PERTURBATIONS OF OPTIMAL AND SUB-OPTIMAL CONTROL PROBLEMS

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Introduction

Computational convenience and measurement limitations prevent the use of exact equations representing physical phenomena. For this reason, it is important to ask under what conditions can we be confident that the approximations that are made will yield solutions which are close in some sense to those of the actual system.

This paper will examine two classes of optimal control problems and define suitable perturbations for which the infimums of the costs will be close for small perturbations. An investigation is also made to determine to what extent do the sets of admissible trajectories corresponding to a certain cost range of the perturbed problems differ from that of the original problem.

Problems similar to the above have been investigated by Markus [7], Cullum [1], and Hermes [5]. Markus considers the stability of a time

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optimal cost in cases where the system and its perturbations are linear, time-invariant and the optimal costs exist. The more general case of a nonlinear system and cost functional is considered by Cullum. She limits her class of perturbations so that the optimal costs exist for the original and perturbed problems. Then, under more restrictive conditions than those considered here, * she shows that perturbations of a given problem yield optimal costs which are close to that of the original problem. Cullums' primary investigation is concerned about the closeness of the optimal trajectories and controls. Hermes's results on these problems are special cases of the results of Cullum.

The investigations of these researchers suffer from a common defect in that they require the existence of an optimal control for the original as well as the perturbed problems. It seems evident, on looking at the available existence results, that "very few" problems have an optimal solution. Furthermore, because of computational difficulties, the actual control used is generally only an approximation to the "true" optimal control. In this paper, the existence requirements are removed and instead the stability of sub-optimal controls and trajectories is studied.

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* Cullum requires that the perturbed control constraint sets cover the control constraint sets for the original problem. She also assumes a local controllability condition which in general is difficult to verify. We develop sufficient conditions for the local controllability of perturbed problems.

I. Definition of Stability

Given a system of n equations (in vector notation)

$$\dot{\underline{x}} = \underline{f}(\underline{x}, u, t), \qquad f : R^n \times R^m \times R^1 \to R^n$$

where u(t) is the control parameter which, for any given t can take on values in a given set U(t).

The optimal control problem consists of the following: for a given initial set X_0, a final set X_1, find a measurable control $u(t) \in U(t)$ on some interval $t_0 \leq t \leq t_f$ such that $\underline{x}(t_0) \in X_0$, $\underline{x}(t_f) \in X_1$ and the cost $x^0 = x^0(t_f)$ determined by $x^0 = \int_{t_0}^{t_f} f^0(\underline{x}, u, t)\, dt$ is minimized. For convenience, we will consider the augmented system

$$\dot{x} = f(x, u, t)$$

where $x = (x^0, \underline{x})$, $f(x, u, t) = (f^0(\underline{x}, u, t), \underline{f}(\underline{x}, u, t))$.

An optimal control problem P is then specified by the following data:

$$P = \{f(x, u, t),\ U(t),\ X_0,\ X_1\}$$

Remark: If the target set X_1 is all of R^n, P will be called a free end point problem.

We will define a distance d between P and a perturbation of P, $\hat{P} = \{g(y, v, t),\ V(t),\ Y_0,\ Y_1\}$ and denote it by $d(P, \hat{P})$.

Definition: The problem P will be called stable if given any positive number ϵ, there exists a positive number δ such that

$$|\inf x^0 - \inf y^0| < \epsilon$$

whenever $d(P, \hat{P}) < \delta$.

In the sequel, we will consider subclasses of the problems P and define classes of admissible perturbations under which the problems P are stable.

II. An Approximation Theorem

For the problems P and $\hat{P}$ we define the "velocity sets" of P and $\hat{P}$ at the phase (x, t) as follows:

$$F(x,t) = \{z \mid z = f(x,u,t) \text{ for some } u \in U(t)\}$$

$$G(x,t) = \{z \mid z = g(x,v,t) \text{ for some } v \in V(t)\}$$

In this section we assume that the following conditions are satisfied for the problems P and $\hat{P}$:

(1) $f(x,u,t)$ is continuously differentiable in x, and continuous in u and t

$g(y,v,t)$ is continuously differentiable in y and continuous in u and t

(2) X_0, Y_0 are compact subsets of R^n

(3) $t \in [t_0, t_f] \subset T$ = a fixed finite open time interval in R^1

i.e., $T = (T_0, T_1)$

(4) $\langle x, f(x,u,t) \rangle \leq C[|x|^2 + 1]$ for all $u \in U(t)$, $t \in \overline{T}$

$\langle y, g(y,v,t) \rangle \leq C[|y|^2 + 1]$ for all $v \in V(t)$, $t \in \overline{T}$

where $||$ denotes the Euclidean norm $<,>$ denotes Euclidean inner product and $\overline{T}$ denotes the closure of T

(5) $U(t)$ and $V(t)$ are compact subsets of R^m for each $t \in \overline{T}$, and the maps $t \to U(t)$, $t \to V(t)$ are upper semi-continuous.

