A NONLINEAR DISCRETE SYSTEM EQUIVALENCE OF INTEGRAL PULSE FREQUENCY MODULATION SYSTEMS

by

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SUMMARY

In this paper a study of the effect of IPFM modulation on single input single output feedback control is attempted. For zero input such systems can be reduced to a nonlinear discrete system. The use of Lagrange stability concepts is used for the stability study of such systems.

A step-by-step procedure is devised for the construction of the state trajectories of the IPFM system. This has been applied to a second-order plant where it is shown that instability, asymptotic stability in the large, and asymptotic stability in the Lagrange sense, are exhibited by such systems. It is also shown that in IPFM systems, the periodic oscillation that exists depends on the initial state.

A critical study of the equivalence concepts of such systems are reviewed in this paper and the limitations of the method are pointed out. Further research in this area of feedback modulation is proposed and discussed.

INTRODUCTION

In recent years a few papers have appeared on Pulse Frequency Modulated (PFM) Feedback Systems.[1-6] Such systems appear in engineering applications such as in communications systems (because of noise-filtering properties), in attitude control of space vehicles (minimization of fuel usage for extended space missions), in adaptive control, and in incremental servos (step-up motors) where changes in speed can be affected by changes in pulse rate. Apart from engineering applications, such modulation schemes simulate the main electrical properties of the neuron receptors. Thus, information transmission through the neurons is better explainted by using such models.[7]

The behavior of PFM Feedback Systems is discrete in nature; however, it differs from the conventional sampled-data system in that the sampling rate is not fixed but dependent on a certain threshold. Thus the analysis of such a nonlinear operation is quite difficult. Some attempts have been made to use the describing function for the analysis and stability study of these systems. Such methods are approximate and sometimes one needs an exact measure of the error involved. In some cases, a Lyapunov function has been obtained for the stability study and in other researches a general form of Lyapunov functions has been proposed for certain discontinuous functions.[5]

In this paper a general modulation scheme (Functional Pulse Frequency Modulation, FPFM) is mathematically proposed and from such a form the authors will discuss in detail Integral Pulse Frequency Modulation (IPFM) Feedback Systems. The main contribution is to obtain any equivalent nonlinear discrete system. For such an equivalent system a formulation of the Lagrange stability criterion is obtained. Furthermore, a step-by-step procedure is formulated for the study of certain systems and certain conclusions are obtained. In the course of the discussions we point out the mathematical difficulties that have led some authors (including one of the present authors[3]) to resort to approximate methods. Though the analysis problem of IPFM systems is far from solved, an attempt is made here to clarify some of the problems involved and to point out the possible approaches to future work.

FUNCTIONAL PULSE FREQUENCY MODULATOR (FPFM)

A functional pulse-frequency modulator is defined as a system operating on continuous or piecewise continuous inputs and converting them into sequences of pulses with the following properties:

The shape of the pulse (magnitude and form) is determined a priori. The pulses are numbered by an index p (p integer ≥ 1). The pth pulse is fully characterized by its emission time t_p, its sign ϵ_p ($\epsilon_p = \pm 1$), and a given function P(t) describing its shape. Figure 1 is a block diagram of such a modulator and in Fig. 2a the output sequence of emission is shown for a certain input $x(t)$. If $P(t)$ is a function of time, it is assumed to have a bounded support, i.e., there exists a finite interval (α, β) with $\alpha \leq 0$, $\beta \geq 0$, such that $P(t)$ is identically equal to zero for all $t \notin (\alpha, \beta)$. It should be noted that $P(t)$ can also be a generalized function as $\delta(t)$. In many applications when the shape is of no importance, $P(t)$ is considered as $\delta(t)$. In this work such a case is considered. We now proceed with the characterization of the input-output relation.

The input x to the modulator denoted as $x(t)$ is assumed to be continuous or piecewise continuous for all $t \geq 0$ and equal to zero for all $t < 0$. Let S denote the class of such function. We further consider two, arbitrary, finite numbers θ and θ' and denote by $[x(t); \theta, \theta']$ a point in the space $S \times R \times R$ satisfying $\theta' \geq \theta \geq 0$. A function X is defined, which assigns to every point in this space a real number denoted by $I_x(\theta, \theta')$. The following assumptions are made on $I_x(\theta, \theta')$, as a result of the operation defined by X.

(a) $I_x(\theta, \theta') = I_x(\theta, t) + I_x(t, \theta')$, all $t \in (\theta, \theta')$,

(b) $I_x(\theta, \theta')$ is a continuous function with respect to θ' (when θ is fixed).

It may be noted from the above conditions that $I_x(\theta, \theta) = 0$. Finally, let T (threshold) and μ be two parameters (positive or equal to zero).

The four quantities $(I_x(0, t), P(t), T, \mu)$ fully characterize the input-output relations of the modulator. More precisely, let x(t) be the input signal to the modulator. The corresponding output of the modulator is:

$$m(t) = \mu \sum_{p \geq 1} \epsilon_p P(t - t_p) . \tag{1}$$

If we let $t_0 = 0$, then t_p and ϵ_p are determined in a recursive manner as follows:

$$t_p = \min \{t \mid I_x(t_{p-1}, t) | \} = T , \ t > t_{p-1} \tag{2}$$

$$\epsilon_p = \text{sign} \{I_x(t_{p-1}, t_p)\} . \tag{3}$$

Thus, if x(t) is given, then the modulator output m(t) is uniquely determined. The above definition of the modulator emphasizes its non-linear properties.

