ON FINITE DIMENSIONAL APPROXIMATIONS TO A CLASS OF GAMES

by

J-P Jacob

E. Polak

Memorandum No. ERL-M 211

6 April 1967

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

On Finite Dimensional Approximations to a Class of Games

by

J-P Jacob
IBM Research Laboratory
San Jose, California

E. Polak
University of California
Electronics Research Laboratory
Berkeley, California

INTRODUCTION

Games defined on subsets of general linear topological spaces can be viewed as a natural extension of classical games in R^n. However, our interest in such games stems from differential games with open or closed loop strategies. As a result, we shall confine most of our arguments to certain relevant spaces only, viz. spaces of Lipschitzian functions from R^q into R^n, with the topology of uniform convergence on compacta. Although at first glance, this may seem to be an unnecessary restriction, it is more than justified by the strength of the results which it brings within our reach.

The research reported herein was supported in part by the National Aeronautics and Space Administration under Grant NsG-354. Supplement 2, 3, and 4.

When a game is defined on subsets of finite dimensional spaces, it is often possible to obtain a solution by means of nonlinear programming algorithms, but there are virtually no algorithms for solving games defined on abstract spaces. Consequently, the purpose of this paper is to construct a theory of sequences of approximation games, defined on finite dimensional spaces, and hence solvable, whose solutions converge to a solution of the original game. We shall show that for games defined on spaces of Lipschitzian functions such approximations always exist and, furthermore, we shall give an algorithm for their construction.

To simplify exposition we adopt the following logical notation: $(\forall\ x)_A$ is to be read as "for all x in A," $\exists$ is to be read "there exists" and s.t. is an abbreviation for "such that."

I. APPROXIMATIONS TO GAMES

We begin by considering games in a general setting, unencumbered by the complex structure of the problem that we want to consider eventually.

1 Definition: Let $\mathcal{X}$, $\mathcal{Y}$ be two Hausdorff, locally convex linear topological spaces. Let X, Y, be compact convex subsets of $\mathcal{X}$, $\mathcal{Y}$, respectively. Finally, let $\mathcal{F}$ be a real valued continuous function on $X \times Y$, convex on X for every $y \in Y$ and concave in Y for every $x \in X$. We shall call the problem of finding an $\bar{x} \in X$ and a $\bar{y} \in Y$ such that for all $x \in X$ and $y \in Y$

2 $$\mathcal{F}(\bar{x}, y) \leq \mathcal{F}(\bar{x}, \bar{y}) \leq \mathcal{F}(x, \bar{y})$$

a <u>convex-concave game</u>.† We shall denote the game by the triplet $\mathcal{G} = (X, Y, \mathcal{F})$ and we shall call any pair $(\bar{x}, \bar{y})$ satisfying (2) a <u>solution</u>.

The existence of solutions to games of the type described above is guaranteed by Ky Fan's theorem (Ref. [1]), which states: "Let E, F be two Hausdorff, locally convex, linear topological spaces. Let H, K be compact convex subsets of E and F, respectively. Let f be a real-valued continuous function defined on $H \times K$. If for each $(x_0, y_0) \in H \times K$ the sets

$$\{x \in H \mid f(x, y_0) = \min_{x' \in H} f(x', y_0)\} \tag{3}$$

$$\{y \in K \mid f(x_0, y) = \max_{y' \in K} f(x_0, y')\} \tag{4}$$

are convex, then there exists a pair $(\bar{x}, \bar{y}) \in H \times K$ such that

$$\min_{x \in H} \max_{y \in K} f(x, y) = \max_{y \in K} \min_{x \in H} f(x, y) = f(\bar{x}, \bar{y})\text{''} \tag{5}$$

We shall now show that it follows immediately from the Ky Fan theorem that convex-concave games always have a solution.

---

† This definition of a game is somewhat more restricted than the one usually encountered.

6 **Theorem:** Let $\mathcal{G} = (X, Y, \mathfrak{F})$ be a convex-concave game. Then it has a solution $(\bar{x}, \bar{y})$.

