A PERTURBATIONAL ANALYSIS OF NORTON-TYPE CONSTANT RESISTANCE NETWORKS

by

C. A. Desoer and K. K. Wong

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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Summary—It is shown that in order that a Norton high-pass low-pass complementary ladder network be constant resistance, all its reactive elements must be linear. The proof requires a novel perturbational analysis of nonlinear networks which gives precise bounds, in terms of the input, of the effect of the nonlinearities. This type of perturbational analysis can be considered to be a contribution to the identification problem.

1. INTRODUCTION

It has previously been shown [1] that every constant resistance network made of linear, passive, time-invariant elements can be modified so that its elements may become time-varying and yet it maintains its constant resistance property. It has also been shown [1] that, under certain conditions, the elements can become nonlinear and time-varying and yet the constant resistance property can be preserved. Many constant-resistance networks satisfy these conditions. However, Norton's high-pass low-pass complementary ladders furnish examples of networks which do not obey the conditions. Since these conditions are merely sufficient, it is still conceivable that some, or all, of the reactive elements can be nonlinear and yet the parallel ladders can maintain their constant-resistance property. The purpose of this paper is to prove that this is not possible. This result has been announced previously [2].

In order to obtain a valid proof, new and original techniques for probing nonlinear networks had to be developed. It is believed that our perturbational analysis is of far greater interest than the answer to the above problem. In particular, it may have important uses in identification theory; indeed, a careful study of the reasoning below will show that we solve an identification problem which may be formulated as follows: consider a high-pass low-pass complementary ladder-type constant-resistance network; suppose its topology is specified as well as the nature of its elements (i.e., whether they are resistors, inductors, or capacitors), and suppose that the network is zero-state equivalent to a one-ohm resistor, determine the characteristics of its elements. We show that if the elements are assumed to be time-invariant, they must necessarily be linear.

2. ANALYSIS

The easiest way to make our method clear is to interlace the general analysis with one specific example. In this way the notation is easy to understand. We pick as our example the Norton-type constant-resistance network shown in Fig. 1. With charges and fluxes as network variables, the state description of the network is:

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{\varphi}_4 \\ \dot{\varphi}_5 \\ \dot{\varphi}_6 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & -\frac{1}{2} & 0 & \frac{4}{3} & 0 & 0 \\ -\frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & -2 \\ -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} e , \qquad (1)$$

and the port description is:

$$i = -\frac{3}{2} q_1 - \frac{1}{2} q_2 + \frac{4}{3} \varphi_4 + \frac{2}{3} \varphi_5 + e . \qquad (2)$$

More generally, Eqs. (1) and (2) are of the form

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\,\underset{\sim}{x} + \underset{\sim}{b}\,e \tag{1'}$$

$$i = \langle \underset{\sim}{c}, \underset{\sim}{x} \rangle + e \tag{2'}$$

Suppose that the reactive elements are now nonlinear: more precisely, let their q - v and φ - i characteristics be

$$\begin{aligned} i_{C_1} &= \tfrac{3}{2} f_1(q_1) & i_{L_4} &= \tfrac{4}{3} f_4(\varphi_4) \\ i_{C_2} &= \tfrac{1}{2} f_2(q_2) & i_{L_5} &= \tfrac{2}{3} f_5(\varphi_5) \\ i_{C_3} &= \tfrac{3}{4} f_3(q_3) & i_{L_6} &= 2f_6(\varphi_6) \end{aligned} \quad . \tag{3}$$

We shall make the following assumptions on the characteristics for $j = 1, 2, \ldots, 6$:

F1. $f_j \in C^2$ (i.e., the second derivative of f_j exists and is continuous), with $f_j(0) = 0$,

F2. f_j is a strictly monotonically increasing function, and $0 < f_j'(\cdot) < \infty$,

F3. $f_j : R \xrightarrow{\text{onto}} R$.

Let us call the original linear network $\mathcal{N}$ and the modified network $\tilde{\mathcal{N}}$ (see Figs. 1 and 2). With the nonlinear characteristics specified in Eq. (3), the state and port equations can be shown to be of the form [1]

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\, \underset{\sim}{f}(\underset{\sim}{x}) + \underset{\sim}{b}\, e\ , \tag{4}$$

$$i = \langle \underset{\sim}{c},\ \underset{\sim}{f}(\underset{\sim}{x}) \rangle + e\ , \tag{5}$$

where $\underset{\sim}{A}$, $\underset{\sim}{b}$, and $\underset{\sim}{c}$ are the same as in Eqs. (1') and (2'), and $\underset{\sim}{f}(\underset{\sim}{x}) \triangleq \left(f_1(q_1),\ f_2(q_2),\ f_3(q_3),\ f_4(\varphi_4),\ f_5(\varphi_5),\ f_6(\varphi_6)\right)'$. For our perturbational analysis, we choose $e(t) = E + \tilde{e}(t)$, where E is an arbitrary real constant and $\tilde{e}(\cdot)$ is small, i.e., $\|\tilde{e}\| \triangleq \sup_t |\tilde{e}(t)|$ is a small number. In other words, E is a bias and $\tilde{e}$ is a small signal. Let $\underset{\sim}{x} = \underset{\sim}{X} + \tilde{\underset{\sim}{x}}$, with $\underset{\sim}{X}$ being a constant n-vector satisfying the equation

$$\underset{\sim}{A}\, \underset{\sim}{f}(\underset{\sim}{x}) = -\underset{\sim}{b}\, E\ , \tag{6}$$

be the zero-state response of Eq. (4) to the input $e = E + \tilde{e}$. The assumptions on $\underset{\sim}{f}$ imply that for any E, Eq. (6) has one and only one solution; indeed, as we shall see later, the linear network $\mathcal{N}$ is asymptotically stable, hence $\underset{\sim}{A}$ is nonsingular; furthermore, assumptions F2 and F3 imply that $\underset{\sim}{f}(\cdot)$ is a one-to-one mapping of R^n onto R^n. As a consequence of F1, Eq. (4) can be expressed as

