BOUNDS ON THE RESPONSES OF NONLINEAR CONTROL SYSTEMS

by

Ralph Iwens

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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ABSTRACT

It is shown how a frequency domain criterion can be used to obtain meaningful bounds on the responses of nonlinear feedback systems. If the system is disturbed by a known input or nonzero initial condition from its state of equilibrium, bounds are obtained on overshoot and settling time as the system returns to its state of equilibrium. The method presented also permits to compute bounds for responses to bounded inputs which do not bring the system to a state of equilibrium. The bounds have a graphical interpretation in the Nyquist-plane which is similar in concept to M-circles for linear systems. This graphical interpretation can be used advantageously in the design of nonlinear feedback systems.

I. Introduction

The problem of stability of nonlinear control systems has received much attention in recent years. Of great interest in the theory of nonlinear systems is not only their stability but also the transient behavior of a system as it returns to its state of equilibrium. While in the theory of linear feedback systems the various methods for testing stability have also been widely exploited to obtain qualitative as well as quantitative information on the transient behavior of the system (e.g., root locus methods, Nyquist plots, Bode plots), hardly any use has been made in this respect of V. M. Popov's[1, 2] relatively new and simple frequency domain stability criterion for nonlinear systems. Only Naumov and Tsypkin[3] have recently shown how the information gained from the Popov stability test can be used in the design of compensating networks for stabilizing unstable systems, and how one can determine, what they call, "the degree of stability" of a system, which gives the rate at which the transient dies out. Naumov and Tsypkin do not obtain a bound on the settling time or overshoot.

Very recently Siljak[4, 5] published a number of papers on the transient behavior of nonlinear systems. His method combines linearization techniques, describing function methods and analysis in the parameter plane. The nonlinearity must be described precisely, cannot be time-varying and must be suitable for linearization. Frequently such assumptions are not satisfied.

This paper shows how an extended version of the V. M. Popov Theorem can be used to obtain meaningful bounds on the responses of nonlinear feedback systems and how this information can aid in the design of these systems. The nonlinear element of the system may be time-varying and must only be described to the extent that it is contained within a certain sector in its input-output plane.

II. Description of System and Notation

The system under consideration is the single input, single output unity feedback system S shown in Fig. 1.

Assumption 1. The nonlinear, time-varying element N is characterized by a piecewise continuous, single-valued function $\varphi(\sigma, t)$, $\sigma \in (-\infty, \infty)$, $t \in [0, \infty)$, such that

$$a \leq \frac{\varphi(\sigma, t)}{\sigma} \leq b < \infty, \qquad \forall \sigma \neq 0, \quad \forall t \geq 0 \tag{1}$$

and

$$\varphi(0, t) = 0, \quad \forall t \geq 0 \quad . \tag{2}$$

Denote the output of N by

$$u(t) = \varphi[\sigma(t), t] \quad . \tag{3}$$

Assumption 2. The linear plant is assumed to be nonanticipative, time-invariant and completely controllable and observable and is characterized by its transfer function $W(s)$. $W(s)$ is a rational fraction in s with its numerator polynomial of lower degree than the denominator. $W(s)$ has poles only in the left half s-plane (principal case) or has some poles on the $j\omega$-axis (particular cases). $z(t)$ is the zero input response of the linear plant.

Notation. A system S satisfying the above assumptions for specific numbers a and b will be referred to as a system $S \in N(a, b, t)$.

Notation. The system S_L is the linear companion system of the system S with the nonlinear element N replaced by a constant linear gain K. For a specific gain K the output of S_L is denoted by $y_K(t)$; $e_K(t) = r(t) - y_K(t)$ is called the error of the linear companion system S_L.

III. Preliminaries

Our aim is to establish bounds on the responses of the system S when it is disturbed from its state of equilibrium, which is the zero state. This disturbance may be a non-zero initial state, a load disturbance at the output, or a deliberate disturbance in form of an input. Of particular interest is the transient behavior of the system as it returns to the equilibrium point after the disturbance has occurred. However, the results can also be used to obtain bounds on the responses to bounded inputs when the system does not return to its equilibrium point at the origin. Since the only assumption about the time-varying nonlinearity is that it be contained in a sector $[a, b]$, some of the bounds are accordingly conservative if compared with the transient response of the linear companion system, especially if the sector containing the nonlinearity is large. This should not be surprising since the system is quite free in its possible behavior. However, with relatively little effort useful bounds on the transient response can be obtained. With somewhat more computational effort these bounds can be improved considerably.

The bounds that will be established are based on the following theorem which is a special case of the Main Lemma established by Bergen, Iwens and Rault in Ref. 6. A statement, similar to that in Ref. 6, can be found in Ref. 7.

Theorem 1. If for a principal case of the system $S \in N(0, k, t)$ there exists a positive δ and an $\alpha \geq 0$ such that for all $\omega \geq 0$ the inequality

$$\mathrm{Re}\{W(j\omega - \alpha)\} + \frac{1}{k} \geq \delta > 0 \qquad \text{(P)}$$

holds and $W(s - \alpha)$ has all its poles in the left half s-plane, then the following inequality is satisfied for all $t \geq 0$:

$$\left(\int_0^t e^{2\alpha\tau} u^2(\tau)\, d\tau\right)^{1/2} \le \frac{1}{\delta}\left(\int_0^t e^{2\alpha\tau}[r(\tau) - z(\tau)]^2\, d\tau\right)^{1/2} . \qquad (L)$$

The proof of Theorem 1 follows directly from the proof of the Main Lemma in Ref. 6.

The following theorem plays an important role in obtaining the main results of this paper.

