RIEMANN-STIELTJES APPROXIMATIONS OF STOCHASTIC INTEGRALS

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1. Introduction

Let $x(\omega, t)$ $t \geq 0$ be a separable Brownian motion defined on a fixed, but as yet unspecified, probability space $(\Omega, \mathcal{A}, \mathcal{P})$. Because a Brownian motion is almost surely of unbounded variation, integrals of the form

$$(1) \qquad I(\Phi) = \int_0^1 \Phi(\omega, t)\, d_t\, x(\omega, t)$$

require special definition. One definition, and until recently the only definition, is that due to Ito, and will be referred to as the <u>stochastic integral</u> in this paper. The definition of a stochastic integral proceeds as follows: [1, Chap. 9, 2 Chap. 7].

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Let $\Phi(\cdot, \cdot)$ satisfy

(A) Φ is a (ω, t) function measurable with respect to $\mathcal{A} \times \mathcal{B}$ and for each t $\Phi(\cdot, t)$ is $\mathcal{A}_t$ measurable, where $\mathcal{A}_t$ is the smallest sub-σ-algebra of ω sets with respect to which $\{x(\omega, s), \ s \leq t\}$ are all measurable, and $\mathcal{B}$ is the σ-algebra of one-dimensional Lebesgue measurable sets.

(B) $$\int_0^1 |\Phi(\omega, t)|^2 \, dt < \infty \quad \text{for almost all } \omega$$

or

(B') $$\int_0^1 E|\Phi(\omega, t)|^2 \, dt < \infty.$$

The stochastic integral is first defined for Φ functions which are step functions in t for almost all ω by the Riemann sum

(2) $$I(\Phi) = \sum_{\nu=1}^{N} \Phi_\nu(\omega) \left(x(\omega, t_{\nu+1}) - x(\omega, t_\nu)\right).$$

For more general Φ, let Φ_n be a sequence of step functions such that

$$\int_0^1 |\Phi(\omega, t) - \Phi_n(\omega, t)|^2 \, dt \xrightarrow[n \to \infty]{} 0 \text{ almost all } \omega$$

or

$$\int_0^1 E|\Phi(\omega,t) - \Phi_n(\omega,t)|^2 \, dt \xrightarrow[n\to\infty]{} 0$$

according to whether (B) or (B') is satisfied. The stochastic integral $I(\Phi)$ is then defined as the limit in probability (resp. limit in quadratic mean) of $I(\Phi_n)$.

While the definition of a stochastic integral is entirely self-consistent, it need not have any connection with ordinary integrals. Indeed, as is shown by the familiar example [1, p. 444].

$$(3) \qquad \int_0^1 x(\omega,t)\, d_t\, x(\omega,t) = \frac{1}{2}\left[x^2(\omega,1) - x^2(\omega,0)\right] - \frac{1}{2},$$

a calculus based on the stochastic integral cannot be entirely compatible with that corresponding to ordinary integrals which must surely yield $\int_0^1 x(t)\,dx(t) = \frac{1}{2}[x^2(1) - x^2(0)]$. These considerations motivated Stratonovich [3] to suggest a symmetrized definition for (1), which resulted in a calculus compatible with ordinary calculus. In a similar vein we have suggested in earlier papers [4, 5] that in applications one is frequently concerned with the limit of a sequence of Riemann-Stieltjes integrals resembling a stochastic integral but with a sequence of "smooth" approximations $\{x_n(\omega,t)\}$ replacing the Brownian motion $x(\omega,t)$. It was found that this limit, when it exists, differs in general from the stochastic integral having the same form. For example, if $\{x_n(\omega,t)\}$ have piecewise continuous t derivatives, then clearly

$$\int_0^1 x_n(\omega, t)\, d_t\, x_n(\omega, t) = \frac{1}{2}\, [x_n^2(\omega, 1) - x_n^2(\omega, 0)]$$

$$\xrightarrow[n \to \infty]{} \frac{1}{2}\, [x^2(\omega, 1) - x^2(\omega, 0)]$$

which differs from (3) by a "correction term" equal to 1/2. These earlier papers [4, 5] established the relationship between the limits of such sequences of Riemann-Stieltjes integrals and the corresponding stochastic integrals. However, these results as well as those of Stratonovich [3] were restricted to two special cases:

(a) $\Phi(\omega, t) = F(x(\omega, t), t)$

(b) $\Phi(\omega, t) = F(y(\omega, t), t)$, and $y(\omega, t)$ is a diffusion process process related to $x(\omega, t)$ through a stochastic differential equation.