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\, \underset{\sim}{f}(\underset{\sim}{X} + \tilde{\underset{\sim}{x}}) + \underset{\sim}{b}\, E + \underset{\sim}{b}\, \tilde{e}$$

$$= \underset{\sim}{A}\, \underset{\sim}{f}(\underset{\sim}{X}) + \underset{\sim}{A}\, \underset{\sim}{f}'(\underset{\sim}{X})\, \tilde{\underset{\sim}{x}} + \underset{\sim}{g}(\underset{\sim}{x}) + \underset{\sim}{b}\, E + \underset{\sim}{b}\, \tilde{e}\ ,$$

where $\underset{\sim}{g}(\tilde{\underset{\sim}{x}})$ depends on E and contains only the higher order terms in $\tilde{\underset{\sim}{x}}$. Using Eq. (6), we have

$$\dot{\tilde{\underset{\sim}{x}}} = \underset{\sim}{A}\, \underset{\sim}{f}'(\underset{\sim}{X})\, \tilde{\underset{\sim}{x}} + \underset{\sim}{g}(\tilde{\underset{\sim}{x}}) + \underset{\sim}{b}\, \tilde{e} \,. \tag{7}$$

Equation (7) is the perturbational equation about the constant <u>operating point</u> $\underset{\sim}{X} \triangleq (Q_1, Q_2, Q_3, \Phi_4, \Phi_5, \Phi_6)'$. We shall show that

(i) for all E, the operating point $\underset{\sim}{X}$ is a.s.i.l. (asymptotically stable in the large [3]),

(ii) the perturbational analysis is first used to show that all the elements of the low-pass ladder are linear, and

(iii) the constant resistance assumption implies that all reactive elements of both ladders are linear.

(i) <u>Operating Point Stability</u>. Let us show that the operating point is a.s.i.l. Once this is established it will follow that if $e(t) = E\, u(t)$ (where $u(\cdot)$ is the unit step), then for all E,

$$\lim_{t \to \infty} \underset{\sim}{x}(t) = \underset{\sim}{X} \,.$$

Given any E, the operating point $\underset{\sim}{X}$ is determined by Eq. (6); let us translate the coordinates of the characteristics according to

$$\hat{\underset{\sim}{x}} \triangleq \underset{\sim}{x} - \underset{\sim}{X}, \quad \hat{\underset{\sim}{f}}(\hat{\underset{\sim}{x}}) \triangleq \underset{\sim}{f}(\underset{\sim}{X} + \hat{\underset{\sim}{x}}) - \underset{\sim}{f}(\underset{\sim}{X}) \;;$$

hence, Eq. (4) (with $e(t) = E\, u(t)$) becomes

$$\dot{\hat{\underset{\sim}{x}}} = \underset{\sim}{A}\, \hat{\underset{\sim}{f}}(\hat{\underset{\sim}{x}}) . \tag{8}$$

The three assumptions on $\underset{\sim}{f}(\cdot)$ are also satisfied by $\hat{\underset{\sim}{f}}(\cdot)$. With q's and φ's as state variables, the matrix $\underset{\sim}{A}$ is expressed as the product of two square matrices

$$\underset{\sim}{A} = \underset{\sim}{H}\, \underset{\sim}{D} ,$$

where $-\underset{\sim}{H}$ is the hybrid resistive matrix, and $\underset{\sim}{D}$ is a diagonal matrix for the ladder networks considered here. In our example, $\underset{\sim}{D} = \text{diag.}\,(S_1, S_2, S_3, \Gamma_4, \Gamma_5, \Gamma_6) = \text{diag.}\left(\frac{3}{2}, \frac{1}{2}, \frac{3}{4}, \frac{4}{3}, \frac{2}{3}, 2\right)$. $\underset{\sim}{H}$ can be interpreted as the A-matrix of the network $\mathcal{N}$ with all reactive elements equal to unity. As a candidate for a Lyapunov function, we define $V(\cdot)$ as:

$$V(\hat{\underset{\sim}{x}}) = \sum_{i=1}^{6} d_i \int_0^{\hat{x}_i} \hat{f}_i(s)\, ds , \tag{9}$$

where d_i is the ith diagonal element of $\underset{\sim}{D}$. V is the energy stored in the network $\tilde{\mathcal{N}}$. We assert that V is a Lyapunov function for Eq. (8) and that $\hat{\underset{\sim}{x}} = \underset{\sim}{0}$, its only equilibrium point, is a.s.i.l. To prove it we observe that

1. (a) $V(\underline{0}) = 0$,

   (b) $V(\hat{\underset{\sim}{x}}) > 0$ for $\hat{\underset{\sim}{x}} \neq 0$ (by F1 and F2) ,

   (c) $V \to \infty$ as $\|\hat{\underset{\sim}{x}}\| \to \infty$ (by F3) ;

2. along any zero-input trajectory, the time derivative of V is given by

$$\dot{V}(\hat{\underset{\sim}{x}}) = \sum_{i=1}^{6} d_i \, \hat{f}_i(\hat{x}_i) \, \dot{\hat{x}}_i = \langle \hat{\underset{\sim}{f}}(\hat{\underset{\sim}{x}}), \, \underset{\sim}{D} \, \underset{\sim}{A} \, \hat{\underset{\sim}{f}}(\hat{\underset{\sim}{x}}) \rangle \; . \tag{10}$$

Note that $\hat{\underset{\sim}{x}} \neq \underset{\sim}{0} \Leftrightarrow \hat{\underset{\sim}{f}}(\hat{\underset{\sim}{x}}) \neq \underset{\sim}{0}$. It remains to show that V is negative definite. Consider the original linear network $\mathcal{N}$ and its zero-input response, i.e., the solution of

$$\dot{\underset{\sim}{y}} = \underset{\sim}{A} \, \underset{\sim}{y} \; . \tag{11}$$

The stored energy $\mathcal{E}_s(t)$ in $\mathcal{N}$ is given by

$$\mathcal{E}_s(t) = \frac{1}{2} \sum_{i=1}^{6} d_i \, y_i^2(t) = \frac{1}{2} \langle \underset{\sim}{y}(t), \, \underset{\sim}{D} \, \underset{\sim}{y}(t) \rangle \; . \tag{12}$$