Theorem 2. Consider a principal case of the system $S \in N(0, k, t)$. Define the quantity

$$p(\alpha, \delta, t) = \frac{1}{\delta}\left(\int_0^t e^{2\alpha\tau} w^2(\tau)\, d\tau\right)^{1/2}\left(\int_0^t e^{-2\alpha(t-\tau)} [r(\tau) - z(\tau)]^2\, d\tau\right)^{1/2} . \qquad (4)$$

where $w(t)$ is the impulse response corresponding to $W(s)$. If there exists an $\alpha \ge 0$ and a $\delta > 0$ such that

$$\text{(i)} \quad \mathrm{Re}\{W(j\omega - \alpha) + \frac{1}{k}\} \ge \delta > 0, \quad \forall \omega \ge 0 \qquad (5)$$

(ii) $W(s - \alpha)$ has all its poles in the left half s-plane,

then $\sigma(t)$ is bounded by the inequality

$$r(t) - z(t) - p(\alpha, \delta, t) \le \sigma(t) \le r(t) - z(t) + p(\alpha, \delta, t) . \qquad (6)$$

Before proving Theorem 2 it is noted that condition (ii) of Theorem 2 assures that

$$\int_0^t e^{2\alpha\tau} w^2(\tau)\, d\tau < \infty , \quad \forall t \ge 0 . \qquad (7)$$

Moreover, it follows directly from the Nyquist criterion that condition (ii) of the theorem is implied by condition (i) if the locus of $W(j\omega - \alpha)$ lies entirely in the finite complex plane. For at the value of α where the locus of $W(j\omega - \alpha)$ leaves the finite complex plane, at least one pole of $W(s)$ has been shifted onto the $j\omega$-axis. In effect one must then only satisfy condition (i) of Theorem 2 provided $W(j\omega - \alpha)$ lies only in the finite complex plane.

Furthermore we note the following properties of $p(\alpha, \delta, t)$:

(a) $p(\alpha, \delta, t) \geq 0, \quad \forall t \geq 0, \ \alpha \geq 0, \ \delta > 0$ (8)

(b) For $[r(t) - z(t)] \in L_2(0, \infty)$ and $\alpha = 0$, $p(\alpha, \delta, t)$ is uniformly bounded for $t \in [0, \infty)$ and
$\lim_{t\to\infty} p(\alpha, \delta, t) = \text{constant} < \infty$.

(c) For $[r(t) - z(t)] \in L_2(0, \infty) \cap L_\infty(0, \infty)$ and $\alpha > 0$
$p(\alpha, \delta, t)$ is uniformly bounded for $t \in [0, \infty)$ and
$\lim_{t\to\infty} p(\alpha, \delta, t) = 0$. This follows directly from the Schwarz inequality and the Riemann-Lebesgue lemma as shown in Appendix A.

(d) For $[r(t) - z(t)] \in L_\infty(0, \infty)$ and $\alpha = 0$, $\lim_{t\to\infty} p(\alpha, \delta, t) \to \infty$.

(e) For $[r(t) - z(t)] \in L_\infty(0, \infty)$ and $\alpha > 0$,
$p(\alpha, \delta, t) \leq \text{constant} < \infty, \ \forall t \geq 0$.

From properties (d) and (e) it can be concluded that, in general, bounds for responses to inputs belonging to $L_\infty(0, \infty)$ can only be obtained if $\alpha > 0$.

Proof of Theorem 2. The system $S \epsilon N(0, k, t)$ (see Fig. 1) is described by the equation

$$\sigma(t) = r(t) - z(t) - \int_0^t w(t-\tau)\, u(\tau)\, d\tau \tag{9}$$

or equivalently

$$\sigma(t) = r(t) - z(t) - \int_0^t e^{\alpha(t-\tau)}\, w(t-\tau)\, e^{-\alpha(t-\tau)}\, u(\tau)\, d\tau \tag{10}$$

From (10) it follows by the Schwarz inequality that

$$r(t) - z(t) - \left(\int_0^t e^{2\alpha\tau}\, w^2(\tau)\, d\tau\right)^{1/2} e^{-\alpha t}\left(\int_0^t e^{2\alpha\tau}\, u^2(\tau)\, d\tau\right)^{1/2}$$

$$\leq \sigma(t) \leq r(t) - z(t)$$

$$+\left(\int_0^t e^{2\alpha\tau}\, w^2(\tau)\, d\tau\right)^{1/2} e^{-\alpha t}\left(\int_0^t e^{2\alpha\tau}\, u^2(\tau)\, d\tau\right)^{1/2} \tag{11}$$

Using inequality (L) of Theorem 1 and also equation (4), inequality (11) becomes

$$r(t) - z(t) - p(\alpha, \delta, t) \leq \sigma(t) \leq r(t) - z(t) + p(\alpha, \delta, t) \tag{12}$$

Q.E.D.

IV. Main Results

The results of Theorem 2 are restrictive in the sense that they only apply to principal cases of the system S and that the time-varying nonlinearity must be contained in a sector of the type [0, k]. It is desirable to obtain results which are equally well applicable to principal and particular cases and where the time-varying nonlinearity is contained in a sector $[a, b]$, where $b > 0$ and a need not necessarily be positive. In order to apply Theorem 2 to the system $S \in N(a, b, t)$ the system is first transformed into an equivalent system $\tilde{S} \in N(0, b - a, t)$ by the following change of variable:

$$\tilde{\varphi}(\sigma, t) = \varphi(\sigma, t) - a\sigma \tag{13}$$

The system $\tilde{S}$ is shown in Fig. 2. To obtain $\tilde{S}$ substitute equation (13) into equation (9) and take Laplace transforms. The corresponding quantities of the two equivalent systems are then related by

$$\tilde{W}(s) = \frac{W(s)}{1 + a W(s)} \tag{14}$$

$$\tilde{z}(t) = \mathcal{L}^{-1}\left[\frac{Z(s)}{1 + a W(s)}\right] \tag{15}$$

$$\tilde{r}(t) = \mathcal{L}^{-1}\left[\frac{R(s)}{1 + a W(s)}\right] \tag{16}$$

where $\mathcal{L}$ denotes the Laplace transform operator and $Z(s)$ and $R(s)$ are the Laplace transforms of $z(t)$ and $r(t)$ respectively. In this transformation it is assumed that the number a is such that $\tilde{W}(s)$ has all its poles in the left half s-plane, which does not mean that $W(s)$ must have all its poles in the left half s-plane. Observe that the transformation (13) does not affect the quantity $\sigma(t)$; also, for $a = 0$ the systems $S \in N(a, b, t)$ and $\tilde{S} \in N(0, b - a, t)$ are identical.