Proof: Since $\mathfrak{F}$ is convex-concave, it follows that for each $(x_0, y_0) \in X \times Y$ the sets

7 $$\{x \in X \mid \mathfrak{F}(x, y_0) = \min_{x' \in X} \mathfrak{F}(x', y_0)\}$$

8 $$\{y \in Y \mid \mathfrak{F}(x_0, y) = \max_{y' \in Y} \mathfrak{F}(x_0, y')\}$$

are convex. Hence by Ky Fan's theorem there exists a pair $(\bar{x}, \bar{y}) \in X \times Y$ such that

9 $$\min_{x \in X} \mathfrak{F}(x, \bar{y}) = \max_{y \in Y} \mathfrak{F}(\bar{x}, y) = \mathfrak{F}(\bar{x}, \bar{y})$$

But (9) is equivalent to (2), and hence it follows that $(\bar{x}, \bar{y})$ is a solution.

We now restrict ourselves to games in which the sets X, Y are countably infinite dimensional. We shall try to construct a solution to such a game $\mathcal{G} = \{X, Y, \mathfrak{F}\}$ by constructing a sequence of approximating games $\mathcal{G}_i = \{X_i, Y_i, \mathfrak{F}\}$ with $X_i \subset X$, $Y_i \subset Y$, such that for $i = 1, 2, \cdots$, X_i and Y_i are finite dimensional, convex, compact sets. Finite dimensional, convex-concave approximating games can be solved by nonlinear programming algorithms and under suitable assumptions, we shall show that their solutions converge to a solution of the original game. The following theorem clarifies this question.

10 <u>Theorem:</u> Consider the convex-concave game $\mathcal{G} = \{X, Y, \mathcal{F}\}$. For $i = 1, 2, \cdots$, let $X_i \subset X$, $Y_i \subset Y$ be compact convex sets such that $\overline{\bigcup_{i=1}^{\infty} X_i} = X$ and $\overline{\bigcup_{i=1}^{\infty} Y_i} = Y$. If for any $x \in X$ and $y \in Y$ there exist sequences $\{x_i\}$, $\{y_i\}$, with $x_i \in X_i$, $y_i \in Y_i$, $i = 1, 2, 3, \cdots$, such that $x_i \to x$, $y_i \to y$, then any convergent subsequence of $\{(\bar{x}_i, \bar{y}_i)\}$, a sequence of solutions to the convex-concave games $\mathcal{G}_i = \{X_i, Y_i, \mathcal{F}\}$, converges to a solution $(\bar{x}, \bar{y})$ of the game $\mathcal{G}$.

<u>Proof:</u> First we observe that, by Theorem (6), the original game $\mathcal{G}$ and the approximating games $\mathcal{G}_i$ have solutions for all $i = 1, 2, \cdots$. Now, suppose that $\{(\bar{x}_i, \bar{y}_i)\}$ is a sequence of solutions to the games $\mathcal{G}_i$, $i = 1, 2, \cdots$, and that $\{(\bar{x}_{i_k}, \bar{y}_{i_k})\}$ is a subsequence converging to $(\bar{x}, \bar{y})$. Let (x, y) be any point in $X \times Y$ and let $\{x_{i_k}\}$, $\{y_{i_k}\}$ be sequences such that $x_{i_k} \in X_{i_k}$, $y_{i_k} \in Y_{i_k}$ and $x_{i_k} \to x$, $y_{i_k} \to y$. The existence of such sequences is assured by the assumptions of the theorem. Then, since $(\bar{x}_{i_k}, \bar{y}_{i_k})$ is a solution of $\mathcal{G}_{i_k}$, we have that

11 $$\mathcal{F}(\bar{x}_{i_k}, y_{i_k}) \leq \mathcal{F}(\bar{x}_{i_k}, \bar{y}_{i_k}) \leq \mathcal{F}(x_{i_k}, \bar{y}_{i_k})$$

Since $\mathcal{F}$ is a continuous function on $X \times Y$, by letting $i_k \to \infty$ in (11), we obtain

12 $$\mathcal{F}(\bar{x}, y) \leq \mathcal{F}(\bar{x}, \bar{y}) \leq \mathcal{F}(x, \bar{y})$$

for any $x \in X$, $y \in Y$, which proves that $(\bar{x}, \bar{y})$ is a solution to $\mathcal{G}$.