The rate of decrease of the stored energy in $\mathcal{N}$ is, by Tellegen's theorem [4], the power dissipated in the resistors, and it is given by

$$\frac{d}{dt} \mathcal{E}_s(t) = \langle \underset{\sim}{y}(t), \, \underset{\sim}{D} \, \dot{\underset{\sim}{y}}(t) \rangle = \langle \underset{\sim}{y}(t), \, \underset{\sim}{D} \, \underset{\sim}{A} \, \underset{\sim}{y}(t) \rangle \; . \tag{13}$$

Thus, along any zero-input trajectory of Eq. (11)

$$\langle \underset{\sim}{y}(t), \underset{\sim}{D} \underset{\sim}{A} \underset{\sim}{y}(t) \rangle < 0 ,$$

except at isolated instants when it is zero. Since with $e(\cdot) = 0, \mathcal{N}$ is a hinged network consisting of two ladders with a common ground, we can consider the energy dissipation of each ladder independently. Since each ladder is a linear, passive, time-invariant, series-L shunt-C (or series-C shunt-L) ladder, it follows that:

(a) all modes are coupled to the terminating resistor, § and

(b) the current through the resistor is of the form

$$\sum_i p_i(t) e^{\lambda_i t} ,$$

where $p_i(\cdot)$ are polynomials.

Hence each resistor dissipates energy at a positive rate, except at some isolated instants when it is zero. Therefore, $\dot{V} < 0$ in Eq. (10) for almost all $\underset{\sim}{x} \neq \underset{\sim}{0}$. Thus V satisfies all the conditions required to establish a.s.i.l. We conclude then that any operating point, which is uniquely established by E, is a.s.i.l.

Remark. For such a series-L shunt-C ladder structure, if any one (or more) of the reactances has a nonmonotone characteristic, then the operating points on those parts of the characteristic with negative

§ This can be proved directly by writing the state equations and showing that with the resistor voltage as output, all states are observable.

slopes are (locally) unstable. For if one or more of the reactances were negative, there would be at least one exponentially growing mode as can be seen by the continued fraction expansion test for the characteristic polynomial of the corresponding A-matrix. This means that such an operating point cannot be physically established.

(ii) Perturbational Analysis. The first use of our perturbational analysis will be to show that more than half of all the reactive elements in $\tilde{\mathcal{N}}$ must in fact be linear if the constant resistance property is to be retained. Before the results of the perturbation can be used, we must first show that, under small signal inputs, the response of the actual nonlinear network is very close to that of the perturbational linear network. To do this we make use of the following

Lemma.* Consider the differential equation

$$\dot{x} = G x + g(x) + u , \tag{14}$$

where x, g, and u are n-vectors, and G is an $n \times n$ constant matrix, and also $g(0) = 0$. Assume that:

(a) every solution of $\dot{x} = G x$ approaches zero as $t \to \infty$,

(b) for any $\delta > 0$ there is an $\epsilon > 0$ such that $|x_1|, |x_2| \leq \delta$ implies that $|g(x_1) - g(x_2)| \leq \epsilon \, |x_1 - x_2|$,**

* The Lemma is proved in the Appendix.

** $|\cdot|$ is the Euclidean norm, and $\|x\| \triangleq \sup_t |x(t)|$.

(c) furthermore, if $\delta \to 0$, ϵ can be chosen so that $\epsilon \to 0$.

(In particular, conditions (b) and (c) will be satisfied if $\underset{\sim}{g}(\cdot)$ is of second order.)

Call $\underset{\sim}{x}(\cdot)$ the zero-state solution of Eq. (14) to the input $\underset{\sim}{u}$. Call $\underset{\sim}{x}_0(\cdot)$ the zero-state solution of

$$\dot{\underset{\sim}{x}}_0 = \underset{\sim}{G}\,\underset{\sim}{x}_0 + \underset{\sim}{u} \tag{15}$$

to the <u>same</u> input. Equation (15) is the linear approximation to Eq. (14) about the operating point $\underset{\sim}{x} = \underset{\sim}{0}$. Under these conditions, for sufficiently small $\|\underset{\sim}{u}\|$,

1. $$\frac{\|\underset{\sim}{x} - \underset{\sim}{x}_0\|}{\|\underset{\sim}{x}_0\|} \leq \frac{\epsilon M}{1 - \epsilon M}, \tag{16}$$

where $$M \triangleq \int_0^{\infty} |e^{-\underset{\sim}{G}t}|\, dt . \tag{17}$$

2. As $\|\underset{\sim}{u}\| \to o$, $\epsilon \to 0$.

This last conclusion is extremely important for our purposes. It says that by taking the "small signal" sufficiently small, we can make the ratio $\|\underset{\sim}{x} - \underset{\sim}{x}_0\| / \|\underset{\sim}{x}_0\|$ as small as we wish. Now $\|\underset{\sim}{x} - \underset{\sim}{x}_0\|$ is the peak value of the difference between the response of the nonlinear system (Eq. (14)) and that of the linear perturbational system (Eq. (15)); and $\|\underset{\sim}{x}_0\|$ is the peak value of the response of the linear perturbational system. This means that the relative error introduced by replacing $\underset{\sim}{x}$ by $\underset{\sim}{x}_0$ can be made arbitrarily small by taking $\|\underset{\sim}{u}\|$ sufficiently small. In particular, if all components of $\underset{\sim}{u}$ are sinusoidal and at the same frequency, then, as the amplitudes of these sinusoids go together to zero, the response of the nonlinear system is a curve whose peak deviation (from the sinusoidal response $\underset{\sim}{x}_0$) becomes arbitrarily small compared to the amplitude of the sinusoidal response $\underset{\sim}{x}_0$.