Theorem 2 may now be applied to the system $\tilde{S} \in N(0, b - a, t)$ and an inequality for the quantity $\sigma(t)$ can be obtained. But $\sigma(t)$ has not been affected by the transformation and is thus also the $\sigma(t)$ of the original system $S \in N(a, b, t)$. Hence one obtains that the output $y(t) = r(t) - \sigma(t)$ of the original system $S \in N(a, b, t)$ is bounded by the inequality

$$r(t) - [\tilde{r}(t) - \tilde{z}(t)] - \tilde{p}(\alpha, \delta, t) \leq y(t) \leq r(t) - [\tilde{r}(t) - \tilde{z}(t)] + \tilde{p}(\alpha, \delta, t) \tag{17}$$

where now

$$\tilde{p}(\alpha, \delta, t) = \frac{1}{\delta}\left(\int_0^t e^{2\alpha\tau} \tilde{w}^2(\tau)\, d\tau\right)^{1/2} \left(\int_0^t e^{-2\alpha(t-\tau)} [\tilde{r}(\tau) - \tilde{z}(\tau)]^2\, d\tau\right)^{1/2} \tag{18}$$

By Theorem 2 inequality (17) is valid if $\tilde{W}(s - \alpha)$ has all its poles in the left half s-plane and if there exists a positive δ and an $\alpha \geq 0$ such that for all $\omega \geq 0$

$$\mathrm{Re}\left\{\frac{W(j\omega - \alpha)}{1 + a W(j\omega - \alpha)} + \frac{1}{b - a}\right\} \geq \delta > 0 . \tag{19}$$

Observe now that by (15) and (16)

$$\tilde{r}(t) - \tilde{z}(t) = e_a(t)$$

where $e_a(t)$ is the error of the linear companion system S_L with constant gain $K = a$ and with the same initial conditions as the system $S \in N(a, b, t)$. Then inequality (17) becomes

$$y_a(t) - \tilde{p}(\alpha, \delta, t) \leq y(t) \leq y_a(t) + \tilde{p}(\alpha, \delta, t) \tag{20}$$

or equivalently,

$$|y(t) - y_a(t)| \leq \tilde{p}(\alpha, \delta, t) \tag{21}$$

where $y_a(t)$ is the output of the linear companion system S_L with gain $K = a$ in response to the same input $r(t)$ and with the same initial condition as for the nonlinear system.

From (19) it is evident that for narrow sectors $[a, b]$, i.e., $b - a$ is small, δ becomes large, and thus $\tilde{p}(\alpha, \delta, t)$ is small (see (18)), and the response of the nonlinear time-varying system differs little from the response of the linear time-invariant system. In the limit as $(b - a) \to 0$, the quantity $\delta \to \infty$, and thus for all $t \geq 0$, $\tilde{p}(\alpha, \delta, t) \to 0$ and $y(t) \to y_a(t)$. This was expected, since $(b - a) \to 0$ implies that the time-varying nonlinearity in the sector $[a, b]$ has become a constant gain $K = a$. Therefore the function $\tilde{p}(\alpha, \delta, t)$ may be looked upon as a "penalty function" for the uncertainty introduced by the time-varying nonlinearity of which it is only known that it is contained in a sector $[a, b]$.

It is clear now that the numerical value of δ plays an important role in the bound (20). We shall now derive a geometric construction in the $W(j\omega)$-plane that permits us to read δ directly from the Nyquist plot.

δ-Circles

From (19) we obtain

$$\mathrm{Re}\left\{1 + a\overline{W} + bW + ab|W|^2\right\} \geq \delta\, |1 + aW|^2\, (b - a) \qquad (22)$$

Let $W = X + jY$. Then,

$$1 + [a + b - 2\delta ab - 2\delta a^2]\, X + a[b - a\delta(b - a)]\, (X^2 + Y^2) \geq \delta(b - a) \qquad (23)$$

We now put the constraint on δ that for given a and b

$$b - a\delta(b - a) > 0 \qquad (24)$$

For $a > 0$ (23) reduces then to

$$\left[X + \frac{1}{2}\,\frac{a+b-2a\delta(b-a)}{ab-a^2\delta(b-a)}\right]^2 + Y^2 \geq \left[\frac{1}{2}\,\frac{a-b}{ab-a^2\delta(b-a)}\right]^2 \tag{25}$$

which for fixed a and b represents a family of circles with their centers on the real axis, of which each circle corresponds to a different δ. These circles are called δ-circles. In order to satisfy inequality (19) for a certain δ, for $a > 0$ the locus of $W = X + jY$ must stay outside the corresponding δ-circle; for $a < 0$, the inequality sign in (25) is inverted and the locus of $W = X + jY$ must stay inside the corresponding δ-circle. The family of circles in (25) is more conveniently described by their intersections with the real axis which determine each circle uniquely. The intersections are given by

$$X_1 = -\frac{1}{a} \tag{26}$$

and

$$X_2 = -\frac{1-\delta(b-a)}{b-a\delta(b-a)} \tag{27}$$

Note that X_1 is independent of δ.

The constraint $b - a\delta(b-a) > 0$ becomes now meaningful, since when $b - a\delta(b-a) = 0$, $|X_2| \to \infty$. It can be checked that for $b - a\delta(b-a) < 0$ a different set of circles is obtained. As far as the locus of $W = X + jY$ is concerned, we must only interchange the words "inside" and "outside". However, for the type of complex valued functions $W(j\omega)$ we are concerned with, this other set of circles is of no interest.

The previous discussion can now be summarized in the statement of the following theorem which is the main result of this paper.