This paper extends the results of [4, 5] in considering more general integrands $\Phi(\omega, t)$, while retaining the idea of approximating the Brownian motion by differentiable processes. It will be shown that the "correction term" between the limit of a sequence of Riemann-Stieltjes integrals and the corresponding stochastic integral can be expressed in terms of the Frechét differential of $\Phi(\cdot, t)$. In those special cases where the earlier results [3, 4, 5] apply, results of this paper reduce accordingly.

2. A Statement of the Problem

For integrands of the form $\Phi(\omega, t) = F(x(\omega, t), t)$ or $\Phi(\omega, t) = F(y(\omega, t), t)$, an approximation of $x(\omega, t)$ by $x_n(\omega, t)$ induces automatically an approximation $\Phi^{(n)}(\omega, t) = F(x_n(\omega, t), t)$ or $\Phi^{(n)}(\omega, t) = F(y_n(\omega, t), t)$. One of the difficulties in extending our earlier results [4, 5] is that it is unclear as how $\Phi(\omega, t)$ is to be affected in general by an approximation of the Brownian motion. Roughly speaking, the dependence of $\Phi(\omega, t)$ on the sample function $x(\omega, \cdot)$ must be kept the same, while $x(\omega, \cdot)$ undergoes an approximation. The approach taken here in overcoming this difficulty is to choose the basic space Ω in such a way that approximating the sample functions of the Brownian motion is equivalent to approximating elements of Ω, thus inducing an approximation of $\Phi(\omega, t)$ in a natural way.

Let $\Omega = C[0,1]$ be the space of all continuous real valued functions defined on $[0,1]$, and denote by $x(\omega, t)$ the value of ω at t. Let $\mathcal{A}$ be the σ-algebra of Borel (= Baire) sets with respect to the (uniform) topology induced by the norm

$$(4) \qquad ||\omega|| = \max_{0 \leq t \leq 1} |x(\omega, t)|$$

It is well known [6, 7] that the finite dimensional distributions of a standard Brownian motion (Gaussian, zero-mean, $\text{cov}(s,t) = \min(s,t)$) can be uniquely extended to a measure $\mathcal{P}$ on $(\Omega, \mathcal{A})$, and this is the Wiener measure. Defined in this way, $x(\omega, t)$ is necessarily separable. In what follows, we denote by $\mathcal{B}$ the class of Lebesgue measurable sets and $\mu(\cdot)$ the Lebesgue measure. Almost surely (a.s.) shall mean either for all (ω, t) except a set of $\mathcal{P} \times \mu$ measure zero, or for all ω except a set of $\mathcal{P}$ measure zero; which one it is is always clear from the context. Now, let $\Phi(\omega, t)$ satisfy the following hypotheses.

H_1: Φ is a complex valued (ω, t) function measurable with respect to $\mathcal{A} \times \mathcal{B}$ and for each t $\Phi(\cdot, t)$ is $\mathcal{A}_t$ measurable, where $\mathcal{A}_t \subset \mathcal{A}$ is the smallest σ-algebra with respect to which $\{x(\omega, s),\ s \leq t\}$ are all measurable.

H_2: For each $(\omega, t) \in \Omega \times [0,1]$, there exists a unique continuous linear functional $F(\cdot, \omega, t)$ on Ω such that

(5) $$|\Phi(\omega+\omega', t) - \Phi(\omega, t) - F(\omega'; \omega, t)| \leq K||\omega'||^{1+\alpha}(1 + ||\omega||^{\beta} + ||\omega||^{\beta})$$

where K, α, β are finite positive constants independent of ω, ω', t. The linear functional $F(\cdot, \omega, t)$, which is necessarily the Fréchet differential of $\Phi(\cdot, t)$ at ω, admits the Riesz representation

(6) $$F(\omega'; \omega, t) = \int_0^1 x(\omega', s)\, d_s f(s; \omega, t)$$

where $f(\cdot, \omega, t)$ has bounded variation.