We now go back to the network $\tilde{\mathcal{N}}$. To help visualization let us write the state and port equations for $\tilde{\mathcal{N}}$:

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{\varphi}_4 \\ \dot{\varphi}_5 \\ \dot{\varphi}_6 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & -\frac{1}{2} & 0 & \frac{4}{3} & 0 & 0 \\ -\frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & -2 \\ -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} f_1(q_1) \\ f_2(q_2) \\ f_3(q_3) \\ f_4(\varphi_4) \\ f_5(\varphi_5) \\ f_6(\varphi_6) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} e \qquad (18)$$

$$i = \left(-\frac{3}{2}, -\frac{1}{2}, 0, \frac{4}{3}, \frac{2}{3}, 0\right) \begin{vmatrix} f_1(q_1) \\ f_2(q_2) \\ f_3(q_3) \\ f_4(\varphi_4) \\ f_5(\varphi_5) \\ f_6(\varphi_6) \end{vmatrix} + e \qquad (19)$$

About any operating point $\underset{\sim}{X}$, the perturbational equation takes the form of Eq. (7). Note that by assumption F2, $\underset{\sim}{f}'(\underset{\sim}{X})$ is a diagonal matrix with positive diagonal entries. It is easy to see that Eq. (7) is the state equation of a network having the same topology as $\mathcal{N}$, but each reactance of $\mathcal{N}$ is replaced by a nonlinear one whose characteristic is $f_i'(X)\tilde{x}_i$ plus a higher order nonlinear term, whose collective contribution to the state equations is in the term $\underset{\sim}{g}(\underset{\sim}{x})$. Let us now consider the linear network with the A-matrix $\underset{\sim}{\tilde{A}}$, where $\underset{\sim}{\tilde{A}} \triangleq \underset{\sim}{A}\,\underset{\sim}{f}'(\underset{\sim}{X})$. As already discussed in (i), all modes of such network are exponentially decaying; hence all eigenvalues of $\underset{\sim}{\tilde{A}}$ have negative real parts. It follows that there is a finite positive number M such that

$$\int_0^{\infty} |e^{\underset{\sim}{\tilde{A}}t}|\, dt = M < \infty . \qquad (20)$$

Condition (a) of the lemma is thus satisfied. Now since $\underset{\sim}{g} \in C^1$ (because $\underset{\sim}{f} \in C^2$), $\underset{\sim}{g}(\underset{\sim}{0}) = \underset{\sim}{0}$ and $\underset{\sim}{g}'(\underset{\sim}{0}) = \underset{\sim}{0}$, it follows that for δ positive and sufficiently small, $\|\tilde{x}_1\|$, $\|\tilde{x}_2\| \leq \delta$ implies that

$$g(\tilde{x}_1) = g(\tilde{x}_2) + g'(\tilde{x}_2)(\tilde{x}_1 - \tilde{x}_2) + o(\|\tilde{x}_1 - \tilde{x}_2\|) . \quad (21)$$

Thus $\|\tilde{x}_1\|$, $\|\tilde{x}_2\| \leq \delta$ implies that

$$\|g(\tilde{x}_1) - g(\tilde{x}_2)\| \leq \epsilon \|\tilde{x}_1 - \tilde{x}_2\| , \quad (22)$$

where the positive number ϵ may be taken as

$$\epsilon = 2 \max_{|x| \leq \delta} |g'(x)| . \quad (23)$$

Furthermore, the continuity of $g'(\cdot)$ and $g'(0) = 0$ guarantee that $\epsilon \to 0$ as $\delta \to 0$. Thus all the assumptions of the lemma are satisfied. Therefore for $\|\tilde{e}\| \triangleq \sup_t |\tilde{e}(t)|$ sufficiently small, the zero-state response of the nonlinear network described by Eq. (7) is very close to that of the linear network described by

$$\dot{z} = \tilde{A} z + b \tilde{e} . \quad (24)$$

More precisely, the lemma states that the relative error committed by replacing $\tilde{x}$ by z can be made arbitrarily small provided $\|\tilde{e}\|$ is taken sufficiently small. We now assert that <u>the nonlinear network described by Eq. (7) will be constant resistance only if the linear network $\mathcal{N}_p$ described by Eq. (24) is constant resistance</u>. Indeed, let us prove it by contradiction. Suppose $\mathcal{N}_P$ were not constant resistance; then for some input $\tilde{e}_1$, the input current would differ from $\tilde{e}_1$. Consider a large number N, and suppose that $\tilde{e}_1/N$ is applied to the nonlinear network (described by Eq. (7)) and to $\mathcal{N}_P$. However large N is, the difference between $\tilde{e}_1/N$ and the input current to $\mathcal{N}_P$ will exhibit a fixed relative error (with respect to N). Since as $N \to \infty$ the response of the nonlinear network tends to that of $\mathcal{N}_P$ with a vanishing relative error, then the input current of the nonlinear network cannot be equal to $\tilde{e}_1/N$. Hence the nonlinear network is not constant resistance. Therefore we have shown that if the nonlinear network (Eq. (7)) is constant resistance, then $\mathcal{N}_P$ (described by Eq. (24)) is also constant resistance. Let us recall that, in the linear case, the network is constant resistance if and only if a predetermined set of ratios is maintained among the values of all the reactive elements [1]. For our example $\mathcal{N}$, if $S_1 = 3k/2$ (with k being any real constant), then $\mathcal{N}$ is constant resistance if and only if $S_2 = k/2$, $S_3 = 3k/4$, $\Gamma_4 = 4k/3$, $\Gamma_5 = 2k/3$, $\Gamma_6 = 2k$. Thus for $\tilde{\mathcal{N}}$ to be constant resistance, the matrix $\underset{\sim}{f}'(\underset{\sim}{X})$ must be equal to the identity matrix times a constant k for all operating points $\underset{\sim}{X}$. For if $\tilde{\mathcal{N}}$ is constant resistance, the perturbation source $\tilde{e}(\cdot)$ must also see a constant resistance at the port. In short, at every operating point, the slopes of all the scalar functions f_j are the same. For $\tilde{\mathcal{N}}$, the operating point which, for any E, is given by Eq. (18), is:

$$\underset{\sim}{X} = \left(Q_1,\ Q_2,\ Q_3,\ \Phi_4,\ \Phi_5,\ \Phi_6\right)'$$

$$= \left(f_1^{-1}(2E/3),\ 0,\ f_3^{-1}(4E/3),\ 0,\ f_5^{-1}(3E/2),\ f_6^{-1}(E/2)\right)' . \qquad (25)$$

Equation (25) shows that the operating points of elements 1, 3, 5, and 6 vary with E, whereas those of elements 2 and 4 stay at the origin regardless of the value of E. (For convenience we refer to elements 2 and 4 by $\tilde{C}_2$ and $\tilde{L}_4$, respectively.) Such a condition always occurs because in high-pass ladder configurations the first series capacitor blocks the d-c path to the subsequent elements. Therefore, we conclude that all reactive elements, except possibly $\tilde{C}_2$ and $\tilde{L}_4$, must be linear, because for each one of them:

(a) at each operating point the slope of the characteristic must be equal to a constant (independent of E) times the slope of the q-v characteristic of $\tilde{C}_2$ at the origin, and

(b) each nonlinear characteristic goes through the origin.