Remarks:

(i) Conditions 2, 3, 4 imply boundedness of the solutions of $\dot{x} = f(x,u,t)$ and $\dot{y} = g(y,v,t)$ i.e.,

$$|x(t)|^2 \leq [|x_0|^2 + 1] \exp[2C(T_1 - T_0)] \leq a^2$$

$$|y(t)|^2 \leq [|y_0|^2 + 1] \exp[2C(T_1 - T_0)] \leq b^2$$

or $x(t) \in B(0,a)$ = a closed ball of radius a about 0, and $y(t) \in B(0,b)$

(ii) Condition 4 may be replaced by the following condition:

(4') there exist functions $\mu \in L_\infty[T_0, T_1]$ and $\gamma \cdot R^1 \to R^1$ such that γ is bounded on bounded sets and $\gamma(s) = 0(s)$ as $s \to \infty$ such that

$$|f(x,u,t)| \leq \mu(t)\,\gamma(|x|) \text{ for all } u \in U(t), \quad t \in \overline{T}$$

$$|g(y,v,t)| \leq \mu(t)\,\gamma(|y|) \text{ for all } v \in V(t), \quad t \in \overline{T}$$

If 4' is satisfied, then there exist constants M and B such that for all x with $|x| \geq B$, $\gamma(|x|) \leq M|x|$. We can then show that $|x(t)| \leq B \exp M \int_{T_0}^{T_1} M(t)\,dt + 1 = a'$ and likewise that $|y(t)| \leq b'$

(iii) Conditions 1 - 4 imply that there exists a constant K such that

$$|f(x,u,t) - f(z,u,t)| \leq K|x-z|$$

and

$$|g(x,v,t) - g(z,v,t)| \leq K|x-z|$$

for all x and z in $B(0,a) \cup B(0,b)$ and $t \in \overline{T}$.

We are now in a position to define a metric on the problem space.

Definition:

$$d_1(P,\hat{P}) = \sup_{\substack{x \in B(0,\underline{a}) \cup B(0,b) \\ t \in \overline{T}}} d_H[F(x,t), G(x,t)] + d_H(X_0, Y_0)$$

where for any two compact subsets A, D of R^n

$$d_H(A,D) = \inf\{\delta \mid A \subset \bigcup_{x \in D} B(x,\delta) \text{ and } D \subset \bigcup_{x \in A} B(x,\delta)\}$$

It is easy to show that $d_1(P, \hat{P})$ is a metric.

Having defined a metric on the subclass of problems satisfying II. (1) - (5) we can now prove the following theorem regarding "admissible trajectories." A trajectory $x(t)$ for $t \in [t_0, t_f] \subset T$ will be called an admissible trajectory for a problem P if $\dot{x}(t) = f(x(t), u(t), t)$ for some measurable $u(t) \in U(t)$ and $\underline{x}(t_0) \in X_0$.

Theorem I.

Let conditions II. (1) - (5) be satisfied for the problem P and its perturbations $\hat{P}$. Then, given any admissible trajectory $x(t)$, $t \in [t_0, t_f] \subset T$ corresponding to the problem P and any $\epsilon > 0$, there exists a $\delta > 0$ such that for every problem P for which $d_1(P, \hat{P}) < \delta$ there exists an admissible trajectory $y(t)$ which satisfies $|x(t) - y(t)| < \epsilon$ for all $t \in [t_0, t_f]$.

Proof: Given any measurable $\dot{x}(t) \in F(x, t)$, $t \in [t_0, t_f] \subset T$ such that $x(t)$ is an admissible trajectory for problem P, let $w(t) = \{w_0(t), w_1(t)) \cdots, w_n(t)\} \in G(x(t), t)$ be defined in the following manner. For each $t \in [t_0, t_f]$ choose $w(t)$ such that $|\dot{x}(t) - w(t)| \leq \delta$, i.e., $w(t) \in B(\dot{x}(t), \delta)$. There exists at least one such value if $d_1(P, \hat{P}) < \delta$. If there is more that one such $w(t)$, for each value of t choose that $w(t)$ for which $w_0(t)$ has the smallest value. This exists since $B(\dot{x}(t), \delta) \cap G(x(t), t)$ is compact for each $t \in [t_0, t_f]$. If $w(t)$ is still not unique, choose that $w(t)$ for which $w_1(t)$ has the smallest value etc.

We shall first prove by induction that each of the functions $w_0(t), w_1(t), \cdots, w_n(t)$ is measurable so that $w(t)$ is measurable. Let us suppose that $w_0(t), \cdots, w_{s-1}(t)$ are measurable (if $s=1$, nothing need be assumed) and let us show that $w_s(t)$ is measurable.

By Lusin's Theorem [4], since $\dot{x}(t), w_0(t), \cdots w_{s-1}(t)$ are measurable, for each $\epsilon_0 > 0$ there exists a closed set $E(\epsilon_0) \subset [t_0, t_f]$ of measure greater than $t_f - t_0 - \epsilon_0$ such that the functions $x(t)$, $w_0(t)$, $\cdots, w_{s-1}(t)$ are continuous. We will show that for any real number a, the set of $t \in E(\epsilon_0)$ for which $w_s(t) \leq a$ is closed.