Finally, IPFM forms a particular class of FPFM for which the "decision function" $I_x(0, t)$ reduces to the integral of the input, i.e.,

$$I_x(0, t) = \int_0^t x(u)\, du . \tag{4}$$

REGULATOR-TYPE IPFM CONTROL SYSTEM

In this paper the IPFM control system shown in Fig. (3) is considered. Its input u(t) is considered to be zero for all negative t. The linear plant is assumed to be of the reciprocal type (i.e., no zeros in its transfer function) and characterized by the input-output relation of the form:

$$\sum_{k=0}^{k=N} a_k \, y^{(k)}(t) = k_0 m(t), \qquad k_0 \geq 0 \,, \tag{5}$$

where a_N can be considered unity.

If we let the normalized threshold and gain be, respectively, $T_r \triangleq \frac{T}{\lambda}$ and $k_r \triangleq k_0\mu$, then Eq. (5) can be written in a vector form as follows:

$$\dot{\underline{Y}} = A\,\underline{Y} + \underline{B}\,k_0\,m(t)\,, \tag{6}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ 0 & & & & & & \\ -a_0 & -a_1 & \cdot & \cdot & \cdot & \cdot & -a_{N-1} \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \underline{Y}(t) = \begin{bmatrix} y(t) \\ y^{(1)}(t) \\ \cdot \\ \cdot \\ \cdot \\ y^{(N-1)}(t) \end{bmatrix} \tag{7}$$

If t_0 is an arbitrary value of t, the solution of Eq. (6) for $t \geq t_0$ is

$$\underline{Y}(t) = e^{A(t-t_0)} \underline{Y}(t_0) + k_0 e^{At} \int_{t_0}^{t} e^{-A\tau} \underline{B}\, m(\tau)\, d\tau . \qquad (8)$$

If the degree of the minimal polynomial of A is denoted by M, then there exists a set of functions $\{\alpha_k(t)\}_{k=0, 1, \ldots, M}$ such that[8]

$$e^{At} = \sum_{k=0}^{M} \alpha_k(t) A^k , \qquad 0 \leq M \leq N - 1 . \qquad (9)$$

Further, if we let $t \in (t_p^+, t_{p+1}^-)$, then

$$m(t) = \epsilon_p \mu\, \delta(t - t_p) . \qquad (10)$$

Using Eqs. (9) and (10) in Eq. (8) and letting $t_0 = t_p^-$, we deduce

$$\underline{Y}(t) = e^{A(t-t_p)} \underline{Y}(t_p^-) + k_0 e^{At} \sum_{k=0}^{M} A^k \underline{B} \int_{t_p^-}^{t} \epsilon_p\, \mu \alpha_k(-\tau) \delta(\tau - t_p)\, d\tau . \qquad (11)$$

Since

$$\int_{t_p^-}^{t} \alpha_k(-\tau)\, \delta(\tau - t_p)\, d\tau = \alpha_k(-t_p) , \qquad (12)$$

and

$$k_0 \mu = k_r , \qquad (13)$$

then noting Eq. (9), we get

$$\underline{Y}(t) = e^{A(t-t_p)} [\underline{Y}(t_p^-) + \epsilon_p k_r \underline{B}] . \qquad (14)$$

If we let $\underline{Y}(t_p^-) \triangleq \underline{Y}_p^-$ and $\underline{Y}(t_p^+) = \underline{Y}_p^+$, then Eq. (14) reduces to:

$$\underline{Y}^+(p) = \underline{Y}_p^- + \epsilon_p k_r \underline{B} . \tag{15}$$

Setting $t = t_{p+1}^-$ in Eq. (14), one obtains the value of $\underline{Y}(t)$ at $t = t_{p+1}^-$ as a function of ϵ_p and t_{p+1}^- as follows:

$$\underline{Y}_{p+1}^- = e^{A(t_{p+1}^- - t_p)} [\underline{Y}_p^- + \epsilon_p k_r \underline{B}] . \tag{16}$$

The above expression is fully characterized if ϵ_p and $t_{p+1}^- - t_p$ are known. Equations (2) and (3) with $I_x(t_p, t) = \int_{t_p}^{t} x(\tau)\, d\tau$ enable us to determine t_{p+1} and ϵ_p as a function of the past history of $y(t)$. This is discussed in the following.