It remains to be shown that $\tilde{C}_2$ and $\tilde{L}_4$ must also be linear.

(iii) Linearity of the High-Pass Ladder. In (ii) we have shown that the low-pass section can only contain linear reactive elements and that the first series capacitor of the high-pass ladder is linear. By using an induction step, we are going to show for the general case that all the remaining reactive elements are linear. Suppose that the first k elements are linear, we shall prove presently that the $(k+1)^{st}$ element is also linear. Refer to Fig. 3. The constant resistance condition implies that the relation between e and j_1 is specified by $Y_1(s)$, the input admittance of the ladder. Similarly, j_k is related to e by a transfer admittance $Y_k(s)$.

Consider the Norton's equivalent circuit of the one-port to the left of the terminals ⓚ, ⓚ' (see Fig. 4). Note that $Y_{eq}(0) = 0$. The source e (in Fig. 3) can be adjusted so that the current source of Fig. 4 is of the form $i = I + \tilde{i}$, where I is a constant and $\tilde{i}$ is small. Since $Y_{eq}(0) = 0$, the d-c current I must go through the nonlinear inductor, and hence the operating point of that inductor is (with notations as in Eqs. (3), (9) and (25)) $\Phi_k' = f_k^{-1}(I/d_k)$. Using the perturbational analysis and a previous reasoning, we conclude that the slope of the function $f_k^{-1}(I/d_k)$ is a constant (independent of I), hence the inductor is linear. Clearly, similar reasoning shows that the series capacitor C_{k+1} (with characteristic $d_{k+1} f_{k+1}(q_{k+1})$) is linear—take the Thévenin's equivalent circuit of the one-port to the left of terminals ⓛ, ⓛ' and adjust e (of Fig. 3) so that the Thévenin's equivalent source is $e_{eq} = E + \tilde{e}$; note that for the present case $Z_{eq}(0) = 0$, hence the nonlinear capacitor C_{k+1} must block the d-c voltage E, etc. This completes the induction step. Consequently the high-pass ladder must have all its elements linear.

This completes the proof that the constant resistance property of $\tilde{\mathcal{N}}$ implies that all its reactive elements are linear.

3. CONCLUSION

Previously [1, 2], we showed that, given a constant resistance network made of linear and time-invariant elements, it is always possible to have the reactive elements become time varying and yet preserve the constant resistance property. A general method in many cases allowed these elements to become nonlinear. This method could not be used on Norton-type constant resistance networks. In this paper it is shown that neither this method nor any other method could do so.

The contribution of this paper can also be considered from the point of view of an identification problem. Given the topology of the network and the nature of the elements, the perturbational method developed in this paper identifies every element of the network as a linear element. Of course, the reactive elements had the slope of their characteristic identified up to a common constant factor. This factor may be interpreted as a frequency normalization factor. In fact, the possibility of having all the reactive elements time varying can be interpreted as having the normalization factor becoming an arbitrary function of time. This identification point of view suggests a new problem: up to now all observations were assumed to be unaffected by noise so we might ask:

what tolerance can one guarantee upon the element characteristics given that the network is constant resistance, and that all measurements are performed in a noise background with a specified power spectrum?

APPENDIX

Proof of Lemma. We have to prove the following facts:

1. $\|\underset{\sim}{x}\|$ is small for $\|\underset{\sim}{u}\|$ sufficiently small, and

$$\|\underset{\sim}{x} - \underset{\sim}{x}_0\| \le \frac{\epsilon M}{1 - \epsilon M} \|\underset{\sim}{x}_0\| ,$$

2. As $\|\underset{\sim}{u}\| \to 0$, $\epsilon \to 0$.

The zero-state solution of Eq. (14) may be expressed as [5]

$$\underset{\sim}{x}(t) = \underset{\sim}{x}_0(t) + \int_0^t e^{\underset{\sim}{G}(t-t')} \underset{\sim}{g}(\underset{\sim}{x}(t'))\, dt' \tag{A1}$$

where $\underset{\sim}{x}_0(\cdot)$ is the zero-state solution of Eq. (15) and is given by

$$\underset{\sim}{x}_0(t) = \int_0^t e^{\underset{\sim}{G}(t-t')} \underset{\sim}{u}(t')\, dt' . \tag{A2}$$

By assumption (a), the positive number M defined in Eq. (17) is finite. From Eq. (A2) we have

$$\|\underset{\sim}{x}_0\| \le \int_0^t |e^{\underset{\sim}{G}(t-t')}|\ \|\underset{\sim}{u}\|\, dt' \le M \|\underset{\sim}{u}\| \tag{A3}$$

for all $t \ge 0$.