Suppose the contrary, then there exists a sequence $\{t_n\}_{n=1}^{\infty} \subset E(\epsilon_0)$ such that t_n converges to $\bar{t}$, and

$$(1) \qquad w_s(t_n) \leq a < w_s(\bar{t})$$

Since $G(x(t), t)$ is upper semicontinuous in x and t with respect to inclusion, $G(x(t), t)$ is uniformly bounded. By the Bolzano-Weierstrass Theorem, a subsequence t_m can be chosen from the t_n such that $w(t_m)$ converges to $\tilde{w}(\bar{t})$ with $w(t_m) \in B(\dot{x}(t_m), \delta) \cap G(x(t_m), t_m)$. Since $\dot{x}(t)$ is continuous for $t \in E(\epsilon_0)$, $B(\dot{x}(t), \delta)$ is upper semicontinuous in t for $t \in E(\epsilon_0)$. Hence, $\tilde{w}(\bar{t}) \in B(\dot{x}(\bar{t}), \delta) \cap G(x(\bar{t}), \bar{t})$.

It follows from (1) and the continuity of the functions $w_i(t)$, $i = 0, \cdots, s-1$ that

$$\tilde{w}_i(\bar{t}) = w_i(\bar{t}) \qquad i = 0, \cdots, s-1$$

and

(2) $$\tilde{w}_s(\bar{t}) \le a < w_s(\bar{t})$$

But (2) implies that $w_s(\bar{t})$ is not the smallest $w_s(\bar{t})$ which lies in $B(\dot{x}(\bar{t}), \delta) \cap G(x(\bar{t}), \bar{t})$. This contradicts the definition of $w_s(t)$ and hence $w_s(t)$ is measurable on $E(\epsilon_0)$. By defining the sequence

$$\begin{aligned} w^n(t) &= w(t) \quad \text{for} \quad t \in E(1/n) \\ &= 0 \quad\quad \text{otherwise}, \end{aligned}$$

it is easy to show that $w(t)$ is measurable for all $t \in [t_0, t_f]$. (see [4], p. 93)

Having shown that $w(t)$ is measurable, a lemma by Fillipov [3] may be used to show the existence of a measurable control $r(t) \in V(t)$ such that $w(t) = g(x(t), r(t), t)$. We now show that if $y(t)$ is the solution of the equation $\dot{y} = g(y, r(t), t)$ for some $\underline{y}(t_0) \in Y_0$, $|x(t) - y(t)| < \epsilon$ for all $t \in [t_0, t_f]$.

$$|x(t) - y(t)| \le \left| \int_{t_0}^{t} [\dot{x}(\tau) - \dot{y}(\tau)]\, d\tau \right| + |x(t_0) - y(t_0)|$$

$$\le \int_{t_0}^{t} |\dot{x}(\tau) - w(\tau)|\, d\tau + \int_{t_0}^{t} |w(\tau) - \dot{y}(\tau)|\, d\tau + |x(t_0) - y(t_0)|$$

(cont'd)

$$\leq \delta(t-t_0) + \int_{t_0}^{t} |g(x(\tau), r(\tau), \tau) - g(y(\tau), r(\tau), \tau)|\, d\tau + |x(t_0) - y(t_0)|$$

$$\leq \delta(t-t_0) + K \int_{t_0}^{t} |x(\tau) - y(\tau)|\, d\tau + |x(t_0) - y(t_0)|$$

Using the Bellman-Gronwall inequality ([9] p. 11) :

$$|x(t) - y(t)| \leq [\delta(t-t_0) + |x(t_0) - y(t_0)|] \exp K(t - t_0)$$

$$\leq [\delta(T_1 - T_0) + |x(t_0) - y(t_0)|] \exp K(T_1 - T_0) .$$

If $d_1(P, \hat{P}) < \delta$, then we can find a $\underline{y}(t_0) \in Y_0$ such that $|x(t_0) - y(t_0)| < \delta$. Hence,

$$|x(t) - y(t)| < \epsilon \qquad \text{for all} \qquad t \in [t_0, t_f] ,$$

where

$$\epsilon = \delta [(T_1 - T_0) + 1] \exp K(T_1 - T_0)$$

Note that this result holds uniformly for all admissible x(t) for problem P, since ϵ is independent of x(t).

<u>Corollary 1</u>

Let P and its perturbations $\hat{P}$ satisfy conditions II. (1) - (5) with the targets sets X_1 and Y_1 being all of R^n. P is then a stable problem.

Proof: Let $I(P)$, $I(\hat{P})$ denote the infimums of the costs for problems P and $\hat{P}$ respectively. Choose an admissible trajectory for the problem P so that $|x^0 - I(P)| < \epsilon/2$. By Theorem I, for $d_1(P, \hat{P})$ sufficiently small, we can find an admissible trajectory $\tilde{y}(t)$ for problem $\hat{P}$ such that $|x^0 - \tilde{y}^0| < \epsilon/2$, which implies that $|I(P) - \tilde{y}^0| < \epsilon$. By symmetry, there exists an admissible trajectory $\tilde{x}(t)$ for problem P such that $|I(\hat{P}) - \tilde{x}^0| < \epsilon$. Thus, $|I(P) - I(\hat{P})| < \epsilon$.

Remark: Theorem 1 clearly holds for fixed time problems, where the fixed time interval lies in T.