We denote a_k^T as the first row of A^k $(0 \leq k \leq M)$ and

$$\underline{Y}_p^+ \triangleq \begin{bmatrix} Y_{p,1}^+ \\ Y_{p,2}^+ \\ \cdot \\ \cdot \\ Y_{p,N}^+ \end{bmatrix} , \tag{17}$$

then Eq. (14) can be written as

$$\underline{Y}(t) = \sum_{k=0}^{M} \alpha_k(t - t_p) \left\langle \underline{a}_k , \underline{Y}_p^+ \right\rangle . \tag{18}$$

By noting the form of A from Eq. (7), we can reduce $\underline{a}_k$ to the vector:

$$\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow (k+1)^{th} \text{ row} \tag{19}$$

hence, Eq. (18) yields

$$\underline{Y}(t) = \sum_{k=0}^{k=M} \alpha_k(t - t_p) Y^+_{p,\,k+1} , \quad t \in (t_p^+ , t_{p+1}^-) . \tag{20}$$

Now, from Eq. (4) and Fig. (3), $I_x(t_p , t) = - \int_{t_p}^{t} \lambda \, Y(\tau) d\tau$ and $\frac{T}{\lambda} = T_r$, then t_{p+1} and ϵ_{p+1} are determined as follows (see Eqs. (2) and (3)):

$$t_{p+1} = \text{Min} \left\{ t \;\middle|\; t > t_p , \; - \int_{t_p}^{t} \sum_{k=0}^{M} \alpha_k(\tau - t_p) \, Y^+_{p,\,k+1} d\tau \right\} = \pm T_r , \tag{21}$$

$$\epsilon_{p+1} = - \text{sign} \left\{ \int_{t_p}^{t_{p+1}} \sum_{k=0}^{k=M} \alpha_k(\tau - t_p) \, Y^+_{p,\,k+1} \, d\tau \right\} . \tag{22}$$

If we define $b_k(t) = - \int_0^t \alpha_k(\tau)\, d\tau$, then

$$- \int_{t_p}^{t} \alpha_k(\tau - t_p)\, d\tau = - \int_0^{t-t_p} \alpha_k(u)\, du = b_k(t - t_p) \,. \tag{23}$$

Therefore, from Eqs. (22) and (23) we obtain

$$t_{p+1} = \text{Min}\left\{ t \mid t > t_p ,\ \left\langle \underline{b}(t - t_p),\ \underline{Y}_p^+ \right\rangle \right\} = \pm T_r \,, \tag{24}$$

$$\epsilon_{p+1} = \text{sign}\left\{ \left\langle \underline{b}(t_{p+1} - t_p),\ Y_p^+ \right\rangle \right\} \quad , \tag{25}$$

where

$$\underline{b} \triangleq \begin{bmatrix} b_0(t) \\ b_1(t) \\ \cdot \\ \cdot \\ b_M(t) \\ 0 \\ 0 \end{bmatrix} \tag{26}$$

The above relation yields a step-by-step procedure for determining the various sequences $\{\epsilon_p\}$ and $\{t_p\}$ or more simply the sequences of vectors $\underline{m}_p = \begin{bmatrix} \epsilon_p \\ t_p \end{bmatrix}$

If $\underline{Y}_0$ is the initial state of the linear plant at $t = 0$, t_1 and ϵ_1 are obtained from (24) and (25) as:

$$t_1 = \text{Min}\left\{ t \mid t > 0,\ \left\langle \underline{b}(t_1),\ \underline{Y}_0 \right\rangle \right\} = \pm T_r \,, \tag{27}$$

$$\epsilon_1 = \text{sign}\left\{ \left\langle \underline{b}(t_1),\ \underline{Y}_0 \right\rangle \right\}. \tag{28}$$

From Eq. (14), we determine $\underline{Y}_1^-$ as follows:

$$\underline{Y}_1^- = e^{At_1} \underline{Y}_0 . \tag{29}$$

From Eq. (15), we obtain $\underline{Y}_1^+$ as

$$\underline{Y}_1^+ = \underline{Y}_1^- + \epsilon_1 k_r \underline{B} . \tag{30}$$

Hence, the vectors $\underline{m}_2$, $\underline{m}_3$, . . . , $\underline{m}_p$ are determined by a similar procedure. The preceding results lead to the definition of the "Equivalent Discrete System." To illustrate the above procedure a second-order example is chosen as follows:

The linear plant is characterized by the following differential equation

$$y^{(2)} + 3cy^{(1)} + 2c^2 y = k_0 m(t) , \tag{31}$$

where c is arbitrary but different from zero.

The matrix "A" of the above system is given:

$$A = \begin{bmatrix} 0 & 1 \\ -2c^2 & -3c \end{bmatrix} \tag{32}$$

the eigenvalues are $\lambda_1 = -c$, and $\lambda_2 = 2c$, hence

$$e^{At} = \alpha_0(t)\, I + \alpha_1(t)\, A , \tag{33}$$

where

$$\left.\begin{aligned} \alpha_0(t) &= 2\, e^{-ct} - e^{-2ct} \\ \alpha_1(t) &= \frac{e^{-ct} - e^{-2ct}}{c} \end{aligned}\right\} \tag{34}$$

For this particular case, Eqs. (24) and (25) are given as:

$$t_{p+1} = \text{Min}\{t \mid t > t_p ,\ \langle \underline{d}(c) ,\ Y_p^+ \rangle\, e^{-c(t-t_p)} + \langle \underline{h}(c) ,\ Y_p^+ \rangle\, e^{-2c(t-t_p)} \} = \pm T_r , \tag{35}$$

$$\epsilon_{p+1} = \text{sign}\{ \langle \underline{d}(c) ,\ Y_p^+ \rangle\, e^{-c(t_{p+1}-t_p)} + \langle \underline{h}(c),\ Y_p^+ \rangle\, e^{-2c(t_{p+1}-t_p)} \}, \tag{36}$$

where $\underline{d}(c)$ and $\underline{h}(c)$ are vectors with two components. They are determined either directly from $\alpha_0(t)$ and $\alpha_1(t)$ or from the definition of the vector $\underline{b}(t)$. Also

$$\underline{Y}_p^+ = \begin{bmatrix} y(t_p^+) \\ y^{(1)}(t_p^+) \end{bmatrix} \tag{37}$$