We are now ready to address ourselves to a specific problem of importance.

II. THE SPACE OF LIPSCHITZIAN FUNCTIONS Λ

The Hausdorff, locally convex, linear topological space with which we shall concern ourselves from now on is the space of Lipschitzian functions with the topology of uniform convergence on compacta. The reason for our interest in this space is that many differential games, to be discussed in the next section, can be treated in the framework we are about to develop.

13 <u>Definition:</u> A function $f: R \to R^p$ is said to be Lipschitzian on $[t_0, \infty)$ if there exists a constant M such that for all t_1, t_2 in $[t_0, \infty)$ we have

14 $$\|f(t_1) - f(t_2)\| \leq M|t_1 - t_2|$$

where $\|\cdot\|$ denotes a norm in R^p.†

15 <u>Definition:</u> We define the set Λ of Lipschitzian functions on $[t_0, \infty)$, $t_0 \geq 0$, as

16 $$\Lambda = \left\{ f:[t_0,\infty) \to R^p \,|\, (\exists\, M)_R (\forall\, t_1, t_2)_{[t_0,\infty)} \|f(t_2) - f(t_1)\| \leq M|t_1 - t_2| \right\}$$

† To extend this definition to functions $f: S \to R^p$, where S is a subset of R^q, simply substitute S for $[t_0, \infty)$ and $\|\cdot\|$ for $|\cdot|$ wherever appropriate.

Under the usual addition of functions, Λ is obviously a linear vector space over the field of real numbers. We shall now define a topology (which will be seen to be the topology of uniform convergence on compacta) under which Λ becomes a Hausdorff, locally convex linear topological space.

17 Definition: (fundamental system of neighborhoods of the origin). For all $n \in I^+$ (the strictly positive integers) and $\mathcal{E} \in R^+$ (the strictly positive real numbers), let

18
$$U(n, \mathcal{E}) = \left\{ f \in \Lambda \,\middle|\, (\forall\, t)_{[t_0, n]} \|f(t)\| \leq \mathcal{E} \right\}$$

19 Definition: (topology in Λ). Let τ be a collection of sets U, contained in Λ, with the property that

20
$$(\forall\, g)_U (\exists\, n)_{I^+} (\exists\, \mathcal{E})_{R^+} \text{ s.t. } \{g + U(n, \mathcal{E})\} \subset U$$

In other words,

21
$$\tau = \{ U \subset \Lambda \mid (\forall\, g)_U (\exists\, n)_{I^+} (\exists\, \mathcal{E})_{R^+} \text{ s.t. } \{g + U(n, \mathcal{E})\} \subset U \}$$

Obviously, τ is a topology for Λ.

22 Lemma: The space (Λ, τ) is a Hausdorff, locally convex, linear topological space, and the topology τ is that of uniform convergence on compacta.

The first part of this lemma is readily established by verifying that the assumptions of Theorems 5.1 and 6.5 of Reference [2] are satisfied. This is a long but completely straightforward exercise, which we omit. If we define the family of sets U by

23 $$U = \{ U(n, \frac{1}{n}) \mid n = 1, 2, \cdots \}$$

then U is seen to be a countable local base for the topology τ. This can now be used to show that τ is the topology of uniform convergence on compacta.

A very important class of subsets of the space Λ, as far as differential games are concerned, is the class made up of sets $\Lambda(L, M)$ defined by

24 $$\Lambda(L, M) = \left\{ f \in \Lambda \mid (\forall\, t_1, t_2)_{[t_0, \infty)} \|f(t_1) - f(t_2)\| \leq M|t_1 - t_2)\| \quad \text{and} \quad \|f(t_0)\| \leq L \right\}$$

We shall now establish that the sets $\Lambda(L, M)$ are compact in (Λ, τ). Consequently, we shall be able to apply Theorem 6 to differential games whose sets of admissible trajectories are closed convex subsets of the sets $\Lambda(L, M)$.