III. A Local Controllability Condition

In order to prove stability results for problems in which the target set is not all of R^n, a local controllability condition must be imposed. We first prove a controllability result which is an extension* of the work of Markus [8].

In this section, we will assume that the problems P and $\hat{P}$ satisfy II. (1) - (5) and in addition, that

(1) $\underline{f}(\underline{x}, u)$ and $\underline{g}(\underline{y}, v)$ do not depend explicitly on t and are continuously differentiable in $(\underline{x}, u)$ and $(\underline{y}, v)$ respectively

(2) $u(t) \in U$ where U is a compact subset of R^m

$v(t) \in V$ where V is a compact subset of R^m

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* Markus proves a controllability result for unperturbed problems about a single point, the origin.

(3) for each point $\underline{x}_1 \in X_1$ there exists a point $w_1 \in U$ such that

(i) $\underline{f}(\underline{x}_1, w_1) = 0$

(ii) U contains $m+1$ vectors $u_1, u_2, \cdots, u_{m+1}$ which span an m-simplex with w_1 in its interior and U also contains $w_1 + \epsilon(u_1 - w_1), \cdots, w_1 + \epsilon(u_{m+1} - w_1)$ for arbitrarily small $\epsilon > 0$, and

(iii) for every such pair $(\underline{x}_1, w_1)$, $\text{rank}[B, AB, A^2B, \cdots, A^{n-1}B] = n$

where

$$A = \frac{\partial \underline{f}}{\partial \underline{x}}(\underline{x}_1, w_1), \qquad B = \frac{\partial \underline{f}}{\partial u}(\underline{x}_1, w_1)$$

We also impose a stronger metric on the problem space, namely

$$d_2(P, \hat{P}) = \sup_{\mathcal{D}} |f(x,u) - g(x,u)| + \sup_{\mathcal{D}} \left| \frac{\partial \underline{f}(\underline{x}, u)}{\partial \underline{x}} - \frac{\partial \underline{g}(\underline{x}, w)}{\partial \underline{x}} \right|$$
$$+ \sup_{\mathcal{D}} \left| \frac{\partial \underline{f}(\underline{x}, u)}{\partial u} - \frac{\partial \underline{g}(\underline{x}, u)}{\partial u} \right| + d_H(X_0, Y_0) + d_H(X_1, Y_1) + d_H(U, V)$$

where $\mathcal{D} = \{(x,u) \mid x \in B(0,a) \cup B(0,b),\ u \in U \cup V\}$

<u>Remark:</u> $d_1(P, \hat{P}) \leq d_2(P, \hat{P})$.

<u>Theorem 2</u>.

If conditions II. (1) - (5) and III. (1) - (3) are fulfilled by P and its perturbations $\hat{P}$, given $\eta > 0$ there exists a $\delta > 0$ such that if

$d_2(P, \hat{P}) < \delta$ then there exist open neighborhoods about X_1, Y_1 such that each point in these neighborhoods can be steered by P and $\hat{P}$ to X_1 and Y_1 respectively in time not exceeding η.

Proof: Consider any point $\underline{x}_1 \in X_1$ and choose w_1 to satisfy III (1-3). Observe that the responses $\underline{x}_L(t)$ of the linear approximating system $\mathcal{L}: \dot{\underline{x}}_L = A(\underline{x}_L - \underline{x}_1) + B(u - w_1)$, $\underline{x}_L(0) = \underline{x}_1$, to controls $u(t) \in U$ for which $|u(t) - w_1| \leq \epsilon < 1$, defined on $0 \leq t \leq 1$ * satisfy the bound $|\underline{x}_L(t) - \underline{x}_1| \leq K_0 \epsilon$ for some $K_0 < \infty$, K_0 independent of ϵ.

Construct an m-simplex $\overline{W}$ about w_1 with vertices $\overline{u}_1, \cdots, \overline{u}_{m+1}$ satisfying III. 3(ii). Let $\mathcal{K}$ be the set of attainability for $\mathcal{L}$ at time $t=1$ for solutions starting at $\underline{x}_1$, with controls $u(t)$ in $\overline{W}$. $\mathcal{K}$ is a convex set which contains $\underline{x}_1$ in its interior. By the theory of bang-bang controls [6], every point of $\mathcal{K}$ can be attained by responses of $\mathcal{L}$ to controls which assume only the $m+1$ values of the vertices of $\overline{W}$. Note that these controls are admissible. We will call this set of measurable controls $U_{\overline{W}}$. Let $\overline{u}_1(t), \cdots, \overline{u}_{n+1}(t)$ be such controls whose corresponding linear responses determine the vertices $\overline{\underline{x}}_{L,1}(1), \cdots, \overline{\underline{x}}_{L,n+1}(1)$ of an n-simplex $\overline{S}$ which contains $\underline{x}_1$ in its interior. Denote the inscibed radius of $\overline{S}$ by $c_1 > 0$.

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* The interval $0 \leq t \leq 1$ is chosen for notational convenience, $0 \leq t \leq \eta/2$ could be chosen as well.