Setting $e^{-c(t-p)} = \Delta_p$ in Eq. (35), we can determine t_{p+1} by solving the following second-degree equation.

$$\langle \underline{d}(c) ,\ \underline{Y}_p^+ \rangle\ \Delta_p + \langle \underline{h}(c) ,\ Y_p^+ \rangle\ \Delta_p^2 = \epsilon_{p+1}\, T_r . \tag{38}$$

A similar equation holds for ϵ_{p+1}. The above equation can be solved exactly for determining the smallest positive root which gives t_{p+1}.

For more general cases of linear plants, the exact solution of higher equations can be quite difficult. However, certain approximations can be made to reduce these higher equations into a second order. For instance, the components of the vector $\underline{b}(t - t_p)$ can be expanded into power series of $(t - t_p)$. By maintaining the constant terms and terms of $(t - t_p)$ and $(t - t_p)^2$, one can determine t_{p+1} and ϵ_{p+1} by solving a second-order equation. It should be noted that the larger the norm of Y_p^+, the better is the approximation.

EQUIVALENT DISCRETE SYSTEM

If $\underline{Y}_p^+$ is defined as $Y(t_p^+)$, then there exists a vector-valued function "F" such that

$$\underline{Y}_{p+1}^+ = F(\underline{Y}_p^+) . \tag{39}$$

<u>Proof</u>: From the definitions of t_{p+1} and ϵ_{p+1} as seen from Eqs. (21) and (22), we deduce that there exist scalar valued functions f and g such that

$$t_{p+1} - t_p = f(\underline{Y}_p^+) \text{ and } \epsilon_{p+1} = g(\underline{Y}_p^+) . \tag{40}$$

Using Eq. (15), we obtain

$$\underline{Y}^{+}_{p+1} = \underline{Y}^{-}_{p+1} + \epsilon_{p+1} k_r \underline{B} . \tag{41}$$

Therefore,

$$Y^{+}_{p+1} = \underline{Y}^{-}_{p+1} + k_r g(\underline{Y}^{+}_{p}) \underline{B} , \tag{42}$$

and from Eq. (16)

$$\underline{Y}^{+}_{p+1} = e^{A[f(Y^{+}_{p})]} (\underline{Y}^{+}_{p}) + k_r \, g(\underline{Y}^{+}_{p}) \underline{B} = \underline{F}(Y^{+}_{p}) . \tag{43}$$

The above equation proves the assertion that there exists a vector-valued function. This function represents the vector equation of a non-linear discrete system.

Definition: Given the IPFM control system considered in Fig. (3) with $u = 0$ and assuming $\underline{Y}_0$ to be the initial value of the state vector of the linear plant, the nonlinear discrete system

$$\underline{Y}^{+}_{p+1} = F(Y^{+}_{p}) , \quad \underline{Y}^{+}_{0} = \underline{Y}_0 \tag{44}$$

is called the "Equivalent Discrete System of the IPFM Control System."

Two properties of the equivalent discrete system are mentioned.

(a) The property $t_{p+1} = t_p + f(Y^{+}_{p})$, indicates that the modulator decides on the emission of a pulse at time t_{p+1} on the basis of the value of the state vector at t^{+}_{p}. Hence, the time interval between two consecutive pulses (p, p+1) varies with Y^{+}_{p} and thus self-adaptive control is exhibited.

(b) The discrete system describes completely the behavior of the output vector of the linear plant. In each interval (t_p^+, t_{p+1}^-), the following equation from Eq. (14) holds:

$$\underline{Y}(t) = e^{A(t-t_p)} \underline{Y}_p^+ . \tag{45}$$

STABILITY OF EQUIVALENT NONLINEAR DISCRETE SYSTEM[10]

In view of the fact that, in general, $\underline{Y}(t)$, for IPFM systems will not tend to $\underline{Y} = 0$ when $t \to \infty$, the concept of asymptotic stability in the Lagrange sense is introduced. The introduction of this stability concept is needed to encompass all the occurring cases.

<u>Definition of Stability</u>: The discrete system $\underline{Y}_{p+1}^+ = F(Y_p^+)$ is considered to be asymptotically stable in the Lagrange sense if there exist two bounded closed sets U and V with $(U \subset V)$ containing $\underline{Y} = 0$, such that the trajectory of this discrete system corresponding to any initial condition $\underline{Y}_0^+ = \underline{Y}_0$, where Y_0 is finite, will eventually enter the set U after a finite time and thereafter remain in V (see Fig. (4)). It should be noted that the definition implies the convergence of the solution to a set whose size is independent of the initial condition $\underline{Y}_0$.

The test of the asymptotic stability in the Lagrange sense for a zero input IPFM control system, based on the equivalent discrete system, can be effected by using the Lyapunov function as follows:

Let $\underline{Y}^{+}_{p+1} = F(\underline{Y}^{+}_{p})$ be the vector difference equation of the equivalent discrete system and $\|Y\|$ denote the Euclidean norm of $\underline{Y}$. Furthermore, let $\underline{Y}$, Y^{*}, $\tilde{Y}$, be the three points in the state space. Finally, for a given bounded set U let $\overline{U}$ designate the complement of this set.