Theorem 3. Consider the system $S \in N(a, b, t)$. Denote $y_a(t)$ as the output of the linear companion system S_L with constant gain $K = a$. Let

$$\tilde{p}(\alpha, \delta, t) = \frac{1}{\delta}\left(\int_0^t e^{2\alpha\tau} \tilde{w}^2(\tau)\, d\tau\right)^{1/2} \left(\int_0^t e^{-2\alpha(t-\tau)} [\tilde{r}(\tau) - \tilde{z}(\tau)]^2\, d\tau\right)^{1/2}$$

where $\tilde{w}(t)$, $\tilde{r}(t)$ and $\tilde{z}(t)$ are as defined in (14) - (16). The response $y(t)$ due to a disturbance of the plant from its state of equilibrium caused by the input $r(t)$ or an initial state corresponding to the zero input response $z(t)$, is then bounded by the inequality

$$|y(t) - y_a(t)| \leq \tilde{p}(\alpha, \delta, t) , \tag{28}$$

if the following conditions are satisfied for α and δ:

1. $\dfrac{W(s - \alpha)}{1 + a\, W(s - \alpha)}$ has all its poles in the left half s-plane

2. $0 < \delta < \dfrac{b}{a(b - a)}$.

3. For all $\omega \geq 0$ the locus of $W(j\omega - \alpha)$ satisfies one of the following conditions:

   (i) for $a > 0$, it lies outside the corresponding δ-circle with center on the real axis of the $W(j\omega)$-plane which intersects the real axis at $X_1 = -(1/a)$ and $X_2 = -\dfrac{1 - \delta(b - a)}{b - a\delta(b - a)}$;

   (ii) for $a = 0$, $\text{Re}\, W(j\omega - \alpha) + \dfrac{1}{b} \geq \delta > 0$;

(iii) for $a < 0$, it lies inside the corresponding δ-circle which has its center on the real axis of the $W(j\omega)$-plane and intersects it at $X_1 = -1/a$ and $X_2 = -\frac{1 - \delta(b-a)}{b - a\delta(b-a)}$.

V. An Example

Consider the servo positioning system $S \in N(a, b, t)$ with $a = 0.5$, $b = 1.0$ and

$$W(s) = \frac{1}{s(s+1)} \tag{29}$$

The system is in the zero state when a unit step input

$$r(t) = 1(t) \tag{30}$$

is applied. From the Nyquist diagrams of $W(j\omega - \alpha)$ for various α's, Fig. 3, it is observed that for several choices of α and δ all the conditions of Theorem 3 are satisfied.* If we are only interested in constant upper and lower amplitude bounds of $y(t)$, we set $\alpha = 0$ and see that the corresponding $\delta = 1.5$. With $\alpha = 0$, $\tilde{p}(\alpha, \delta, t)$ is a bounded, monotonically increasing function. Hence for constant upper and lower amplitude bounds we obtain from (28) that

$$\inf_{0 \le t < \infty} [y_a(t)] - \tilde{p}(0, 1.5, \infty) \le y(t) \le \sup_{0 \le t < \infty} [y_a(t)] + \tilde{p}(0, 1.5, \infty) \tag{31}$$

A fast calculation gives

$$\tilde{p}(0, 1.5, \infty) = 0.817$$

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* Note that in this example encirclement of the cross-hatched circle corresponding to $\delta = 0$ means by the Nyquist criterion that for $\alpha > 0$ all the poles of $(W(s-\alpha)/(1 + aW(s-\alpha))$ are in the left half s-plane.

The infimum of $y_a(t)$ is clearly zero; the supremum of $y_a(t)$ can easily be found by well known techniques of determining the peak overshoot for second order systems in response to a unit step input. It is given by $y_a(t_{max}) = 1.04$. Hence, we have

$$- 0.817 \leq y(t) \leq 1.857, \forall t \geq 0$$

Now choose $\alpha = 0.25$ with the corresponding $\delta = 1$ (see Fig. 3). Other values for α and δ are possible, but since one likes both quantities to be large for a good bound, these values seem a reasonable compromise (observe that α and δ are reciprocally related to each other).

With $\alpha \neq 0$, $\tilde{p}(\alpha, \delta, t) \rightarrow 0$ exponentially as $t \rightarrow \infty$. If one is just interested in the settling time* of the transient of $y(t)$, one need only calculate the value of $\tilde{p}(0.25, 1.0, t)$ for one or two values of t at which one expects the transient of $y(t)$ to have decayed and thus try to confirm one's estimate; that is, one certainly would pick $t \geq T_{as}$, where T_{as} is the settling time of $y_a(t)$. We have, for instance, that

$$\tilde{p}(0.25, 1.0, 4\pi) = 0.134$$

and

$$\tilde{p}(0.25, 1.0, 5\pi) = 0.055$$

Hence, since $T_{as} \approx 3\pi$ sec, we have by (28) that the settling time T_s of $y(t)$ cannot be larger than 5π, i.e.,

$$T_s \leq 5\pi \text{ seconds.}$$

For more, and less conservative, information on the bound of the transient one can plot the entire upper and lower bounds of $y(t)$ as a function

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* The settling time of the response to a step input is the time required for the transient to decrease to approximately 5% of the final value and thereafter remain less than this value.

of time. This is done for $\alpha = 0.25$ and $\delta = 1.0$, i.e., the bounds on y(t) given by

$$y_a(t) - \tilde{p}(0.25, 1.0, t) \le y(t) \le y_a(t) + \tilde{p}(0.25, 1.0, t) \qquad (32)$$

are plotted. We can also plot the bound with $\alpha = 0$ and $\delta = 1.5$, i.e., the bounds on y(t) given by

$$y_a(t) - \tilde{p}(0, 1.5, t) \le y(t) \le y_a(t) + \tilde{p}(0, 1.5, t)\,. \qquad (33)$$

Because of the larger δ, the second bound will give a better amplitude bound. Both bounds, (32) and (33), are plotted in Fig. 4 and then combined to one bound. For comparison purposes the response $y_b(t)$ of the linear companion system S_L with gain K = b (which is the upper bound of the sector [a, b], with a = 0.5, b = 1.0) is also plotted. Considering that we can predict for any time-varying nonlinearity contained in the sector [0.5, 1.0] that the response of the system to a unit step input is confined to lie between the indicated bounds, the results seem to be quite good. Nevertheless, the combined bound shown in Fig. 4 is most likely not the optimum bound that can be obtained by this method. One could, for instance plot an <u>additional</u> bound with an α and δ such that $0.25 < \alpha < 0.375$ and $0 < \delta < 1.0$. Because of the larger α this bound would decay at a faster rate than the one obtained previously; however, due to the smaller δ, its peak would be much higher, so that improvement of the overall combined bound would depend on where the three bounds have their crossovers.