The solution of Eq. (A1) can be obtained by taking the limit of the converging iteration scheme

$$x_{n+1}(t) = x_0(t) + \int_0^t e^{G(t-t')} g(x_n(t'))\, dt' . \tag{A4}$$

Using x_0 as the first term of the approximating sequence, we get from Eq. (A4),

$$x_1(t) = x_0(t) + \int_0^t e^{G(t-t')} g(x_0(t'))\, dt' . \tag{A5}$$

(A3) shows that $\|x_0\|$ can be made as small as we wish by choosing $\|u\|$ sufficiently small; in fact, we may choose $\|u\|$ so small that

$$\|x_0\| \le M \|u\| < 2M \|u\| \le \delta \tag{A6}$$

where δ is the number which is used in assumption (b). By assumption (b), and from Eq. (A5), it follows that

$$\|x_1\| \le M \|u\| + \epsilon M \|x_0\| .$$

Clearly by choosing $\|u\|$ sufficiently small, δ can be made small and by (c), so will ϵ. Consequently by choosing $\|u\|$ sufficiently small we have

$$\|x_1\| \le (1 + \epsilon M) M \|u\| \le \delta \tag{A7}$$

Now from (A3) and (A4), we obtain,

$$\|\underset{\sim}{x}_{n+1}\| \le M \|\underset{\sim}{u}\| + \int_0^t |e^{\underset{\sim}{G}(t-t')}| \; \|\underset{\sim}{g}(\underset{\sim}{x}_n)\| \, dt' . \tag{A8}$$

Let us reason by induction. Suppose that we can choose $\|\underset{\sim}{u}\|$ sufficiently small so that

$$\|\underset{\sim}{x}_n\| \le 2M \|\underset{\sim}{u}\| \le \delta ; \tag{A9}$$

then from (A8),

$$\|\underset{\sim}{x}_{n+1}\| \le M \|\underset{\sim}{u}\| + 2\epsilon M^2 \|\underset{\sim}{u}\| \tag{A10}$$

Since $\epsilon \to 0$ as $\delta \to 0$, we can choose $\|\underset{\sim}{u}\|$ so small such that $\epsilon M < (1/2)$, and consequently,

$$\|\underset{\sim}{x}_{n+1}\| \le 2M \|\underset{\sim}{u}\| . \tag{A11}$$

Thus we have shown that if $\|\underset{\sim}{u}\|$ is sufficiently small, first $\|\underset{\sim}{x}_1\| \le \delta$, and, second, $\|\underset{\sim}{x}_n\| \le \delta$ implies $\|\underset{\sim}{x}_{n+1}\| \le \delta$. Hence for $\|\underset{\sim}{u}\|$ sufficiently small, for all integers $n \ge 1$,

$$\|\underset{\sim}{x}_n\| \le 2M \|\underset{\sim}{u}\| \le \delta . \tag{A12}$$

Furthermore, the sequence $\{\underset{\sim}{x}_n\}$ converges. Now from Eq. (A4), we get, for $n = 1, 2, \ldots$

$$x_{n+1}(t) - x_n(t) = \int_0^t e^{G(t-t')} \left[g(x_n(t')) - g(x_{n-1}(t')) \right] dt'$$

By (A12) and assumption (6)

$$\| x_{n+1} - x_n \| \leq \epsilon M \| x_n - x_{n-1} \| \qquad \text{(A13)}$$

$$n = 1, 2, \ldots$$

With sufficiently small $\|u\|$, $\epsilon M < 1/2$; hence the sequence of functions $\{x_n\}$ is a Cauchy sequence which converges uniformly on $[0, \infty)$ to a necessarily continuous function which we call x. Taking the limit of both sides of Eq. (A4) as $n \to \infty$ we see that x is the zero-state solution of Eq. (14). The difference between the Nth iterate and x_0 can be written as

$$x_N(t) - x_0(t) = \sum_{n=0}^{N-1} \left[x_{n+1}(t) - x_n(t) \right], \qquad \text{(A14)}$$

and by (A5), (A13) and (A14),

$$\| x_N - x_0 \| \leq \sum_{n=0}^{N-1} \| x_{n+1} - x_n \| \leq \sum_{n=1}^{N} (\epsilon M)^n \| x_0 \| \qquad \text{(A15)}$$

Letting $N \to \infty$ we get

$$\|\underset{\sim}{x} - \underset{\sim}{x}_0\| \le \frac{\epsilon M}{1 - \epsilon M} \|\underset{\sim}{x}_0\| \tag{A16}$$

Hence

$$\frac{\|\underset{\sim}{x} - \underset{\sim}{x}_0\|}{\|\underset{\sim}{x}_0\|} \le \frac{\epsilon M}{1 - \epsilon M} \tag{A17}$$

Now as $\|\underset{\sim}{u}\|$ is taken smaller and smaller, $\|\underset{\sim}{x}_0\| \to 0$ (see (A6)), $\|\underset{\sim}{x}_n\| \to 0$ (see (A12)), and $\|\underset{\sim}{x}\| \to 0$ (by the limiting argument); therefore in the estimates we can write $\|\underset{\sim}{g}(\underset{\sim}{x}_n)\| \le \epsilon \|\underset{\sim}{x}_n\|$ and as $\|\underset{\sim}{u}\| \to 0$, $\epsilon \to 0$. Thus with (A17), $\|\underset{\sim}{u}\| \to 0$ implies

$$\frac{\|\underset{\sim}{x} - \underset{\sim}{x}_0\|}{\|\underset{\sim}{x}_0\|} \to 0 \; .$$

Q.E.D.