Stability Theorem: If it is possible to find a continuous function W(Y), two bounded closed sets U and V and two positive constants δ and r such that:

(I) $W(\underline{Y}) = 0$, iff $\underline{Y} = 0$, (46a)

$W(\underline{Y}) \rightarrow +\infty$ when $\underline{Y} \rightarrow +\infty$, (46b)

$W(\underline{Y}) > 0$, $\forall$ $\|Y\| \neq 0$, (46c)

(II) $U \subset V, \underline{Y} = 0 \in U$, (47a)

$\|\tilde{Y} - Y^{*}\| > r$, $\forall$ $\underline{Y}^{*} \in \overline{U}$, $\forall$ $\tilde{Y} \in \overline{V}$, (47b)

$W(\underline{Y}^{*}) - W(\tilde{\underline{Y}}) < 0$, $\forall$ $\underline{Y}^{*} \in \overline{U} \cap V$, $\forall$ $\tilde{Y} \in \overline{V}$, (47c)

(III) $W(\underline{Y}^{+}_{p+1}) - W(Y^{+}_{p}) \leq -\delta$, $\forall$ $\underline{Y}^{+}_{p} \in \overline{U}$, (48a)

$\|\underline{Y}^{+}_{p+1} - \underline{Y}^{+}_{p}\| \leq r$, $\forall$ $\underline{Y}^{+}_{p} \in U$, (48b)

then the discrete system or equivalently the IPFM control system is asymptotically stable in the Lagrange sense.

Proof: Two different cases may occur:

(a) The initial state $\underline{Y}_{0}$ belongs to set U. For this case, if the trajectory of Eq. (44) remains indefinitely in the set U, then the above theorem is proved. This is shown as follows.

Assume that the trajectory remains in U for all p such that $p \leq q$ where q is an arbitrary finite number. Hence, this trajectory will eventually enter the set $\bar{U}$. However, from conditions (48b) and (47b), one can deduce that $\underline{Y}^{+}_{q+1}$ must be in $V \cap \bar{U}$. Furthermore, if condition (47c) and (48a) are used, it is seen that this trajectory cannot enter the set $\bar{V}$. Therefore the theorem is verified.

(b) The initial state $\underline{Y}_0$ belongs to the set $\bar{U}$.

Let W_0 (finite) denote the values of $W(Y_0)$. Assume that the trajectory of Eq. (44) corresponding to the initial condition $\underline{Y}_0$ remains in the set $\bar{U}$ for all values of p such that $0 < p \leq q'$ where q' is an arbitrary finite integer. Using condition (48a) recursively, we obtain

$$W(\underline{Y}^{+}_{q'}) \leq W_0 - q'\delta. \tag{49}$$

From the above equation it is seen that for q' sufficiently large, $W(\underline{Y}^{+}_{q'})$ will be negative. However, from assumption (46c), it is evident that this is impossible. Therefore, the previous assumption is not valid for $q' > \tilde{q}' = \frac{W_0}{\delta}$ and the trajectory must then enter the set U for some $p < \tilde{q}$.

Finally, if we consider the results of A, it is evident that the theorem is demonstrated for the case B .

STATE TRAJECTORIES OF THE EQUIVALENT DISCRETE SYSTEM

In this section we are concerned with the development of a step-by-step procedure for the graphical construction of the state trajectories of the linear plant. This will lead to an investigation of some of the properties of IPFM control systems. This development is based on the following:

Theorem:

Let

$$\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ Y_N \end{bmatrix} \tag{50}$$

be a point in the state space. Furthermore, let $\underline{Y}_p^+$ be given and assume the state trajectory C of the linear plant $\dot{\underline{Y}} = A\underline{Y}$, $(\underline{Y}_0 = \underline{Y}_p^+)$ to be oriented in the direction of increasing "t". Then, $\underline{Y}_{p+1}^-$ is the first intersection of C with the surface defined by:

$$\sum_{k=1}^{k=N} a_k(Y_k - Y_{p,k}^+) = (-1)^{n_{p+1}} \operatorname{sign}(Y_{n_{p+1}})\, a_0\, T_r , \tag{51}$$

where a_k and a_0 are the coefficients given in Eq. (5) and n_{p+1} is the smallest value of k, $1 < k \leq N$, such that the component $Y_{p+1,k}^-$ of Y_{p+1}^-, i.e., the solution of Eq. (51), is different from zero (see Eq. A-7 of Appendix).

Using this theorem with relations (15) and (A-1) of the appendix, i.e.,

$$\underline{Y}^{+}_{p+1} = \underline{Y}^{-}_{p+1} + \epsilon_{p+1} k_r B \,, \tag{52}$$

$$\epsilon_{p+1} = (-1)^{n_{p+1}} \operatorname{sign}(Y^{-}_{p+1,\; n_{p+1}}) \,, \tag{53}$$

we see that we have obtained a step-by-step procedure for finding the trajectories of the equivalent discrete system.