<u>Application to Design Problems</u>

In the design of nonlinear control systems it is usually important to assure specific bounds on the response of the system to a certain class of inputs or a set of initial states. It is obvious from the plots of $W(j\omega - \alpha)$ and the δ-circles of Fig. 3, how one would have to compensate the linear part of the system to obtain better bounds. Cascade lead or lag-lead networks can be used advantageously to bend the loci of $W(j\omega - \alpha)$ away from the small δ-circles to the larger δ-circles. This will result in an improvement of the bounds. If quantitative information on the improvement is desired, one has to recompute the bounds for the compensated linear plant. The order of the linear part of the system has now been increased and the computations of $\tilde{p}(\alpha, \delta, t)$ and $y_a(t)$ are more tedious. In general, for higher order plants the aid of a digital computer is advisable for the calculation of $\tilde{p}(\alpha, \delta, t)$ and $y_a(t)$.

The δ-circles of Fig. 3 appear similar in concept to M-circles for linear systems. It should be kept in mind however, that design by M-circles is only valid for second order linear systems while the δ-circles are valid for any order nonlinear system.

VI. Attainment of Bounds

After having established bounds on the behavior of the system S, the obvious question to ask is, "can these bounds be attained?" That is, is it possible for the response $y(t)$ to touch the bounds at various instants of time? The answer is equivalent to determining when equality holds in the original inequality (6) of Theorem 2, restated here for convenience:

$$r(t) - z(t) - p(\alpha, \delta, t) \le \sigma(t) \le r(t) - z(t) + p(\alpha, \delta, t) \quad (34)$$

where we recall that

$$p(\alpha, \delta, t) = \frac{1}{\delta}\left(\int_0^t e^{2\alpha\tau} w^2(\tau)\, d\tau\right)^{1/2}\left(\int_0^t e^{-2\alpha(t-\tau)}[r(t) - z(t)]^2\, d\tau\right)^{1/2} \quad (35)$$

It is immediate that for $\alpha > 0$, and $r(t) \in L_2(0, \infty) \cap L_\infty(0, \infty)$, equality holds in (34) for $t = 0$ and $t = \infty$. To determine if equality can hold for other values of t, we must investigate the following inequalities which were used to obtain (34).

$$\text{(i)} \quad \left| \int_0^t e^{\alpha(t-\tau)} w(t-\tau) e^{-\alpha(t-\tau)} u(\tau) d\tau \right|$$

$$\leq \left(\int_0^t e^{2\alpha\tau} w^2(\tau) d\tau \right)^{1/2} e^{-\alpha t} \left(\int_0^t e^{2\alpha\tau} u^2(\tau) d\tau \right)^{1/2} \tag{36}$$

$$\text{(ii)} \quad \left(\int_0^t e^{2\alpha\tau} u^2(\tau) d\tau \right)^{1/2} \leq \frac{1}{\delta} \left(\int_0^t e^{2\alpha\tau} [r(\tau) - z(\tau)]^2 d\tau \right)^{1/2} \tag{37}$$

Inequality (36) is the Schwarz inequality for which equality holds if and only if there exists a constant λ such that

$$w(\tau) = \lambda u(\tau) \tag{38}$$

Inequality (37) stems from inequality (L) of Theorem 1. It is easy to check that a necessary condition for equality to hold in (37), is that

$$\text{Re}\left\{ W(j\omega - \alpha) + \frac{1}{k} \right\} = \delta, \ \forall \omega \geq 0^{*} \tag{39}$$

This means that $W(j\omega)$ must be of the form

$$W(j\omega) = c + j \text{ Im } W(j\omega) \tag{40}$$

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* If all the signals appearing in the system were band limited within a frequency band $\omega_1 \leq \omega \leq \omega_2$, then equation (39) would read $\text{Re}\left\{ W(j\omega - \alpha) + \frac{1}{k} \right\} = \delta$, for $\omega_1 \leq \omega \leq \omega_2$. However, we need not consider such signals for even a sinusoidal signal will not produce band limited signals because of start-up transients.

where c is a constant. But subsystems with this kind of a transfer function are not permissible in the system S, since they do not satisfy the condition that the numerator polynomial of W(s) be of lower degree than the denominator. The only exception would be the case when $c = 0$ in equation (40). But this would imply that W(s) is a particular case (e.g., $W(s) = 1/s^n$, n odd) which is not permissible in Theorem 1.

Therefore it can be concluded that the bounds on the transient response of y(t) cannot be attained except at $t = 0$ and as $t \to \infty$. Bounds on the response to a bounded input which does not bring the system to a state of equilibrium are only attained at $t = 0$. Note that in the example of section V the system has been brought to a state of equilibrium by an input which belongs to L_∞. This situation will occur in general whenever $\tilde{r}(t)$ is a member of $L_2 \cap L_\infty$ even though r(t) is not contained in $L_2 \cap L_\infty$. As a matter of fact, r(t) must not even be bounded. All that is required is that the linear companion system S_L has a zero steady-state error for this particular input.[8]

VII. Conclusions

Bounds on the responses of feedback systems containing a time-varying nonlinearity have been obtained from a frequency domain inequality. The nonlinearity must only be described to the extent that it is contained at all times within a certain sector in its input-output plane. The bounds have a graphical interpretation in the Nyquist-plane which is similar in concept to M-circles for linear systems. It is also shown that there is a strong relationship between a nonlinear feedback system and its linear companion system: the smaller the sector that contains the time-varying nonlinearity, the smaller is the absolute value of the difference between the response of the nonlinear system and its linear companion system. The results are obtained by generally applicable tools of functional analysis and can thus be applied to any order system. Indeed, it is easy to show that with certain additional assumptions on the impulse response of the linear subsystem the results are also applicable to infinite dimensional systems. The bounds derived cannot be attained except at $t = 0$, and if the system goes to a state of equilibrium, also at $t = \infty$. Since the bounds are realistic enough to be useful in design problems this is of no consequence.