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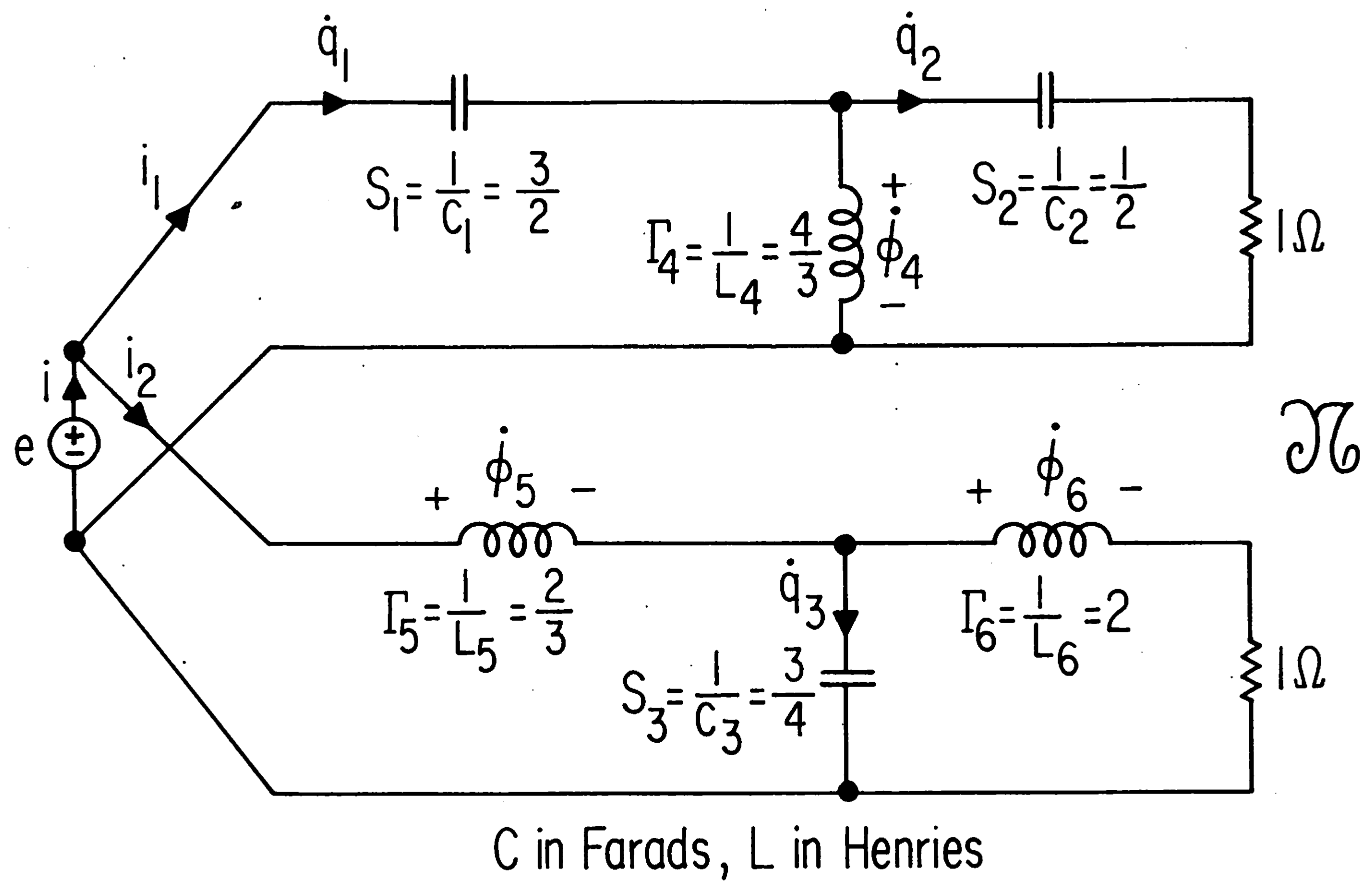

Fig. 1. Example of Norton's high-pass low-pass complementary constant resistance filter pairs.

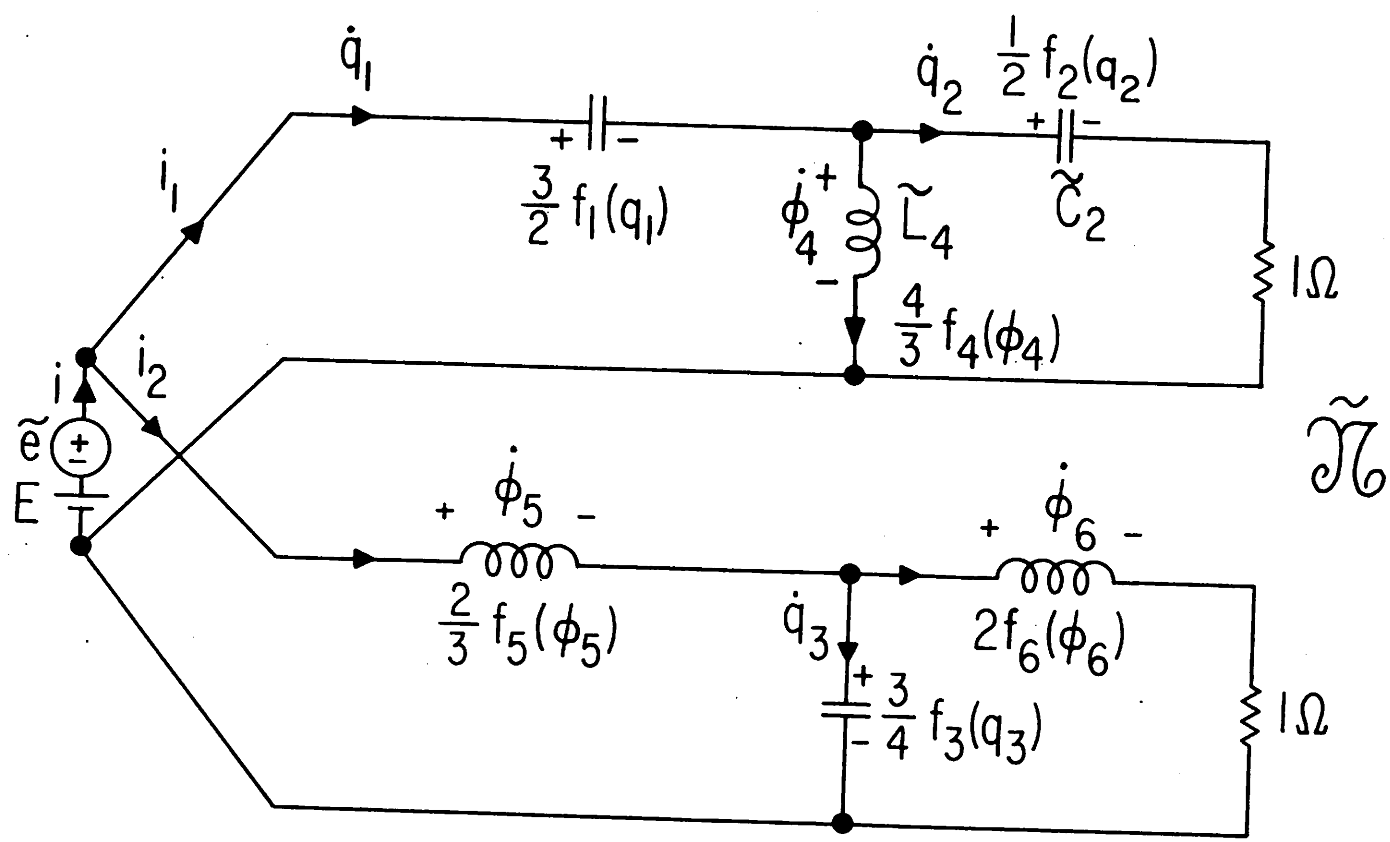

Fig. 2. Modified one-port with nonlinear reactive elements.