Take barycentric coordinates $\alpha = (\alpha_1, \cdots, \alpha_{n+1})$ in $\bar{S}$. Using a lemma by Markus [8] we can obtain a mapping $\bar{u}(t, \alpha)$ of $\bar{S}$ into $U_{\bar{w}}$ which is continuous in the $L_1[0,1]$ topology. Markus also shows that the composite mapping $\alpha \rightarrow \bar{\underline{x}}_L(1, \alpha)$ of $\bar{S}$ into $U_{\bar{w}}$ and $U_{\bar{w}}$ into R^n by $\mathcal{L}$ is the identity map.

We can repeat this entire construction with the simplex W defined by the vertices $w_1 + \epsilon(\bar{u}_1 - w_1), \cdots, w_1 + \epsilon(\bar{u}_{m+1} - w_1)$, $\epsilon > 0$ being arbitrarily small. The control family $u(t, \alpha) = w_1 + \epsilon(\bar{u}(t, \alpha) - w_1)$ determines linear responses $\underline{x}_L(t, \alpha) = \underline{x}_1 + \epsilon(\bar{\underline{x}}_L(t, \alpha) - \underline{x}_1)$. If α are the barycentric coordinates of $S = \underline{x}_1 + \epsilon(\bar{S} - \underline{x}_1)$, we find that $\alpha \rightarrow \underline{x}_L(1, \alpha)$ is the identity map of S into itself.

Since $d_H(U, V) < \delta$, for each vertex $w_1 + \epsilon(\bar{u}_i - w_1)$, $i = 1, \cdots, m+1$ of W there exists a point v_i, $i = 1, 2, \cdots, m+1$ of V such that the euclidean distance between these points is less than δ. Let δ be small enough, so that these points are distinct. Let V_w be the space of measurable controllers which take on values only at these $m+1$ points of V. Define a mapping from U_w into V_w which is continuous in the $L_1[0,1]$ topology as follows: when $u(t, \alpha) \in U_w$ takes on the value $w_1 + \epsilon(\bar{u}_K - w_1)$, let $v(t, \alpha) \in V_w$ take on the value of v_K. Hence, the composite map from S into V_w denoted by $v(t, \alpha)$ is continuous in the $L_1[0,1]$ topology.

Let $\underline{y}(1, \alpha)$ be the solution at $t = 1$ of the differential equation $\dot{\underline{y}} = \underline{g}(\underline{y}, v(t, \alpha))$, $\underline{y}(0) = \underline{x}_1$. It is easy to show that $\underline{y}(1, \alpha)$ is a continuous function of α. We now show that this map approximates the identity map on the boundary of the ball about $\underline{x}_1$ with radius $\epsilon\, c_1$ (which is the inscribed radius of S)

$$|\underline{y}(t,\alpha) - \underline{x}_L(t,\alpha)| \le \int_0^t |\underline{g}(\underline{y}(\tau,\alpha),\ v(\tau,\alpha)) - A(\underline{x}_L(\tau,\alpha) - \underline{x}_1)$$

$$- B(u(\tau,\alpha) - w_1)|\, d\tau$$

$$\le \int_0^t |\underline{g}(\underline{y}(\tau,\alpha),\ v(\tau,\alpha)) - \underline{g}(\underline{x}_L(\tau,\alpha),\ v(\tau,\alpha))|\, d\tau$$

$$+ \int_0^t |\underline{g}(\underline{x}_L(\tau,\alpha),\ v(\tau,\alpha)) - A(\underline{x}_L(\tau,\alpha) - \underline{x}_1)$$

$$- B(u(\tau,\alpha) - w_1)|\, d\tau$$

Define: $\hat{A} = \dfrac{\partial \underline{g}}{\partial \underline{y}}(\underline{x}_1, w_1)$, $\qquad \hat{B} = \dfrac{\partial \underline{g}}{\partial v}(\underline{x}_1, w_1)$

Then, for ϵ, δ small enough,

$$|\underline{y}(t,\alpha) - \underline{x}_L(t,\alpha)| \le K \int_0^t |\underline{y}(\tau,\alpha) - \underline{x}_L(\tau,\alpha)|\, d\tau$$

$$+ \int_0^t |\hat{A}(\underline{x}_L(\tau,\alpha) - \underline{x}_1) + \hat{B}(v(\tau,\alpha) - w_1)$$

$$- A(\underline{x}_L(\tau,\alpha) - \underline{x}_1) - B(u(\tau,\alpha) - w_1)|\, d\tau$$

(cont'd)

$$+ \int_0^t |\underline{g}(\underline{x}_1, w_1)| \, d\tau + \int_0^t C_2(\epsilon)[|\underline{x}_L(\tau, \alpha) - \underline{x}_1|$$

$$+ |v(\tau, \alpha) - w_1|] \, d\tau$$

where

$$\lim_{\epsilon \to 0} C_2(\epsilon) = 0$$

Therefore,

$$|\underline{y}(t, \alpha) - \underline{x}_L(t, \alpha)| \leq K \int_0^t |\underline{y}(\tau, \alpha) - \underline{x}_L(\tau, \alpha)| \, d\tau$$

$$+ \int_0^t |A - \hat{A}| \, |\underline{x}_L(\tau, \alpha) - \underline{x}_1| \, d\tau + \int_0^t |B - \hat{B}| \, |v(\tau, \alpha) - w_1| \, d\tau$$