The proof of the above theorem can be obtained by first showing the following relation:

If $|A|$ is different from zero, then

$$\int_{t_p}^{t} \underline{Y}(\tau)\, d\tau = A^{-1}(e^{A(t-t_p)} - I)\, \underline{Y}^{+}_{p} \,. \tag{54}$$

The above equation can be shown as follows:

$$A^{-1}(e^{A(t-t_p)} - I) = 0 \,, \quad \text{for } t = t_p \,. \tag{55}$$

Furthermore,

$$A^{-1}(e^{A(t-t_p)} - I) = A^{-1} \left\{ \sum_{k=1}^{\infty} \frac{A^k}{k!} (t - t_p)^k \right\} = \sum_{k=1}^{\infty} \frac{A^{k-1}}{k!} (t - t_p)^k \tag{56}$$

and

$$\frac{d}{dt} \{ A^{-1}(e^{A(t-t_p)} - I)\} \; \underline{Y}^{+}_{p} = \frac{d}{dt} \left\{ \sum_{k=1}^{\infty} \frac{A^{k-1}}{k!} (t - t_p)^k \underline{Y}^{+}_{p} \right\}, \tag{57}$$

which is equal to

$$\sum_{k=1}^{\infty} \frac{A^{k-1}}{(k-1)!} \ (t - t_p)^{k-1} \underline{Y}_p^+ = e^{A(t-t_p)} \underline{Y}_p^+ . \tag{58}$$

Since $e^{A(t-t_p)} Y_p^+ = Y(t)$, the assertion in Eq. (54) is proved.

From Fig. 3, and the above relation, we obtain

$$I_e(t_p, t) = - \int_{t_p}^{t} \underline{Y}(\tau) \, d\tau = - v_1^* A^{-1} (e^{A(t-t_p)} - I) \underline{Y}_p^+ , \tag{59}$$

where v_1^* is the transpose of the N vector v_1 :

$$\underline{v}_1 \triangleq \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{60}$$

To prove the theorem, we consider the value t_{p+1} of the (p+1)th emission time. This value satisfies the following relation (see Eqs. (2) and (59)):

$$-v_1^* A^{-1} [e^{A(t_{p+1} - t_p)} - I] = \pm T_r = \epsilon_{p+1} \quad T_r . \tag{61}$$

From the definition of t_{p+1} and ϵ_{p+1}, and using Eq. (54), we obtain

$$\int_{t_p}^{t_{p+1}} \underline{Y}(\tau)\, d\tau = \int_{t_p}^{t_{p+1}} \begin{bmatrix} y(\tau) \\ \tilde{\underline{Y}}(\tau) \end{bmatrix} d\tau = A^{-1}[e^{A(t_{p+1}-t_p)} - I]\, \underline{Y}_p^+ , \tag{62}$$

where

$$\tilde{\underline{Y}}(t) \triangleq \begin{bmatrix} Y_2(t) \\ \cdot \\ \cdot \\ \cdot \\ Y_N(t) \end{bmatrix} = \begin{bmatrix} Y^{(1)}(t) \\ \cdot \\ \cdot \\ \cdot \\ Y^{(N-1)}(t) \end{bmatrix} \tag{63}$$

The above is obtained from $\underline{Y}(t)$ by deleting its first component. Furthermore, if we let

$$\tilde{\underline{T}} \triangleq \int_{t_p}^{t_{p+1}} \tilde{\underline{Y}}(\tau)\, d\tau \triangleq \begin{bmatrix} T_2 \\ \cdot \\ \cdot \\ T_N \end{bmatrix} , \tag{64}$$

then by using Eq. (61), relation (62) can be reduced to:

$$\int_{t_p}^{t_{p+1}} \underline{Y}(\tau)\, d\tau = - \begin{bmatrix} \epsilon_{p+1} T_r \\ \tilde{T} \end{bmatrix} = A^{-1}[e^{A(t_{p+1}-t_p)} - I]\, Y_p^+ . \tag{65}$$

Since $Y_{p+1}^- = e^{A(t_{p+1}^- - t_p)}\, Y_p^+$, Eq. (65) yields

$$-A^{-1}(Y^-_{p+1} - Y^+_p) = \begin{bmatrix} \epsilon_{p+1} T_r \\ \tilde{T} \end{bmatrix}, \tag{66}$$

or

$$Y^-_{p+1} - Y^+_p = -A \begin{bmatrix} \epsilon_{p+1} T_r \\ T_2 \\ \cdot \\ \cdot \\ T_N \end{bmatrix} \tag{67}$$

It should be noted that the solution of the above equation yields the value Y^-_{p+1} at $t = t^-_{p+1}$. Using the form of A given in Eq. (7), we obtain

$$\left.\begin{aligned} Y^-_{p+1,1} &= Y^+_{p,1} - T_2 \\ Y^-_{p+1} &= Y^+_{p,2} - T_3 \\ &\vdots \\ Y^{-1}_{p+1,N-1} &= Y^+_{p,N-1} - T_N \end{aligned}\right\} \tag{68}$$

$$Y^-_{p+1,N} = Y^+_{p,N} + a_0 \epsilon_{p+1} T_r + a_1 T_2 + \ldots + a_{N-1} T_N. \tag{69}$$

Replacing the values of T_2, $T_3 \ldots T_N$ of Eq. (68) into (69), and noting that (see the appendix and note Fig. 3),

$$\epsilon_{p+1} = (-1)^{n_{p+1}} \operatorname{sign}(Y^-_{p+1, n_{p+1}}), \tag{70}$$

we have proved the above theorem.