H_3: $$\int_0^1 |d_s f(s; 0, t)| \leq K < \infty$$

$$|\Phi(0, t)| \leq K < \infty$$

where K may be assumed to be the same as that in (5) with no loss of generality. A function $\Phi(\cdot,\cdot)$ which satisfies H_1, H_2 and H_3 can be shown to satisfy conditions A and B' of the introduction. Hence, the stochastic integral $\int_0^1 \Phi(\omega, t)\, d_t\, x(\omega, t)$ is well defined as a quadratic-mean limit. Furthermore, a sequence $\omega^n(\omega) \in \Omega$ can be so chosen that

$$P_1: \qquad ||\omega^n - \omega|| \xrightarrow[n\to\infty]{} 0$$

P_2: $\quad x(\omega^n, t)$ has piecewise continuous t-derivative

and

$$P_3: \qquad \int_0^1 \Phi(\omega^n(\omega), t)\, d_t\, x(\omega^n(\omega), t) \xrightarrow[n\to\infty]{q.m.}$$

$$\int_0^1 \Phi(\omega, t)\, d_t\, x(\omega, t) + \frac{1}{2} \int_0^1 \Psi(\omega, t)\, dt\ .$$

In P_3, the integral $\int_0^1 \Phi(\omega, t)\, d_t\, x(\omega, t)$ is a stochastic integral, but $\int_0^1 \Phi\, d_t\, x(\omega^n(\omega), t)$ is an ordinary integral because of P_2. The function $\Psi(\omega, t)$ is defined by

$$(7) \qquad \Psi(\omega, t) = f(t^+; \omega, t) - f(t^-; \omega, t)\ .$$

Proposition P_3 is the main result of this paper and extends the results of [4, 5], especially [4].

The details of the proof of our main result is not particularly illuminating as to how the correction term $\frac{1}{2}\int_0^1 \Psi(\omega, t)\, dt$ arises. It may be worthwhile to give a heuristic explanation for it. The Ito definition of a stochastic integral is basically one involving forward difference approximation, i.e.,

$$\int_t^{t+\Delta} \Phi(\omega, t')\, d_{t'}\, x(\omega, t') \sim \Phi(\omega, t)[x(\omega, t+\Delta) - x(\omega, t)].$$

Suppose we consider instead a backward approximation

$$\int_t^{t+\Delta} \Phi(\omega, t')\, d_{t'}\, x(\omega, t') \sim \Phi(\omega, t+\Delta)[x(\omega, t+\Delta) - x(\omega, t)],$$

the difference between the two is $[\Phi(\omega, t+\Delta) - \Phi(\omega, t)][x(\omega, t+\Delta) - x(\omega, t)]$. For a $\Phi(\cdot, \cdot)$ satisfying H_1, H_2, H_3, $\Phi(\omega, t+\Delta) - \Phi(\omega, t) \sim [x(\omega, t+\Delta) - x(\omega, t)]\Psi(\omega, t) + 0(\Delta)$, hence the difference between a forward approximation and a backward approximation is $\sim \Psi(\omega, t)[x(\omega, t+\Delta) - x(\omega, t)]^2 + o(\Delta) \sim \Psi(\omega, t)\Delta$. The factor 1/2 in P_3 represents an average of these two approximations.

3. Proof of the Main Result

First, some simply verifiable consequences of H_1, H_2 and H_3 are stated below.

(8) (a) $|\Phi(\omega, t)| \le K\{1 + ||\omega|| + ||\omega||^{1+\alpha}(1 + ||\omega||^{\beta})\} \le 3K(1 + ||\omega||^{1+\alpha+\beta})$.

(b) Since $x(\omega, t)$ has independent increments and $x(\omega, 0) = 0$ for almost all ω, it follows that [1, p. 363]

(9) $$E\|\omega\|^{\gamma} \leq 8\, E\,|x(\omega, 1)|^{\gamma}, \quad \gamma \geq 1$$

(c) Hence,

(10) $$\left.\begin{array}{l} E\,|\Phi(\omega, t)|^{2} \\ \\ \int_0^1 E|\Phi(\omega, t)|^{2}\, dt \end{array}\right\} \leq M < \infty$$

(d) Therefore, (see [1, pp. 440-441]) there exists a sequence of partitions $\{t_{\nu}^{(n)}\}$ of $[0,1]$ such that if we define