25 <u>Lemma:</u> For every $L \geq 0$ and $M \geq 0$, the set $\Lambda(L, M)$, defined in L, is compact in (Λ, τ).

Proof: First, for any fixed $t \in [t_0, \infty)$, the set $\{f(t) \mid f \in \Lambda(L, M)\} \subset R^p$ is bounded, and hence it has a compact closure. Second, the set of functions $\Lambda(L, M)$ is equicontinuous. Finally, $\Lambda(L, M)$ is a closed subset of the space of continuous functions from R into R^p with the topology of uniform convergence on compacta. Hence, by Ascoli's Theorem (see, for example, page 234 of Ref. [3]), $\Lambda(L, M)$ is a compact subset of Λ.

We now digress to discuss briefly differential games which have motivated our interest in games defined in Lipschitzian function spaces.

III. A CLASS OF DIFFERENTIAL GAMES

The adversaries in a differential game are usually two dynamical systems, referred to as the pursuer and the evader, whose motions are described by differential equations of the form

26 $$\frac{dx(t)}{dt} = h(x(t), u(t), t)$$

where $x(t) \in R^n$ is the state of the dynamical system at time t and $u(t) \in R^m$ is the input at time t. The motion of a system such as (1) is usually constrained by requirements such as that its initial state x_0 at t_0 be a point in a set $X_0 \subset R^n$, that its control u be measurable and bounded, and take values in a fixed set $U \subseteq R^m$ and that for $t_0 \leq t \leq t_f$, with $t_f > t_0$, its trajectories $x(t)$, i.e., the solutions of (26), be confined to a set X in R^n, with $X \supset X_0$.

We designate by Ω the set of all admissible trajectories $x(t)$ defined on $[t_0, t_f]$, i.e., Ω is the set of all trajectories which satisfy all the given constraints. We differentiate between the pursuer and evader by means of the subscripts p and e, respectively.

We now define a differential game.

27 The Differential Game: Given a set of admissible trajectories Ω_p for the pursuer, a set of admissible trajectories Ω_e for the evader, both defined on the same time interval $[t_0, t_f]$ (where $t_f > t_0$ may be infinite), and a payoff function $\mathfrak{J}$ mapping $\Omega_p \times \Omega_e$ into the reals, find an $\bar{x}_p \in \Omega_p$ and an $\bar{x}_e \in \Omega_e$ such that

28 $$\mathfrak{J}(\bar{x}_p, \bar{x}_e) = \min_{x_p \in \Omega_p} \max_{x_e \in \Omega_e} \mathfrak{J}(x_p, x_e) = \max_{x_e \in \Omega_e} \min_{x_p \in \Omega_p} \mathfrak{J}(x_p, x_e)$$

As before, we shall call a point $(\bar{x}_p, \bar{x}_e)$ satisfying (28) a solution (to the differential game (27)).

29 Remark: Intuitively, one may arrive at the above formulation as follows. Assuming that each system will do its "best," the pursuer to intercept and the evader to escape, we will show that they are lead to minimax considerations in the choice of their control laws. To demonstrate this in simple terms, suppose that, due to power and energy linitations, the admissible control laws for the pursuer and evader must restrict their trajectories to sets Ω_p and Ω_e, respectively. Assume, furthermore,

that for each $x_p \in \Omega_p$, $x_e \in \Omega_e$, a real valued cost function $\mathcal{J}(x_p, x_e)$ is defined by the sum of 3 terms, one expressing the "miss" distance after a fixed time T, another expressing the amount of energy used by the pursuer, and a third expressing the negative of the energy used by the evader. With this formulation, it is clear that the pursuer wants to minimize the cost (x_p, x_e), while the evader wants to minimize it.