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Appendix A

It is to be shown that for $[r(t) - z(t)] \in L_2(0, \infty) \cap L_\infty(0, \infty)$ and $\alpha > 0$, $p(\alpha, \delta, t)$ is uniformly bounded for $t \in [0, \infty)$ and $\lim_{t \to \infty} p(\alpha, \delta, t) = 0$.

First let

$$f(t) = [r(t) - z(t)]^2 \tag{41}$$

Since $[r(t) - z(t)] \in L_2(0, \infty) \cap L_\infty(0, \infty)$, it follows that $f(t) \in L_2(0, \infty) \cap L_\infty(0, \infty)$. By definition (4) and using (41)

$$p(\alpha, \delta, t) = \frac{1}{\delta}\left(\int_0^t e^{2\alpha\tau} w^2(\tau)\, d\tau\right)^{1/2}\left(\int_0^t e^{-2\alpha(t-\tau)} f(\tau)\, d\tau\right)^{1/2} \tag{42}$$

By equation (7) the first integral on the right of (42) is bounded for all $t \geq 0$. For the second integral note that by the Schwarz inequality

$$\int_0^t e^{-2\alpha(t-\tau)} f(\tau)\, d\tau \leq \left(\int_0^t e^{-4\alpha(t-\tau)}\, d\tau\right)^{1/2}\left(\int_0^t f^2(\tau)\, d\tau\right)^{1/2}$$

which is uniformly bounded for all $t \in [0, \infty)$ since $f(t) \in L_2(0, \infty)$.

To show that $\lim_{t\to\infty} p(\alpha, \delta, t) = 0$ it is noted that

$$\int_0^t e^{-2\alpha(t-\tau)} f(\tau)\, d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{j\omega + 2\alpha} F(j\omega)\, e^{j\omega t}\, d\omega$$

where $F(j\omega)$ is the Fourier transform of $f(t)$. Since the product of two L_2-functions is in L_1, it follows by the Riemann-Lebesgue lemma[9, 10] that

$$\lim_{t\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{j\omega + 2\alpha} F(j\omega)\, e^{j\omega t}\, d\omega = 0$$

Hence,

$$\lim_{t\to\infty} p(\alpha, \delta, t) = 0 .$$

Appendix B

Calculation of $\tilde{p}(\alpha, \delta, t)$

The following integrals have to be evaluated in order to compute values of $\tilde{p}(\alpha, \delta, t)$ for the example in section V.

$$\text{(i)} \quad \int_0^t e^{2\alpha\tau}\, \tilde{w}^2(\tau)\, d\tau = 4\int_0^t e^{(2\alpha-1)\tau} \sin^2 0.5\tau\, d\tau$$

$$\text{(ii)} \quad \int_0^t e^{-2\alpha(t-\tau)}\, \tilde{r}^2(\tau)\, d\tau = e^{-2\alpha t} \int_0^t e^{(2\alpha-1)\tau} \cdot [\cos 0.5\tau + \sin 0.5\tau]\, d\tau$$

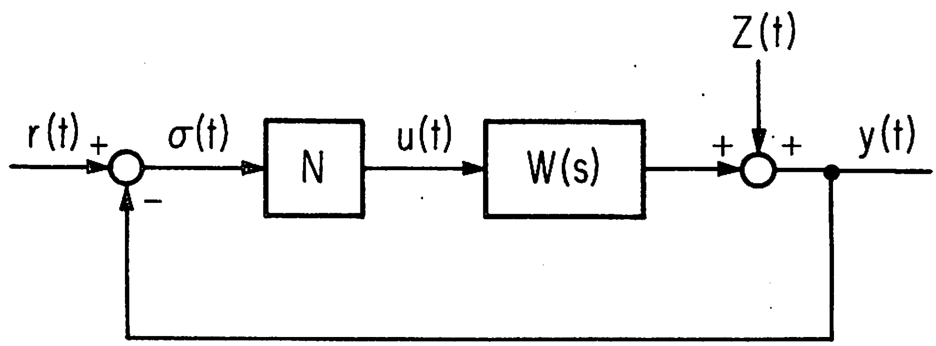
Tables of values of $\tilde{p}(\alpha, \delta, t)$ follow.