(11) $$\alpha_n(t) = \max_{\nu} \{t_{\nu}^{(n)},\ t_{\nu}^{(n)} \leq t\}$$

$$\beta_n(t) = \min_{\nu} \{t_{\nu}^{(n)},\ t_{\nu}^{(n)} > t\}$$

then

(12) $$\max_{0 \leq t \leq 1} [\beta_n(t) - \alpha_n(t)] = \max_{\nu} [t_{\nu+1}^{(n)} - t_{\nu}^{(n)}] \xrightarrow[n\to\infty]{} 0$$

and

(13) $$\int_0^1 E[\Phi(\omega, t) - \Phi(\omega, \alpha_n(t))]^2\, dt \xrightarrow[n\to\infty]{} 0$$

(e) Because $\Phi(\omega, t)$ is $\mathcal{A}_t$ measurable, $x(\omega', s) = x(\omega, s)$ $s \leq t$ implies $\Phi(\omega', t) = \Phi(\omega, t)$. Hence (6) can be written

$$F(\omega', \omega, t) = \int_0^t x(\omega', s)\, d_s\, f(s\,; \omega, t) \tag{14}$$

provided that $f(s\,; \omega, t)$ is made continuous from the right.

(f) Let $\{\varphi_n^t\}$ be any sequence from $\Omega = C[0,1]$ satisfying

$$1 \geq \|\varphi_n^t\| = x(\varphi_n^t, t) \xrightarrow[n\to\infty]{} 0 \tag{15}$$

$$\frac{1}{\|\varphi_n^t\|}\, x(\varphi_n^t, s) \xrightarrow[n\to\infty]{} 0 \qquad s < t \tag{16}$$

then for every $\omega \in \Omega$

$$\frac{1}{\|\varphi_n^t\|}\left[\Phi(\omega + \varphi_n^t, t) - \Phi(\omega, t)\right] = \int_0^t x(\varphi_n^t / \|\varphi_n^t\|, s)\, d_s\, f(s\,; \omega, t) + 0(\|\varphi_n^t\|^\alpha) \xrightarrow[n\to\infty]{} \Psi(\omega, t) \tag{17}$$

(g) Since

$$\frac{1}{\|\varphi_n^t\|}\left|\Phi(\omega + \varphi_n^t, t) - \Phi(\omega, t)\right| \leq \left|\int_0^t x(\varphi_n^t / \|\varphi_n^t\|, s)\, d_s\, f(s\,; \omega, t)\right| + K\|\varphi_n^t\|^\alpha (1 + \|\varphi_n^t\|^\beta + \|\omega\|^\beta) \tag{18}$$

(cont'd.)

$$\leq |\Phi(\varphi_n^t/||\varphi_n^t|| + \omega, t) - \Phi(\omega,t)| + 2K(2+||\omega||^\beta)$$

$$\leq 9K\, 2^{1+\alpha+\beta}(1 + ||\omega||^{1+\alpha+\beta}),$$

it follows by dominated convergence that

$$(19) \qquad \left.\begin{array}{l} E|\Psi(\omega,t)|^2 \\ \\ \int_0^1 E|\Psi(\omega,t)|^2\, dt \end{array}\right\} \leq M < \infty$$

(h) For some sequence of partitions $\{t_\nu^{(n)}\}$, which can be assumed to be the same one as in (d),

$$(20) \qquad \int_0^1 E|\Psi(\omega,t) - \Psi(\omega,\alpha_n(t))|^2\, dt \xrightarrow[n\to\infty]{} 0$$

$\alpha_n(t)$ being defined by (11).

Given a sequence of partitions $\{0 = t_0^{(n)} < t_1^{(n)} \cdots < t_{N_n}^{(n)} = 1\}$ and defining $\alpha_n(t)$, $\beta_n(t)$ as before, we can define a corresponding sequence of polygonal approximations to the Brownian motion as follows: [8] For every $\omega \in \Omega = C[0,1]$ define $\omega^n(\omega)$ by

$$(21) \quad x(\omega^n(\omega),t) = x(\omega,\alpha_n(t)) + \frac{t - \alpha_n(t)}{\beta_n(t) - \alpha_n(t)}[x(\omega,\beta_n(t) - x(\omega,\alpha_n(t))].$$

Now, if, as is the case for (d) and (h) above,

$$\max_{1 \leq t \leq 1} [\beta_n(t) - \alpha_n(t)] \xrightarrow[n\to\infty]{} 0$$

then

$$(22) \quad ||\omega^n(\omega) - \omega|| \leq 2 \sup_{0 \leq t \leq 1} |x(\omega, t) - x(\omega, \alpha_n(t))| \xrightarrow[n\to\infty]{} 0 \qquad \text{a.s.}$$