$$+ \int_0^t \underline{g}(\underline{x}_1, w_1) \, d\tau + \int_0^t |B| \, |u(\tau, \alpha) - v(\tau, \alpha)| \, d\tau$$

$$+ \int_0^t C_2(\epsilon) [|\underline{x}_L(\tau, \alpha) - \underline{x}_1| + |v(\tau, \alpha) - w_1|] \, d\tau$$

$$\leq K \int_0^t |\underline{y}(\tau, \alpha) - \underline{x}_L(\tau, \alpha)| \, d\tau + \delta K_0 \epsilon + \delta(\delta + \epsilon)$$

$$+ \delta + \delta |B| + C_2(\epsilon)[K_0 \epsilon + \delta + \epsilon] \quad \text{for} \quad 0 \leq t \leq 1$$

Using the Bellman-Gronwall lemma:

$$|\underline{y}(1,\alpha) - \underline{x}_L(1,\alpha)| \leq \{\delta[K_0\epsilon + \delta + \epsilon + 1 + |B| + C_2(\epsilon)]$$

$$+ C_2(\epsilon)[K_0\epsilon + \epsilon]\} \exp K$$

Choosing ϵ, δ small enough, we have

$$|\underline{y}(1,\alpha) - \underline{x}_L(1,\alpha)| \leq \frac{c_1}{2}\epsilon$$

By the Brouwer fixed point theorem [2] we conclude that the image of S, by the nonlinear response $\underline{y}(1,\alpha)$ covers an open ball of $\underline{x}_1$, B_{x_1} in R^n.

Now consider the system $\dot{\hat{\underline{y}}} = \hat{\underline{g}}(\hat{y},v) = -\underline{g}(\hat{y},v)$ which also satisfies the conditions of the theorem. Hence there exists an open ball neighborhood $\hat{B}_{x_1}$ of $\underline{x}_1$ covered by the responses of this system starting at $\underline{x}_1$. If $\hat{v}(t)$ steers $\hat{\underline{y}}(t)$ from $\hat{\underline{y}}(0) = \underline{x}_1$ to some point $\hat{\underline{y}}(1)$ in $\hat{B}_{x_1}$, then $v(t) = \hat{v}(1-t)$ steers $\underline{y}(t) = \hat{\underline{y}}(1-t)$ by the original process from $\underline{y}(0) = \hat{\underline{y}}(1)$ to $\underline{y}(1) = \underline{x}_1$. Starting from points in $\hat{B}_{x_1}$ we can reach $\underline{x}_1$ in time 1 by the system $\dot{\underline{y}} = \underline{g}(\underline{y},v)$ with controls $v(t) \in V$. Let $\tilde{B}_{x_1} = B_{x_1} \cap \hat{B}_{x_1}$.

Clearly, this construction can be repeated for each $\underline{x}_1 \in X_1$ by choosing δ appropriately small. Since X_1 is compact, a finite number of the $\tilde{B}_{x_1}$'s will cover X_1. Call this cover N. Let r be the minimum radius of the $\tilde{B}_{x_1}$'s which are elements of N. Let δ be so small that $Y_1 \subset N$ and $\delta < r$. Starting from any point in N, we can reach a point $\underline{x}_1 \in X_1$ in time 1. Since $\delta < r$, we can reach Y_1 from this point in time 1. Thus, starting from N we can reach Y_1 in time 2 and X_1 in time 1 provided δ is sufficiently small.

Remark: Theorem 2 is valid for time varying problems, $\dot{\underline{x}} = \underline{f}(\underline{x}, u, t)$, $u(t) \in U$ etc. if instead of assumption (3) (iii) we require that the linear time-varying system $\dot{\underline{x}} = A(t)(\underline{x} - \underline{x}_1) + B(t)(u - w_1)$ be totally controllable for $t \in T$, where $\underline{x}_1$, w_1 satisfy $\underline{f}(\underline{x}_1, w_1, t) = 0$ for all $t \in T$.

Theorems (1) and (2) can be used to yield a stability result for the case where the target set is not all of R^n.

Corollary 2

If P and its perturbations $\hat{P}$ satisfy the conditions of Theorems 1 and 2 and in addition there exists at least one admissible trajectory x(t), $t \in [t_0, t_f] \subset T$ for problem P such that $\underline{x}(t_f) \in X_1$, then problem P is stable.

Proof: Let I(P), $I(\hat{P})$ denote the infimums of the costs for problems P and $\hat{P}$ respectively. Choose a trajectory $\tilde{x}(t)$ for problem P so that

$$|\tilde{x}^0 - I(P)| < \epsilon/4$$

Let $t_f < T_1$ be the final time for the trajectory $\tilde{x}(t)$. Given any $\hat{\epsilon} > 0$, there exists (by Theorem 1) a $\delta > 0$ and a trajectory $\tilde{y}(t)$ for $\hat{P}$ such that

$$|\tilde{x}(t_f) - \tilde{y}(t_f)| < \hat{\epsilon}$$

Choosing $\hat{\epsilon}$ and δ small enough and using Theorem 2 we can insure that $\tilde{\underline{y}}(t_f)$ can be steered to Y_1 in a time not exceeding ϵ/K, $K = \frac{1}{4} \sup_{\mathcal{D}} |g^0(y,\underline{v})|$. Pick one such trajectory to extend the definition of $\tilde{y}(t)$ for $t > t_f$.

