ON SCALAR PRODUCTS OF SIGNALS PASSING THROUGH MEMORYLESS NONLINEARITIES WITH DELAY

by

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Let x be a signal that is fed through a memoryless time-invariant monotonically increasing nonlinearity. Call y(t) its output, then $y(t) = \varphi(x(t))$. Since φ is monotonically increasing, for all real numbers ξ, ξ'

$$[\varphi(\xi) - \varphi(\xi')]\,(\xi - \xi') \geq 0 . \tag{1}$$

Suppose that, as in Fig. 1, y, the output of the nonlinearity, is advanced or delayed by a fixed amount τ, then it seems natural to expect that

$$\int_{-\infty}^{+\infty} x(t)\, y(t)\, dt \geq \int_{-\infty}^{+\infty} x(t)\, y(t-\tau)\, dt . \tag{2}$$

Such an inequality restricted to a particular class of functions (piece-wise continuous and of bounded support) was proposed by O'Shea in some stability studies [1]. His inequality is correct but his proof is in error. In the following we prove an extended version of the inequality. The result is interesting because it seems intuitively obvious that it should hold in the case of a random process provided one takes expectations of both sides; however, it is far less obvious that it should hold for all samples of a random process. For this reason this inequality should be of interest to specialists in communication theory.

Let us state the assertion precisely. Let $x \in L^2(-\infty, \infty)$. Let φ be a memoryless time-invariant monotonically nondecreasing relation.† If y is also in $L^2(-\infty, \infty)$, then inequality (2) holds for all such x and y's and for all fixed finite τ. Furthermore, if φ is odd (i.e., $(\xi, \eta) \in \varphi \Leftrightarrow (-\xi, -\eta) \in \varphi$), then

$$\int_{-\infty}^{+\infty} x(t)\, y(t)\, dt \geq \int_{-\infty}^{+\infty} |x(t)|\, |y(t-\tau)|\, dt . \tag{3}$$

Proof. (i) Let us establish (2) when x, y are continuous functions on $(-\infty, \infty)$.

We suppose first that x, y are zero outside $[-T, T]$; we observe then that the integrands in (2) are identically zero outside $[-T, T]$. Let us approximate the integrals by Riemann sums, using

† The only requirement on φ is that it be memoryless and that (1) hold; φ need not be a single-valued function. Since φ is now a relation, we write $(x(t), y(t)) \in \varphi$, for all t. And monotonicity implies that if $(x_1, y_1) \in \varphi$, $(x_2, y_2) \in \varphi$ then $(x_1 - x_2)(y_1 - y_2) \geq 0$.

an equally spaced partition of diameter τ/n. If we pick the values of the functions at the left edge of each partition interval, we have to consider the following sum

$$\frac{\tau}{n} \sum_{i=-N}^{N} \left\{ x\left(i \frac{\tau}{n}\right) y\left(i \frac{\tau}{n}\right) - x\left(i \frac{\tau}{n}\right) y\left[(i-n) \frac{\tau}{n}\right] \right\}$$

or, with obvious notations,

$$\frac{\tau}{n} \sum_{i=-N}^{N} \left(x_i \, y_i - x_i \, y_{i-n} \right).$$

Call the sum $R(n)$. Let us reorder the terms of the summation. Call $\pi(\cdot)$ a permutation of the integers such that the two sequences $\left\{ x_{\pi(i)} \right\}$ and $\left\{ y_{\pi(i)} \right\}$ are both <u>monotonically decreasing</u>. Since φ is a monotonically nondecreasing relation, this is possible. For simplicity, call $\{\xi_i\}$ $\{\eta_i\}$ the resulting sequences.

$$R(n) = \frac{\tau}{n} \sum_{j=0}^{N} \left(\xi_j \, \eta_j - \xi_j \, \eta_{\sigma(j)} \right),$$

where $\sigma(j)$ is the index of η induced by the shift of the index i. Consider the sum $\sum \xi_j \, \eta_j$ and, in particular, the sum of two consecutive terms, say $\xi_i \, \eta_i + \xi_{i+1} \, \eta_{i+1}$. Since the pairs $\{\xi_i, \xi_{i+1}\}$ and $\{\eta_i, \eta_{i+1}\}$ are monotonically decreasing sequences

$$\left(\xi_i - \xi_{i+1}\right)\left(\eta_i - \eta_{i+1}\right) \geq 0,$$

$$\left(\xi_i \eta_i + \xi_{i+1} \eta_{i+1}\right) - \left(\xi_i \eta_{i+1} + \xi_{i+1} \eta_i\right) \geq 0.$$

Thus we conclude that given two monotonically decreasing sequences $\{\xi_j\}$, $\{\eta_j\}$, if in the sum $\sum \xi_i \eta_i$, we interchange two consecutive η_i's and leave the ξ_i's unchanged, the sum is not increased. Thus starting with the sum

$$\sum_j \xi_j \eta_j ,$$

by successive interchanges of consecutive η_j's, we can bring $\eta_{\sigma(N)}$ from its original term to the last term without increasing the sum. In the resulting sum consider all terms but the last: again the ξ's and the η's are monotonically decreasing, therefore we can repeat the procedure. Therefore $R(n) \geq 0$ for all n. Since the integral of interest is the limit of $R(n)$ as $n \to \infty$, inequality (2) is established for continuous functions whose support is $[-T, T]$. Then by letting $T \to \infty$, inequality (2) holds for all continuous x, y for which the integrals make sense.

(ii) The conclusion is still valid when $x, y \in L^2(-\infty, \infty)$. Indeed, the mapping $\psi : L^2 \times L^2 \to R$, which maps

$$(x, y) \to \int \left[x(t)\, y(t) - x(t)\, y(t-\tau)\, dt \right],$$

is continuous (in the domain we use the product L^2-topology). Since continuous functions are dense in L^2, inequality (2), which is true for continuous functions, will remain true by continuity for functions in L^2.

(iii) In addition, let φ be odd, i.e., $(x, y) \in \varphi$ implies $(-x, -y) \in \varphi$. This with monotonicity gives $xy = |x| \, |y| \geq 0$. Thus, if $(x, y) \in \varphi$, then $(|x|, |y|) \in \varphi$. Consequently (2) implies (3).

Q.E.D.

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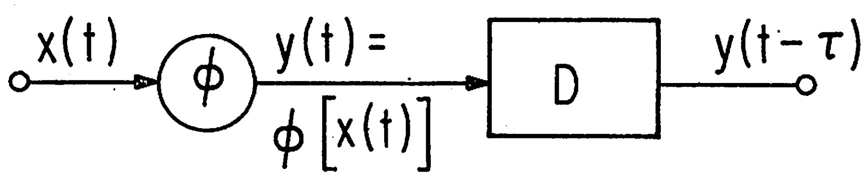
References

[1] R. P. O'Shea, "A combined frequency-time domain stability criterion for autonomous continuous systems," IEEE Trans. on Automatic Control, Vol. AC-11, pp. 477-484, July 1966.

LIST OF FIGURES


x(t)
φ
y(t) =
φ [x(t)]
D
y(t − τ)