Our main result can now be stated as

<u>Theorem.</u> Let $\Phi(\omega, t)$ satisfy H_1, H_2 and H_3. Then, there exist a sequence of partitions of $[0,1]$ and a corresponding sequence of polygonal approximations $\omega^n(\omega)$ defined by (21) such that

$$(23) \quad \int_0^1 \Phi(\omega^n(\omega), t)\, d_t\, x(\omega^n(\omega), t) \xrightarrow[n\to\infty]{q.m.} \int_0^1 \Phi(\omega, t)\, d_t\, x(\omega, t) + \frac{1}{2} \int_0^1 \Psi(\omega, t)\, dt$$

where the first integral on the right hand side is a stochastic integral (but because of (21) the left hand side is an ordinary integral).

Proof: According to (d) and (h) we can always choose a sequence of partitions so that (12), (13) and (20) are satisfied. Because of (13) and the definition of a stochastic integral

$$(24) \quad \sum_{\nu=1}^{N_n} \Phi(\omega, t_{\nu-1}^{(n)}) [x(\omega, t_\nu^{(n)}) - x(\omega, t_{\nu-1}^{(n)})]$$

$$= \int_0^1 \Phi(\omega, \alpha_n(t))\, d_t\, x(\omega^n(\omega), t) \xrightarrow[n\to\infty]{q.m.} \int_0^1 \Phi(\omega, t)\, d_t\, x(\omega, t)$$

Hence, we only need to prove

$$(25)\quad F_n(\omega) = \int_0^1 [\Phi(\omega^n(\omega), t) - \Phi(\omega, \alpha_n(t))]\, d_t\, x(\omega^n(\omega), t) \xrightarrow[n\to\infty]{q.m.} \frac{1}{2}\int_0^1 \Psi(\omega, t)\, dt$$

Now, let $\xi_n(\omega, t) \in \Omega$ be defined by

$$(26)\quad x(\xi_n(\omega, t), s) = x(\omega^n(\omega), \min(s, \alpha_n(t))) \qquad 0 \leq s \leq 1$$

and rewrite (25) as

$$(27)\quad F_n(\omega) = \int_0^1 [\Phi(\omega^n(\omega), t) - \Phi(\xi_n(\omega, t), t)]\, d_t\, x(\omega^n(\omega), t)$$

$$+ \int_0^1 [\Phi(\xi_n(\omega, t) - \Phi(\omega, \alpha_n(t))]\, d_t\, x(\omega^n(\omega), t)$$

The integral of the second integral is $\mathcal{A}_{\alpha_n(t)}$ measurable and

$$(28)\quad E\left\{ [x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))]^k \middle| \mathcal{A}_{\alpha_n(t)} \right\} = \begin{cases} 0, & k = 1 \\ \beta_n(t) - \alpha_n(t), & k = 2 \end{cases}$$

Therefore,

$$(29)\quad E\left| \int_0^1 [\Phi(\xi_n(\omega, t), t) - \Phi(\omega, \alpha_n(t))]\, d_t\, x(\omega^n(\omega), t) \right|^2$$

(cont'd.)

$$= E\left\{\sum_{\nu}\sum_{\mu}\left[\frac{x(\omega,t_\nu)-x(\omega,t_{\nu-1})}{t_\nu - t_{\nu-1}}\right]\left[\frac{x(\omega,t_\mu)-x(\omega,t_{\mu-1})}{t_\mu - t_{\mu-1}}\right]\right.$$

$$\left.\int_{t_{\nu-1}}^{t_\nu} dt \int_{t_{\mu-1}}^{t_\mu} ds\,[\Phi(\xi_n(\omega,t),t) - \Phi(\omega,t_{\nu-1})][\Phi(\xi_n(\omega,s),s) - \Phi(\omega,t_{\mu-1})]\right\}$$

$$= E\left\{\sum_{\nu}\sum_{\mu} E\left[\cdot \,\middle|\, \mathcal{A}_{\max(t_{\nu-1},t_{\mu-1})}\right]\right\}$$

$$= E\left\{\sum_{\nu}\frac{1}{(t_\nu - t_{\nu-1})}\left|\int_{t_{\nu-1}}^{t_\nu}[\Phi(\xi_n(\omega,t),t) - \Phi(\omega,t_{\nu-1})]\,dt\right|^2\right\}$$