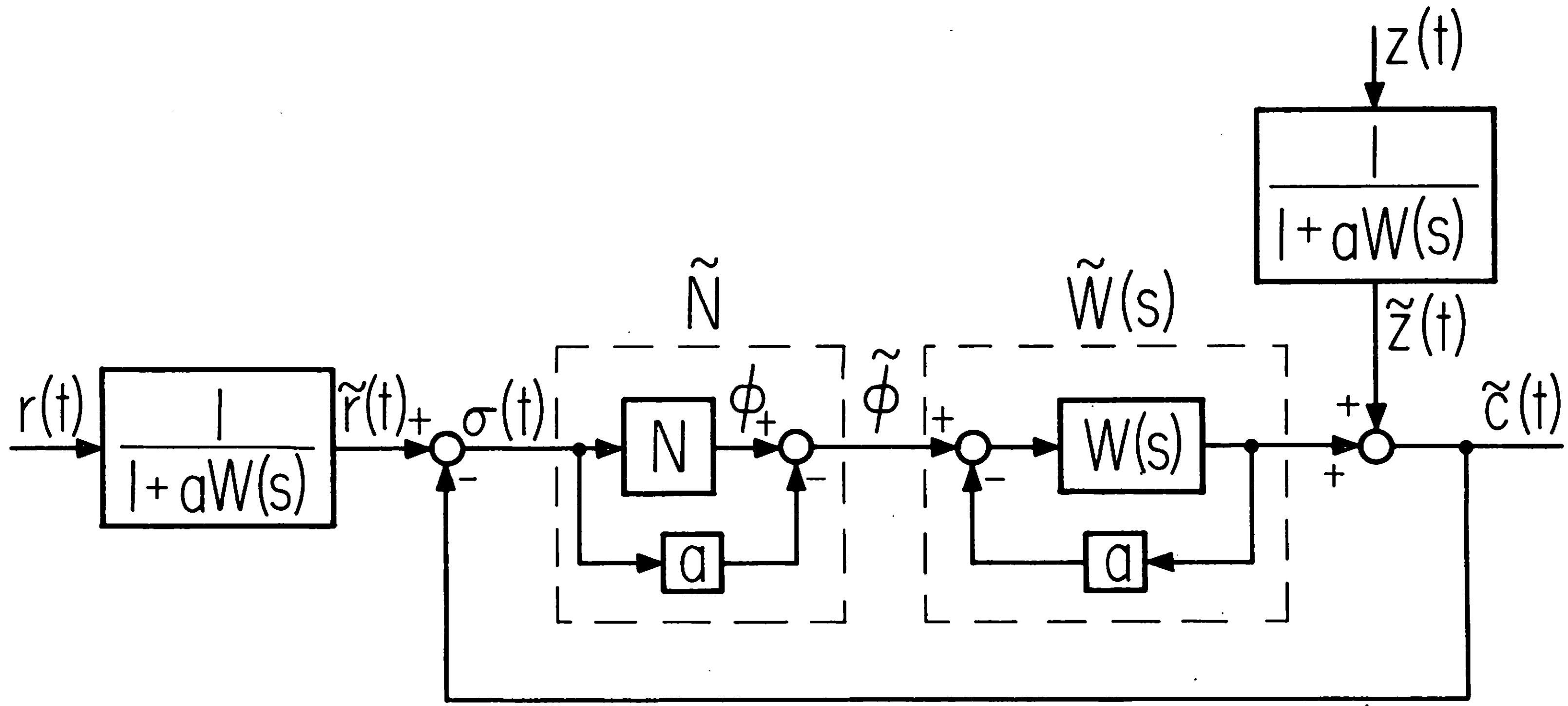
Tables of Values of $\tilde{p}(\alpha, \delta, t)$

t	$\left(\int_0^t \tilde{w}^2(\tau)\,d\tau\right)^{1/2}$	$\left(\int_0^t \tilde{r}^2(\tau)\,d\tau\right)^{1/2}$	$\tilde{p}(0, 1.5, t)$
0	0	0	0
0.5π	0.61	1.09	0.443
1.0π	0.938	1.22	0.76
1.5π	0.995	1.223	0.814
2.0π	$\approx$ 1.00	1.225	0.817
2.5π	1.00	1.225	0.817
3.0π	1.00	1.225	0.817
∞	1.00	1.225	0.817

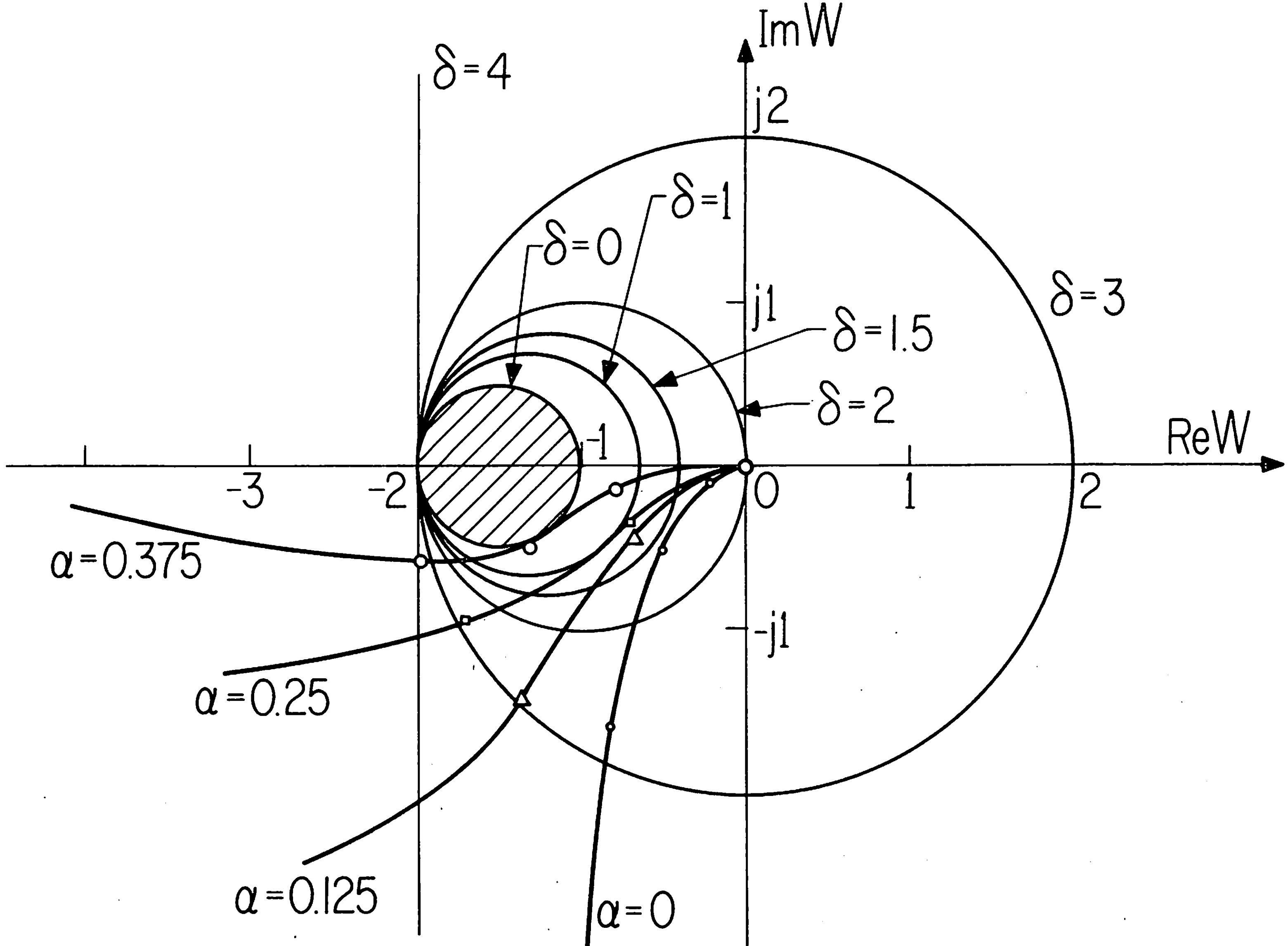
t	$\left(\int_0^t e^{0.5\tau}\,\tilde{w}^2(\tau)\,d\tau\right)^{1/2}$	$\left(\int_0^t e^{-0.5(t-\tau)}\,\tilde{r}^2(\tau)\,d\tau\right)^{1/2}$	$\tilde{p}(0.25, 1.0, t)$
0	0	0	0
0.5π	0.825	0.88	0.725
1.0π	1.48	0.732	1.085
1.5π	1.728	0.489	0.845
2.0π	1.752	0.328	0.575
2.5π	1.756	0.235	0.412
3.0π	1.777	0.15	0.266
4.0π	1.786	0.075	0.134
5.0π	1.789	0.033	0.055
∞	1.789	0.00	0.00