Finally, it may be noted that for all $\underline{Y}$ which do not have their first component equal to zero, the surface described by Eq. (51) reduces to two half-hyperplanes H_1 and H_2 as follows:

$$H_1 \rightarrow \sum_{k=1}^{k=N} a_k (Y_k - Y_{p,k}^+) = -a_0 T_r , \quad Y_1 > 0 , \tag{71}$$

$$H_2 \rightarrow \sum_{k=1}^{k=-N} a_k (Y_k - Y_{p,k}^+) = a_0 T_r , \quad Y_1 < 0 . \tag{72}$$

Examples:

To illustrate the application of the above procedure, assume that the linear plant is characterized by

$$y^{(2)} + a_1 y^{(1)} + a_0 y = k_0 m(t) . \tag{73}$$

For the system shown in Fig. 3, the construction of the state trajectory of Y(t) follows.

For $Y_1 > 0$ and $Y_1 < 0$, H_1 and H_2 from Eq. (71) and (72) reduce respectively to

$$a_1(Y_1 - Y_{p,1}^+) + (Y_2 - Y_{p,1}^+) = -a_0 T_r , \tag{74}$$

$$a_1(Y_1 - Y_{p,1}^+) + (Y_2 - Y_{p,2}^+) = a_0 T_r . \tag{75}$$

The above equations are of parallel straight lines with slopes of $-a_1$. They intersect, in the phase plane Y_1, Y_2, the line $G(Y_1 = Y^+_{p,1})$ at $Y^+_{p,2} - a_0 T_r$ and $Y^+_{p,1} + a_0 T_r$, respectively. If $\underline{Y}(0) = \underline{Y}^+_p$ is known, the point $\underline{Y}^-_{p+1}$ is easily obtained as shown in Fig. 5.

From $\underline{Y}_0$, we obtain $\underline{Y}^-_1$ by applying theorem (51). We obtain Y^+_1 from Y^-_1 by using relation (15) with

$$\epsilon_1 = (-1)^{n_1+1} \operatorname{sign}(Y^-_{1,n_1}) \text{ (see appendix and Fig. 3).}$$

Similarly, we obtain $\underline{Y}^+_2$, Y^+_3, . . . Y^+_p, and thus a step-by-step procedure is easily developed.

Three different cases of a second-order linear plant are given for the application of the method.

(a) $y^{(2)} + y = k_0 m(t)$

with $k_0 = 1$, $\lambda = 1$, $T = 1$, $\mu = 1$, $k_r = 1$, $T_r = 1$.

The linear plant is unstable, and from Fig. 6 the zero input IPFM control system is also unstable.

(b) $y^{(2)} + 3y^{(1)} + 2y = k_0 m(t)$

with $k_0 = 4$, $\lambda = 1$, $T = 1$, $\mu = 1$, $k_r = 4$, $T_r = 1$.

The linear is strictly stable, and from Fig. 7 the zero input IPFM control system is asymptotically stable in the large.

(c) $y^{(2)} + 0.5y^{(1)} + y = k_0 m(t)$

with $k_0 = 4$, $\lambda = 1$, $T = 1$, $\mu = 1$ $k_r = 4$, $T_r = 1$.

The linear system is stable, and from Fig. 8 the zero input IPFM control system is asymptotically stable in the Lagrange sense.

It should be noted from these examples that when periodic oscillations are obtained, their characteristics depend upon the initial state. In the case of a first-order system (which has been investigated thoroughly but, for brevity consideration, is not presented here), the relation of the characteristics of eventual periodic oscillations to the initial state of the plant may be obtained exactly. This emphasizes the main aspect of the oscillations in IPFM systems.

CONCLUSIONS

The concept of a nonlinear equivalent discrete system has been introduced and investigated. The results obtained in the case of a plant with no zeros in its transfer function can be extended to more general plants. It leads to the only approach for describing the local behavior of the regulator-type IPFM control system. The results that can be achieved by the equivalence are the exact analytical solution of the first-order case and the geometrical construction of the trajectories in higher order systems. In view of the difficulty of analyzing IPFM systems, these results are important. It is of interest to note that more general results on the global behavior of IPFM control have also been obtained. They are not discussed here but will be presented in future work.

It is clearly seen that there are practical limits to the equivalent method presented in this paper. In general, these are due to the difficulty in obtaining a closed-form expression for the equivalent discrete system as well as a suitable candidate for the Lyapunov function. However, by replacing the modulator by an equivalent gain, as can be done in certain cases, we can obtain candidates for the Lyapunov function more easily. It should be noted that the computer can also be adapted to test the different conditions of Lagrange stability.

Finally, the graphic method presented in this paper is exact in its concept, and thus offers a way of checking earlier results which were based on approximate methods such as the describing function. This is needed when the previous methods fail or more accurate results are sought.