$$\leq \int_0^1 E\,|\Phi(\xi_n(\omega,t),t) - \Phi(\omega,\alpha_n(t))|^2\,dt$$

$$\leq 4\left\{\int_0^1 E|\Phi(\xi_n(\omega,t),t) - \Phi(\omega,t)|^2\,dt\right.$$

$$\left. + \int_0^1 E|\Phi(\omega,t) - \Phi(\omega,\alpha_n(t))|^2\,dt\right\} \xrightarrow[n\to\infty]{} 0$$

by virtue of dominated convergence and (13). Thus, (25) reduces to

$$(30) \quad \int_0^1 [\Phi(\omega^n(\omega),t) - \Phi(\xi_n(\omega,t),t)]\,d_t\,x(\omega^n(\omega),t) \xrightarrow[n\to\infty]{q.m.} \frac{1}{2}\int_0^1 \Psi(\omega,t)\,dt$$

From H_2, (14) and (26) we can write

(31) $\Phi(\omega^n(\omega), t) - \Phi(\xi_n(\omega, t), t)$

$$= \left[\frac{x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))}{\beta_n(t) - \alpha_n(t)}\right] \int_{\alpha_n(t)}^{t} (s - \alpha_n(t)) d_s\, f(s ; \xi_n(\omega, t), t)$$

$$+ \left|x(\omega, \beta_n(t) - x(\omega, \alpha_n(t))\right|^{1+\alpha} G_n(\omega, t)$$

$$= \left[\frac{x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))}{\beta_n(t) - \alpha_n(t)}\right] \int_{\alpha_n(t)}^{t} [f(t ; \xi_n(\omega, t), t) - f(s ; \xi_n(\omega, t), t)] ds$$

$$+ \left|x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))\right|^{1+\alpha} G_n(\omega, t)$$

$$= \left[\frac{x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))}{\beta_n(t) - \alpha_n(t)}\right] (t - \alpha_n(t))\, \Psi(\xi_n(\omega, t), t)$$

$$+ [x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))]\, H_n(\omega, t)$$

$$+ \left|x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))\right|^{1+\alpha} G_n(\omega, t)$$

where $|G_n(\omega, t)|$, $|H_n(\omega, t)|$ are both dominated by $K'(1 + \|\omega\|^{1+\gamma})$ $\gamma > 0$, $H_n(\omega, t)$ is $\mathcal{Q}_{\alpha_n(t)}$ measurable and $\xrightarrow[n\to\infty]{} 0$ a.s. Hence, it is easy to show that (30) reduces to

(32) $$\int_0^1 \left[\frac{x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))}{\beta_n(t) - \alpha_n(t)}\right]^2 (t - \alpha_n(t))\, \Psi(\xi_n(\omega, t), t)\, dt$$

(cont'd.)

$$\xrightarrow[n\to\infty]{q.m.} \frac{1}{2} \int_0^1 \Psi(\omega, t)\, dt$$

or

$$(33) \quad \int_0^1 \left\{ \frac{[x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))]}{\beta_n(t) - \alpha_n(t)} - 1 \right\} \left[\frac{t - \alpha_n(t)}{\beta_n(t) - \alpha_n(t)} \right] \Psi(\xi_n(\omega, t), t)\, dt$$

$$+ \int_0^1 \left[\frac{t - \alpha_n(t)}{\beta_n(t) - \alpha_n(t)} \right] [\Psi(\xi_n(\omega, t), t) - \Psi(\omega, \alpha_n(t))]\, dt$$

$$+ \frac{1}{2} \int_0^1 [\Psi(\omega, \alpha_n(t)) - \Psi(\omega, t)]\, dt \xrightarrow[n\to\infty]{q.m.} 0 .$$

Denoting the three integrals in (33) by I_1, I_2 and I_3, we find that because $\Psi(\xi_n(\omega, t), t)$ is $\mathcal{Q}_{\alpha_n(t)}$ measurable and

$$(34) \quad E \left\{ \frac{[x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))]^2}{\beta_n(t) - \alpha_n(t)} - 1 \,\middle|\, \mathcal{Q}_{\alpha_n(t)} \right\} = 0$$

by using arguments similar to those of (29), we can show that

$$E\, I_1^2 \leq 2 \max_{0 \leq t \leq 1} [\beta_n(t) - \alpha_n(t)] \int_0^1 E\, |\Psi(\xi_n(\omega, t), t)|^2\, dt \xrightarrow[n\to\infty]{} 0$$