Hence,

$$\begin{aligned} |I(P) - \tilde{y}^0| &\leq |I(P) - \tilde{x}^0| + |\tilde{x}^0 - \tilde{y}^0| \\ &\leq \epsilon/4 + |\tilde{x}^0 - \tilde{y}^0(t_f)| + |\tilde{y}^0(t_f) - \tilde{y}^0| \\ &\leq \epsilon/4 + \hat{\epsilon} + \left| \int_{t_f}^{t_f+\epsilon/K} g^0(\tilde{\underline{y}}, \tilde{v})\, dt \right| \\ &\leq \epsilon/4 + \hat{\epsilon} + \epsilon/4 = \epsilon/2 + \hat{\epsilon} \end{aligned}$$

Let $\delta > 0$ be so small that $\hat{\epsilon} < \epsilon/2$, we then have

$$|I(P) - \tilde{y}^0| \leq \epsilon$$

Likewise we can show that for δ sufficiently small there exists a trajectory $\hat{x}(t)$ for problem P such that

$$|I(\hat{P}) - \hat{x}^0| \leq \epsilon$$

Thus, $|I(P) - I(\hat{P})| \leq \epsilon$ for δ sufficiently small.

It is easy to find examples which violate the conditions of Theorem 2 and which are unstable.

Example: Let P be specified by

$$\dot{x}^0 = 1$$

$$\dot{x}^1 = u \quad , \quad x^1(0) = 1$$

$$\dot{x}^2 = X^2 \quad , \quad x^2(0) = 0$$

$$U = [-1, 1] \,, \quad X_1 = \{(0, 0)\}$$

with $\hat{P}$ given by

$$\dot{y}^0 = 1$$

$$\dot{y}^1 = v \quad , \quad y^1(0) = 1$$

$$\dot{y}^2 = y^2 \quad , \quad y^2(0) = \delta$$

$$U = [-1, 1] \quad Y_1 = \{(0, 0)\}$$

Notice that $\inf x^0 = 1$, but $\hat{P}$ has infinite cost, since the target point cannot be reached in finite time.

IV. Stability with Regard to Trajectories

A desirable feature of optimal control problems would be that the sets of trajectories for a given range of costs of the perturbed problems are close to the of the original problem. To make this motion precise, we define

(i) $\mathcal{X}(P,\epsilon) = \{ z(t) | z(t)$ is an admissible trajectory for problem P satisfying all boundary conditions and $z^0\text{-inf } z^0 \leq \epsilon \}$

(ii) $d(\mathcal{X}(P,\epsilon), \mathcal{X}(\hat{P},\hat{\epsilon}))$ = the Hausdorff distance between the sets $\mathcal{X}(P,\epsilon)$ and $\mathcal{X}(\hat{P},\hat{\epsilon})$ relative to the following distance between elements of $\mathcal{X}(P,\epsilon)$ and $\mathcal{X}(\hat{P},\hat{\epsilon})$: let $x(t)$, $t \in [t_{0_x}, t_{f_x}] = T_x$ be an element of $\mathcal{X}(P,\epsilon)$ and $y(t)$, $t \in [t_{0_y}, t_{f_y}] = T_y$ be an element of $\mathcal{X}(\hat{P},\hat{\epsilon})$ then,

$$||x(t) - y(t)||_* = \sup_{t \in T_x \cap T_y} |x(t) - y(t)| + |t_{f_x} - t_{f_y}| + |t_{0_x} - t_{0_y}|$$

(iii) $d((P,\epsilon), (\hat{P},\hat{\epsilon})) = d_i(P,\hat{P}) + |\epsilon - \hat{\epsilon}|$ where $d_i(P,\hat{P}) = d_1(P,\hat{P})$ for free end point problems satisfying the hypotheses of Theorem 1 $d_i(P,\hat{P}) = d_2(P,\hat{P})$ for fixed end point problems satisfying the the hypotheses of Theorems 1 and 2.

With these definitions, we would like that $\mathcal{X}(P,\epsilon)$ considered as a mapping of sets into sets be continuous in ϵ and P with respect to the corresponding metrics. Unfortunately, this is in general not true.

<u>Example 1:</u> Let P be specified by

$$\dot{x}^0 = -x^1 + \sqrt{2\epsilon}\,, \quad x^0(0) = 0$$

$$\dot{x}^1 = u \quad , \quad x^1(0) = 0 \qquad , \quad X_1 = R^n$$

$$U = [-1,1], \; t \epsilon \, (-2\sqrt{2\epsilon}\,,\; 2\sqrt{2\epsilon}\,)$$

and $\hat{P}$ is specified by:

$$\dot{y}^0 = -y^1 + \sqrt{2\epsilon} \qquad y^0(0) = 0$$

$$\dot{y}^1 = v \qquad y^1(0) = -\delta\,, \quad \delta > 0 \qquad , \; Y_1 = R^n$$

$$V = [-1, 1]\,, \qquad t \epsilon \, (-2\sqrt{2\epsilon}\,,\; 2\sqrt{2\epsilon}\,)$$

Then, $\inf x^0 = \inf y^0 = 0$. Consider the trajectory $x^1 = t$, $0 \le t \le \sqrt{2\epsilon}$ generated by $u(t) = 1$, $0 \le t \le \sqrt{2\epsilon}$ with cost $x^0 = x^0(\sqrt{2\epsilon}\,) = \epsilon$. Observe that for all $\delta > 0$, we cannot find an admissible trajectory $y(t)$ for problem $\hat{P}$ such that $\|x(t) - y(t)\|_* < \epsilon$ and $y^0 - \inf y^0 < \epsilon$.