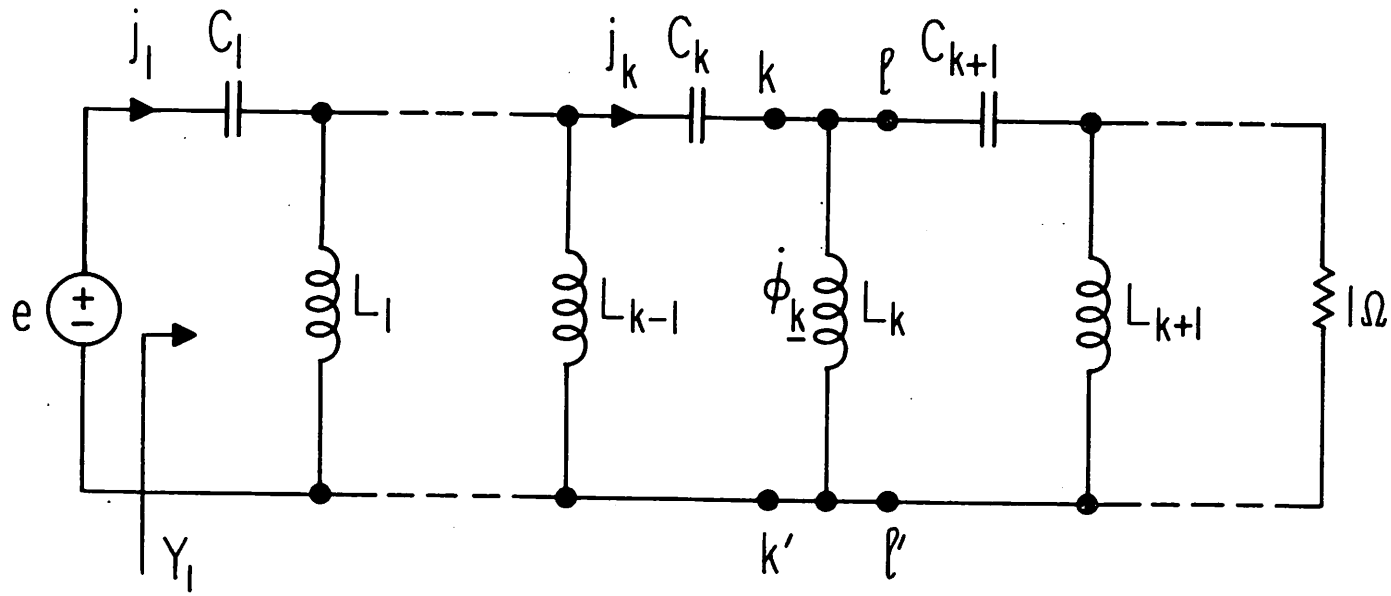

Fig. 3. The general high-pass ladder considered in (iii).

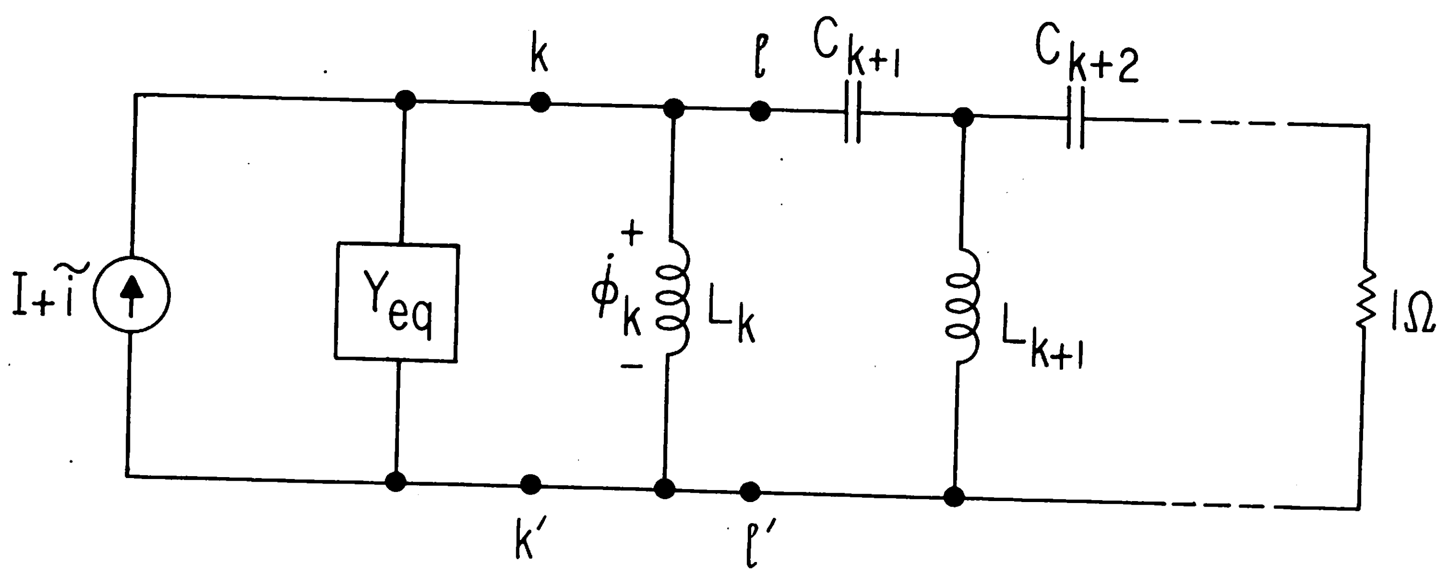

Fig. 4. The ladder of Fig. 3 after the one-port to the left of ⓚ, ⓚ′ is replaced by its Norton's equivalent.