The last integral I_3 in (33) converges to zero in quadratic mean because of (20). Thus, it only remains to prove

$$(35)\quad \int_0^1 \left[\frac{t-\alpha_n(t)}{\beta_n(t)-\alpha_n(t)}\right] [\Psi(\xi_n(\omega,t),t) - \Psi(\omega,\alpha_n(t))]\,dt \xrightarrow[n\to\infty]{q.m.} 0$$

which can be further reduced to

$$(36)\quad \int_0^1 \left[\frac{t-\alpha_n(t)}{\beta_n(t)-\alpha_n(t)}\right] [\Psi(\xi_n(\omega,t)t) - \Psi(\omega,t)]\,dt \xrightarrow[n\to\infty]{q.m.} 0$$

To prove (36), we note that from (f) we can find for every t in $[0,1]$ a sequence $\{\varphi_n^t\}$ satisfying (15) and (16) and in addition

$$(37)\quad ||\varphi_n^t|| \geq \operatorname*{Sup}_{0\leq s\leq 1} |\beta_n(s) - \alpha_n(s)|^{1/3}$$

so that for almost all ω

$$(38)\quad \Psi(\xi_n(\omega,t),t) - \left[\frac{\Phi(\xi_n(\omega,t)+\varphi_n^t,t) - \Phi(\xi_n(\omega,t),t)}{||\varphi_n^t||}\right] \xrightarrow[n\to\infty]{} 0$$

$$(39)\quad \Psi(\omega,t) - \frac{\Phi(\omega+\varphi_n^t,t) - \Phi(\omega,t)}{||\varphi_n^t||} \xrightarrow[n\to\infty]{} 0$$

Further, because $x(\omega,s)$ is a Brownian motion, we have

$$(40)\quad \frac{\operatorname*{Max}_{0\leq s\leq t} |x(\xi_n(\omega,t),s) - x(\omega,s)|}{||\varphi_n^t||}$$

$$\leq \frac{\underset{0 \leq s \leq 1}{\text{Max}} |x(\omega^n(\omega), s) - x(\omega, s)|}{\underset{0 \leq t \leq 1}{\text{Max}} |\beta_n(t) - \alpha_n(t)|^{1/3}}$$

$$\leq \frac{2 \underset{0 \leq s \leq 1}{\text{Sup}} |x(\omega, s) - x(\omega, \alpha_n(s))|}{\underset{0 \leq t \leq 1}{\text{Max}} |\beta_n(t) - \alpha_n(t)|^{1/3}}$$

$$\leq 2 \underset{0 \leq s \leq 1}{\text{Sup}} \left\{ \frac{|x(\omega, s) - x(\omega, \alpha_n(s))|}{|s - \alpha_n(s)|^{1/3}} \right\} \xrightarrow[n \to \infty]{} 0 \qquad \text{a.s.}$$

Thus, for all t and almost all ω

$$\frac{\Phi(\xi_n(\omega, t) + \varphi_n^t, t) - \Phi(\omega + \varphi_n^t, t)}{\|\varphi_n^t\|} \xrightarrow[n \to \infty]{} 0 \tag{41}$$

$$\frac{\Phi(\xi_n(\omega, t), t) - \Phi(\omega, t)}{\|\varphi_n^t\|} \xrightarrow[n \to \infty]{} 0 \tag{42}$$

Whence

$$\Psi(\xi_n(\omega, t), t) - \Psi(\omega, t) \xrightarrow[n \to \infty]{} 0 \qquad \text{a.s.} \tag{43}$$

and (36) follows by dominated convergence (using the bounds provided by (18)). The proof for the theorem is now complete.

Corollary. Under the hypothesis of the theorem, a sequence of partitions exists for which

$$(44) \qquad \int_0^1 \Phi(\omega^n(\omega), t)\, d_t\, x(\omega^n(\omega), t) \xrightarrow[n\to\infty]{a.s.} \int_0^1 \Phi(\omega, t)\, d_t\, x(\omega, t) + \frac{1}{2} \int_0^1 \Psi(\omega, t)\, dt$$

Proof. This result is obvious since every q.m. convergent sequence has an a.s. convergent subsequence with the same limit.