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APPENDIX

Relation Between the Sign of the Emission Sequence $\{\epsilon_p\}$ and the Input (or its Derivative to the Modulator at Time $\{t_p\}$

In this appendix the following relation, which is used in the text, will be derived:

$$\epsilon_{p+1} = (-1)^{(n_{p+1}-1)} \operatorname{sign}(x^{-}_{p+1, n_{p+1}}). \qquad \text{(A-1)}$$

We assume that x(t) is infinitely differentiable, has bounded derivatives for all value of t in the interval (t_p, t_{p+1}), and has a finite number of zeros in each finite interval, i.e.,

$$|x^{(n)}(t)| \leq M_n, \qquad \forall t \notin S_t(T, x(t)), \ \forall_n, \qquad \text{(A-2)}$$

where

$S_t(T, x(t))$ represents the emission time sequences.

Lemma A.1: There exists a positive number α satisfying $0 < \alpha \leq t_{p+1} - t_p$, such that for all t belonging to the interval $(t_{p+1} - \alpha, t_{p+1})$, x(t) has the sign of ϵ_{p+1}.

The above will be proved by contradiction. Assume that no such α exists (assumption a).

Let $\{\theta_k\}$ be the sequence of zeros of x(t) in the interval $[t_p, t_{p+1}]$ and $\theta_{max} = \operatorname{Max}\{\theta_k | \theta_k \in \{\theta_k\}, t_{p+1} - \theta_k > 0\}$. In the interval (θ_{max}, t_{p+1}), $x(t)$ has a constant sign and by assumption (a), has the sign $-\epsilon_{p+1}$. Therefore, $\int_{\theta_{max}}^{t} x(\tau)\, d\tau$ has the sign of $-\epsilon_{p+1}$ in the same interval. Without loss of generality, assume that $\epsilon_{p+1} = 1$. The input-output relation of the modulator, $\int_{t_p}^{t} x(\tau)\, d\tau$, reaches the threshold T for the first time at $t = t_{p+1}$.

Hence,

$$\int_{t_p}^{\theta_{max}} x(\tau)\, d\tau < T, \tag{A-3}$$

and

$$\int_{t_p}^{t_{p+1}} x(\tau)\, d\tau = \int_{t_p}^{\theta_{max}} x(\tau)\, d\tau + \int_{\theta_{max}}^{t_{p+1}} x(\tau)\, d\tau. \tag{A-4}$$

The above is less than

$$T - (t_{p+1} - \theta_{max})\, x(\xi), \text{ where } \theta_{max} < \xi < t_{p+1}. \tag{A-5}$$

Therefore,

$$\int_{t_p}^{t_{p+1}} x(\tau)\, d\tau \leq T - \delta, \ \delta > 0. \tag{A-6}$$

The above inequality contradicts the fact that a positive pulse has been emitted at $t = t_{p+1}$, therefore, α can be chosen arbitrary in the interval $(0, t_{p+1} - \theta_{max}]$ and the lemma is proved.

Proceeding to prove Eq. (A-1), we denote $x^-_{p+1,n} = x^{(n-1)}(t^-_{p+1})$ and n_{p+1} is the smallest value of $n \geq 1$ such that $x^-_{p+1,n}$ is different from zero. More precisely, we have

$$\left.\begin{aligned} x^-_{p+1,n} &= 0, \qquad 1 \leq n < n_{p+1} \\ x^-_{p+1,n_{p+1}} &\neq 0, \quad \text{otherwise.} \end{aligned}\right\} \tag{A-7}$$

For any $t \in (t_{p+1} - \alpha, t_{p+1})$, a Taylor's expansion for $x(t)$ can be found.

$$x(t) = \sum_{k=1}^{k=M} \frac{(t - t^-_{p+1})^k}{k!} x^{(k)}(t^-_{p+1}) + \frac{(t - t^-_{p+1})^{n+1}}{(n-1)!} x^{(n+1)}[c_t], \quad c_t \in [t, t^-_{p+1}] . \tag{A-8}$$

Using Eq. (A-7), we reduce the above to

$$x(t) = \frac{(t - t^-_{p+1})^{(n_{p+1}-1)}}{(n_{p+1} - 1)!} x^-_{p+1,n_{p+1}} + \frac{(t - t^-_{p+1})^{(n_{p+1})}}{(n_{p+1})!} x^{(n_{p+1})}(c_t) . \tag{A-9}$$

If we let

$$\hat{x}(t) = \frac{x(t)}{\dfrac{(t - t^-_{p+1})^{(n_{p+1} - 1)}}{(n_{p+1} - 1)!}} \tag{A-10}$$

then use Eq. (A-9), we obtain

$$\hat{x}(t) = x^-_{p+1,\, n_{p+1}} + \frac{(t - t^-_{p+1})}{n_{p+1}}\, x^{(n_{p+1})}(c_t) \,. \qquad \text{(A-11)}$$

Since $x^{(n_{p+1})}(c_t)$ is bounded by $M_{n_{p+1}}$, the value of $t - t_{p+1}$ can be taken small enough so that $\tilde{x}(t)$ can have the sign of $x^-_{p+1,\, n_{p+1}}$ in Eq. (A-11).

Using the sign of the expression of $\tilde{x}(t)$ in Eq. (A-10), Lemma A.1, and the fact that $t - t^-_{p+1}$ is negative, we obtain

$$\epsilon_{\bar{p+1}} = (-1)^{(n_{p+1} - 1)} \operatorname{sign}(x^-_{p+1,\, n_{p+1}}) \qquad \text{(A-12)}$$

which proves the theorem.