List of Figures

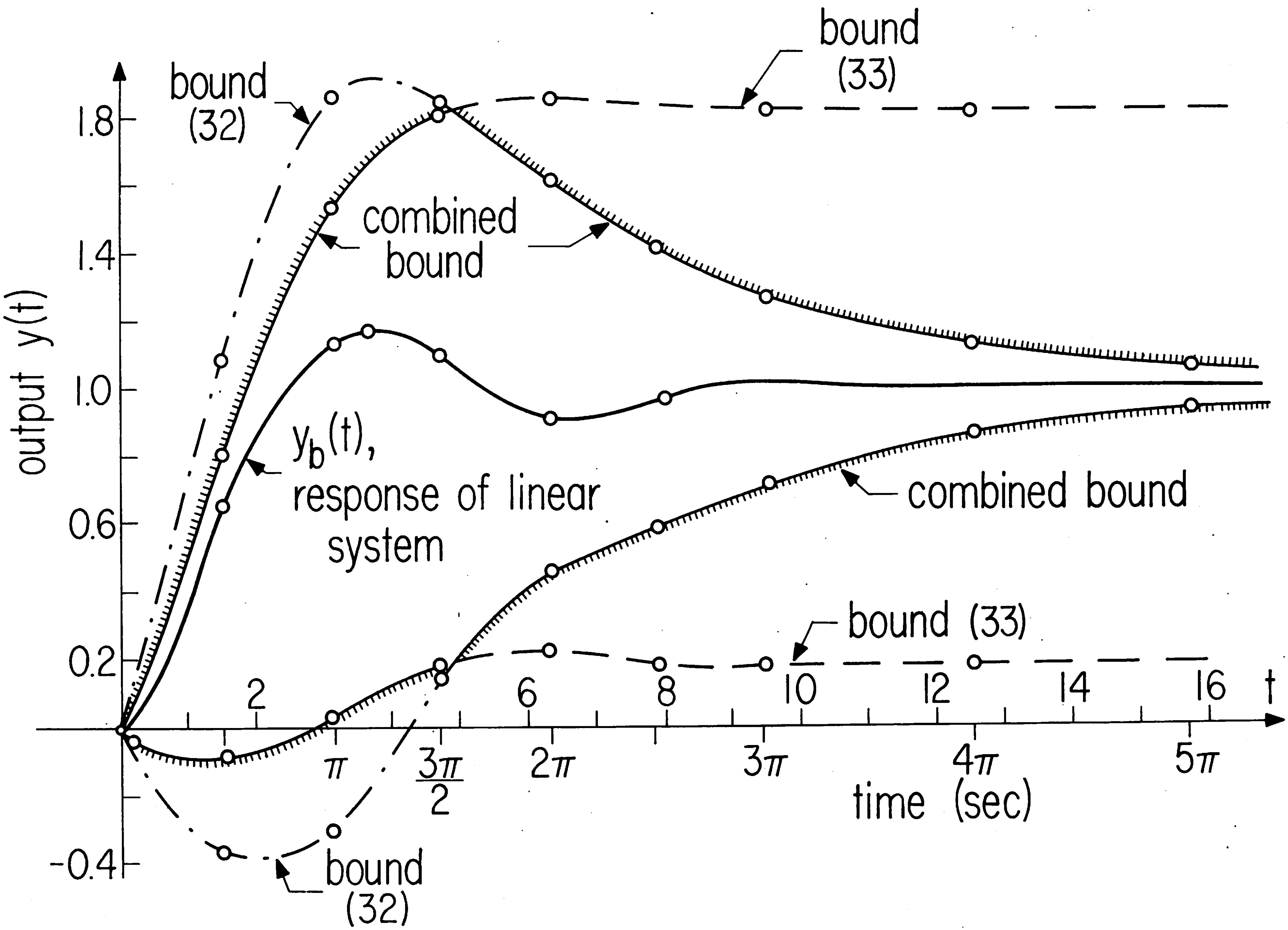

Z(t)
r(t)
+
σ(t)
N
u(t)
W(s)
+
+
y(t)
−



z(t)
1
1+aW(s)
Ñ
W̃(s)
z̃(t)
r(t)
r̃(t)
σ(t)
N
φ
φ̃
W(s)
c̃(t)
a
a
1
1+aW(s)



ImW
j2
j1
-j1
ReW
δ=4
δ=1
δ=0
δ=3
δ=1.5
δ=2
-1
-3
-2
0
1
2
α=0.375
α=0.25
α=0.125
α=0

bound (33)
bound (32)
combined bound
$y_b(t)$, response of linear system
combined bound
bound (33)
bound (32)
output y(t)
1.8
1.4
1.0
0.6
0.2
-0.4
2
6
8
10
12
14
16
t
π
3π/2
2π
3π
4π
5π
time (sec)