4. Examples and Applications

First, consider a class of examples corresponding more or less to the situation in [3,4,5]. Let

$$(45) \qquad \Phi(\omega, t) = M(y(\omega, t), t)$$

where

$$(46) \qquad y(\omega, t) = \int_0^t v(\omega, s)\, d_s\, x(\omega, s)$$

is a stochastic integral and $M(y, t)$ is twice y-differentiable. It is easy to show that if $v(\cdot, \cdot)$ satisfies H_1, H_2 and H_3 then so does $\Phi(\cdot, \cdot)$. Furthermore, by virtue of (17)

$$(47) \qquad \Psi(\omega, t) = \lim_{n\to\infty} \frac{1}{||\varphi_n^t||} [\Phi(\omega + \varphi_n^t, t) - \Phi(\omega, t)]$$

$$= v(\omega, t)\, M'(y(\omega, t), t) \qquad \left(M'(y, t) = \frac{\partial M(y, t)}{\partial y} \right)$$

Much weaker conditions on $v(\cdot,\cdot)$ also suffice to yield (47), but this fact would require a more lengthy discussion. Applying the main theorem to the example considered earlier (see (3)), we find

$$(48) \quad \int_0^1 x(\omega^n(\omega), t)\, d_t\, x(\omega^n(\omega, t)) \xrightarrow[n\to\infty]{q.m.} \int_0^1 x(\omega, t)\, d_t\, x(\omega, t) + \frac{1}{2}\int_0^1 dt$$

$$= \frac{1}{2}\left[x^2(\omega, t) - x^2(\omega, 0)\right]$$

as it should.

From the point of view of many physical problems, application of stochastic integral to differential equations is important. It is well known [1, pp. 273-291] that under suitable conditions on $\sigma(\cdot,\cdot)$ and $m(\cdot,\cdot)$, the following stochastic differential equation has a unique solution:

$$(49) \quad d_t\, y(\omega, t) = m(y(\omega, t), t)\, dt + \sigma(y(\omega, t), t)\, d_t\, x(\omega, t).$$

Here, a solution $y(\cdot, t)$ is interpreted as an $\mathcal{Q}_t$ measurable function satisfying

$$(50) \quad y(\omega, t) = y(\omega, 0) + \int_0^t m(y(\omega, s), s)\, ds + \int_0^t \sigma(y(\omega, s), s)\, d_s\, x(\omega, s)$$

where the last integral is a stochastic integral. Let

$$(51) \qquad \Phi_t(\omega, s) = \sigma(y(\omega, s), s), \qquad s \le t$$

$$= 0 \qquad s > t$$

then in view of our discussion preceding (47), we can expect that under suitable conditions on $\sigma(\cdot,\cdot)$

$$(52) \quad \int_0^1 \Phi_t(\omega^n(\omega), s)\, d_s\, x(\omega^n(\omega), s) \xrightarrow[n\to\infty]{q.m.} \int_0^t \sigma(y(\omega, s), s)\, d_s\, x(\omega, s)$$

$$+ \frac{1}{2} \int_0^t \sigma'(y(\omega, s), s)\, \sigma(y(\omega, s), s)\, ds$$

This was the basic motivation of the results given in [4,5]. If, as in the references [4, 5], we define

$$(53) \quad y_n(\omega, t) = y(\omega, 0) + \int_0^t m(y_n(\omega, s), s)\, ds$$

$$+ \int_0^t \sigma(y_n(\omega, s), s)\, d_s\, x(\omega^n(\omega), s)$$

where $\omega^n(\omega)$ is defined by (21), then even if $y_n(\omega, t)$ has a limit as $n \to \infty$, the limit is not the solution of (50). Rather, we expect the limit $\hat{y}(\omega, t)$ to satisfy

$$(54) \quad \hat{y}(\omega, t) = y(\omega, 0) + \int_0^t m(\hat{y}(\omega, s), s)\, ds + \int_0^t \sigma(\hat{y}(\omega, s), s)\, d_s\, x(\omega, s)$$

$$+ \frac{1}{2} \int_0^t \sigma(\hat{y}(\omega, s), s)\, \sigma'(\hat{y}(\omega, s)\, ds\ .$$

Our main theorem can be used to prove (54). However, the conditions given in [4] on $\sigma(\cdot,\cdot)$ need to be strengthened to accomodate H_2.

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