Now suppose that the pursuer arbitrarily selects some trajectory $\bar{x}_p \in \Omega_p$. Then, regardless of the evader's choice, the pursuer is assured of the "cost" being at most

30 $$\max_{x_e \in \Omega_e} (\bar{x}_p, x_e) = (\bar{x}_p, \bar{x}_e) .$$

Since the pursuer is trying to minimize the cost, he should, of course, select his trajectory $\bar{x}_p$ so as to minimize $\mathcal{J}(\bar{x}_p, \bar{x}_e)$. Hence he should select an $\bar{x}_p$ for which

31 $$\mathcal{J}(\bar{x}_p, \bar{x}_e) = \min_{x_p \in \Omega_p} \max_{x_e \in \Omega_e} \mathcal{J}(x_p, x_e) .$$

Observe that $\mathcal{J}(\bar{x}_p, \bar{x}_e)$ is an upper bound on the cost for the pursuer. Furthermore, it is the lowest upper bound on the pursuer's cost.

Similarly, we argue that the evader should select an $\bar{\bar{x}}_e$ with the property that, for some $\bar{\bar{x}}_p$,

32 $$\mathcal{J}(\bar{\bar{x}}_p, \bar{\bar{x}}_e) = \max_{x_e \in \Omega_e} \min_{x_p \in \Omega_p} \mathcal{J}(x_p, x_e)$$

Furthermore, if equality (28) holds it becomes evident that <u>both</u> pursuer and evader are doing their best simultaneously. We now quote conditions (see [4]) under which the sets of admissible trajectories are closed in the space of continuous functions with the topology of uniform convergence on compacta.

33 <u>Theorem:</u> Consider the system

34 $$\frac{dx(t)}{dt} = h(x(t), u(t), t)$$

where $h : R^n \times R^m \times R \to R^n$ is a continuous mapping. Let U be a continuous mapping from $[t_0, \infty)$ into 2^{R^m} such that for every $t \in [t_0, \infty)$, the set $U(t)$ is compact, and let $\mathcal{U}$ denote the set of all admissible controls, i.e., the set of all measurable functions $u : [t_0, \infty) \to R^m$ such that for every $t \in [t_0, \infty)$, $u(t) \in U(t)$. Let $\mathcal{X}$ be the set of all admissible trajectories x of (34) starting at a given point x_0, i.e., $x(t) = x(t; x_0, u)$, with $x(t_0; x_0, u) = x_0$, and $u \in \mathcal{U}$. Then

(a) If there exists a locally integrable function $k : R \to R$ and finite numbers M and N such that

35 $$\|f(x, u, t) - f(x', u, t)\| \leq k(t)\|x - x'\|$$

and $\|f(x, u, t)\| \leq k(t)[M + N\|x\|]$ for all x, x' in R^n, $u \in \bigcup_{t \in [t_0, \infty)} U(t)$ and $t \in [t_0, \infty)$, then for every control $u \in \mathcal{U}$ and every $x_0 \in R^n$ there exists a unique trajectory $x(t; x_0, u)$ of (6), defined on $[t_0, \infty)$ such that $s(t_0, x_0, u) = x_0$.

(b) Suppose that the hypothesis of (a) are satisfied. The set Ω of admissible trajectories starting at a given point x_0 is closed in the topology of uniform convergence on compacta if and only if for every attainable phase (x', t') the set $F(x', t')$ is convex, where (x', t') is said to be an attainable phase if there exists an $x \in \Omega$ such that $x(t') = x'$ and $F(x', t') = \{h(x', u, t') \mid u \in U(t')\}$.

Thus, whenever a differential system satisfies the conditions of the preceding theorem, its set of admissible trajectories Ω, starting at a given point x_0, is closed in the space of continuous functions with the topology of uniform convergence on compacta. If, in addition, for some real L, M, the set Ω is contained in $\Lambda(L, M)$, defined in (24), then Ω is obviously compact in the space of Lipschitzian functions Λ.

We now give two examples when this is true.