<u>Example 2:</u>

Let P be specified by

$$\dot{x}^0 = 1 \quad , \quad x^0(0) = 0$$

$$\dot{x}^1 = u \quad , \quad x^1(0) = 0$$

$$U = [-1, 1]\,, \qquad X_1 = \{1\} \cup \{1+c\},$$

c is a positive constant.

Then, it is easily seen that there exists no $\delta > 0$ such that if $|c - \epsilon| < \delta$, $\epsilon < c$ then $d(\mathcal{X}(P,c), \mathcal{X}(P,\epsilon)) < c$.

Example 3: Let P be a free end point, fixed time problem specified by:

$$\dot{x}^0 = -2|u| + 1 \quad , \quad x^0(0) = 0$$

$$\dot{x}^1 = u \quad , \quad x^1(0) = 0$$

$$U = [-1/\sqrt{2}, 1] \quad , \quad \text{final time is } 1$$

with $\hat{P}$ specified by

$$\dot{y}^0 = -2|v| + 1 \quad , \quad y^0(0) = 0$$

$$\dot{y}^1 = v \quad , \quad y^1(0) = 0$$

$$V = [-1/\sqrt{2} + \delta, 1] \quad , \quad \delta > 0$$

$$\inf x^0(1) = \inf y^0(1) = 0$$

Consider the trajectory $x(t) = -1/t\sqrt{2}$, $0 \leq t \leq 1$. Clearly, there exists no $\delta > 0$ such that if $d((P, 1/2), (\hat{P}, 1/2)) \leq \delta$ then there exists an admissible trajectory $y(t)$ for problem $\hat{P}$ such that $||x(t) - y(t)|| \leq 1/10$ and $y^0 - \inf y^0 \leq 1/2$.

The simplicity of these counter examples suggests that it will be extremely difficult to find conditions under which $\chi(P, \epsilon)$ is continuous. A weaker type of stability with regard to trajectories can however be easily demonstrated through the use of Theorems 1 and 2.

Theorem 3

Let the conditions of Theorem 1 (Theorem 2) be satisfied for the free end point (fixed end point) Problem P. Given $\epsilon_2 > \epsilon_1 > 0$, $\epsilon_3 > 0$ there exists a $\delta > 0$ such that $\mathcal{X}(P, \epsilon_1)$ lies in an ϵ_3 neighborhood of $\mathcal{X}(\hat{P}, \epsilon_2)$ and $\mathcal{X}(\hat{P}, \epsilon_1)$ lies in an ϵ_3 neighborhood of $\mathcal{X}(P, \epsilon_2)$ provided $d_i(P, P) < \delta$.

Proof: By Theorem 1 (Theorem 2), for any admissible trajectory $x(t) \in \mathcal{X}(P, \epsilon_1)$ there exists a $y(t)$ for problem $\hat{P}$ satisfying all boundary conditions and a $\delta > 0$ such that

$$||x(t) - y(t)||_* < \min\left(\frac{\epsilon_2 - \epsilon_1}{2}, \ \epsilon_3\right)$$

provided $d_i(P, \hat{P}) < \delta$. Note that δ can be chosen independent of $x(t)$.

Moreover,

$$|y^0 - \inf y^0| \leq |y^0 - \inf x^0| + |\inf x^0 - \inf y^0|$$

$$\leq |y^0 - x^0| + |x^0 - \inf x^0| + |\inf x^0 - \inf y^0|$$

$$\leq \frac{\epsilon_2 - \epsilon_1}{2} + \epsilon_1 + \frac{\epsilon_2 - \epsilon_1}{2}$$

$$\leq \epsilon_2 .$$

Hence, $y(t) \epsilon \mathcal{X}(\hat{P}, \epsilon_2)$ and therefore $\mathcal{X}(P, \epsilon_1)$ is in an ϵ_3 neighborhood of $\mathcal{X}(\hat{P}, \epsilon_2)$. By a similar argument, $\mathcal{X}(\hat{P}, \epsilon_1)$ is in an ϵ_3 neighborhood of $\mathcal{X}(P, \epsilon_2)$.

Conclusions

In this paper we have attempted to explore the dependence of the solutions to suboptimal control problems upon the formulation of the problem. We have indicated that terget requirements contribute heavily to the instability of the posed problems, i.e., if a system is not locally controllable, instability may often result. It is the feeling of the authors therefore, that as far as possible, target requirements should be replaced by modifying the cost function because this usually makes the problem stable and furthermore one is rarely interested in meeting target requirements exactly.

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