 \frac{\partial}{\partial a_z}\right]^n \int_{V_k} \frac{e^{-j\vec{k}\cdot(\bar{r}-\bar{r}_0)}\,dV_k}{(p - \vec{k}\cdot \underline{\underline{a}}\cdot \vec{k})_{\underline{\underline{a}}=I}}$$

$$= \frac{(-1)^n}{n!} \left(\delta_2 \frac{\partial}{\partial a_y} + \delta_3 \frac{\partial}{\partial a_z}\right)^n \phi_0[p, I; \bar{r}/\bar{r}_0] \,.$$

APPENDIX C

The function $k_1^{2n} \nu^{2n} \frac{\partial^{2n}}{\partial x^{2n}} \phi_{2n}(p, I; \bar{r}/\bar{r}_0)$

$$k_1^{2n} \nu^{2n} \frac{\partial^{2n}}{\partial x^{2n}} \phi_{2n} = (-1)^n \frac{k_1^{2n} \nu^{2n}}{(2\pi)^3} \int_{V_k} \frac{e^{-j\vec{k} \cdot (\bar{r}-\bar{r}_0)} k_x^2 \, dV_k}{[k_1^2 - |\vec{k}|^2]^{2n+1}}$$

$$= (-j)^{2n} \frac{k_1^{2n} \nu^{2n}}{(2\pi)^3 \, 2n!} \frac{\partial^{2n}}{\partial b^{2n}} \int_{V_k} \frac{e^{-j\vec{k} \cdot (\bar{r}-\bar{r}_0)} \, dV_k}{[k_1^2 - |\vec{k}|^2 + bk_x]_{b=0}} \quad .$$

Let $k_x = k_x' + \frac{b}{2}$, then

$$\frac{(-j)^{2n} k_1^{2n} \nu^{2n}}{(2\pi)^3 \, 2n!} \frac{\partial^{2n}}{\partial b^{2n}}$$

$$\int_{V_k} \frac{e^{-j[k_x'(x-x_0) + k_y(y-y_0) + k_z(z-z_0)]} \, e^{-j\frac{b}{2}(x-x_0)}}{[k_1^2 + \frac{b^2}{4} - k_x'^2 - k_y^2 - k_z^2]_{b=0}} \, dk_x' \, dk_y \, dk_z$$

$$= \frac{(-j)^{2n} k_1^{2n} \nu^{2n}}{2n!} \frac{\partial^{2n}}{\partial b^{2n}} \left\{ \phi_0 \, (k_1^2 + \frac{b^2}{4}), \, I, \, \bar{r}/\bar{r}_0 \quad e^{-j\frac{b}{2}(x-x_0)} \right\}_{b=0} .$$

REFERENCES

1. F. V. Bunkin, (1957), On radiation in anisotropic media, J. Esp. Theoret. Phys. (U. S. S. R.) 32, pp. 338-346.

2. P. C. Clemmow, (1962), On the theory of radiation from a source in a magnetoionic medium, Electromagnetic Theory and Antennas, Proceedings of Symposium held at Copenhagen, Denmark, Part I, pp. 461-475.

3. M. S. Lighthill, Introduction to Fourier Analysis and Generalized Functions, Cambridge University Press, 1958.

4. K. K. Mei and J. G. Van Bladel, (1963), Scattering by perfectly-conducting rectangular cylinders, IEEE Trans. on Antennas and Propagation, Vol. AP-11, pp. 185-192.

5. R. Mitra and G. A. Deschamps, (1962), Field solution for a dipole in an anisotropic medium, Electromagnetic Theory and Antennas, Proceedings of Symposium held at Copenhagen, Denmark, Part I, pp. 495-512.