36 <u>Example.</u> Suppose that h is uniformly bounded by some positive constant B. Then, it is readily seen that, for any U:

(i) the assumptions of part (a) of Theorem (33) are satisfied; hence the set Ω is well defined;

(ii) $\Omega \subset \Lambda(B, \|x_0\|)$;

(iii) if $F(x', t')$ is convex for every attainable phase (x', t'), then, by part (b) of Theorem (33), Ω is closed in the space C of continuous functions with the topology of uniform convergence on compacta, and it now follows that Ω is closed in $\Lambda \subset C$, with the induced topology.

37 Example: Suppose that $h(x,u,t) = Ax + Bu$, where A, B are constant matrices, and suppose that all the eigenvalues of A have negative real parts. If for all $t \in [t_0, \infty)$, $U(t) = W$, a fixed compact set, then again it is easy to show that the set Ω of admissible trajectories starting at a given point x_0 is compact in (Λ, τ).

The set Ω will be convex when, say, $h(x,u,t)$ is of the form

38 $$h(x,u,t) = Ax + g(u)$$

where A is a constant matrix, and $g: R^m \to R^n$ is a continuous mapping, $U(t) = W$, a fixed compact set such that $g(W)$ is convex.

Finally, as an example of a convex-concave payoffs consider

39 $$J(x_p, x_e) = \left\langle x_p(T) - x_e(T),\ Q[x_p(T) - x_e(T)] \right\rangle - \left\langle x_e, R x_e \right\rangle$$

which can be interpreted as follows. The fight between pursuer and evader is to be started at the time T, when the game is over, and at that time the evader would like to be as close to home (the origin) as possible. Note that the first summand of S represents the terminal distance between pursuer and evader, the matrix Q (positive semi-definite) having been introduced to enable us to consider, for example, physical distance as opposed to state-space distance. Similarly, in the second term, the matrix R (positive semi-definite) was introduced to enable the evader to minimize, some, but not all, of his state components (it may not be advantageous to be travelling at near zero velocity when the fight starts). Now, if $(Q - R)$ is positive semi-definite, then J is convex-concave.

40 Remark: This section has been mainly devoted to differential games played on the trajectories of both players. However, many important differential games are played on sets of admissible controls or control laws. Now, if the controls $u(t)$ are assumed to be Lipschitzian functions (we mean here open-loop controls) then all our results apply automatically. Incidently, games played on open loop control sets are usually simpler than games played on trajectory sets of differential systems with inputs. The reason for this is that it is usually easier to establish compactness and convexity of a given set of control functions than of the set of resulting trajectories. Examples of games played on open-loop control laws and methods for their solution can be found in Reference [5].

For games played on spaces of feedback control laws u, i.e., control laws u such that $u(t) = u(x(t))$, we extend definition (13) to read: a function of $R^q \to R^p$ is Lipschitzian if there exists an $M \in R^+$ such that

41 $$(\forall\, t_1, t_2) \quad \|f(t_1) - f(t_2)\| \leq M \|t_1 - t_2\|_{R^q} ,$$

and our results become again automatically applicable provided the control laws in the game are Lipschitzian. (This follows from the fact that exactly as in Section II, we can define Λ^q to be a space of Lipschitzian functions defined on R^q and construct a corresponding fundamental system of neighborhoods $U^q(n, \varepsilon)$ and a topology τ^q. Theorem (6) then extends to these games.)

An interesting class of games played on feedback control laws is the one in which the control laws are linear since these are easy to implement.

IV. APPROXIMATIONS TO LIPSCHITZIAN GAMES

We now return to convex-concave games, defined on spaces of Lipschitzian functions. Thus, by a Lipschitzian, convex-concave game we shall mean a convex-concave game, defined as in (1), in which $\mathfrak{X}, \mathcal{Y}$ are spaces of Lipschitzian function, with the topology of uniform convergence on compacta.

We now introduce the finite dimensional approximations to Lipschitzian convex-concave games.

42 Definition: Let $\mathcal{G} = \{X, Y, \mathfrak{F}\}$ be a Lipschitzian convex-concave game. Let $\{x_1, x_2, \ldots, x_k\}$ be a set of elements in X, let $\{y_1, y_2, \ldots, y_\ell\}$ be a set of elements in Y, and let $co\{x_i\}$, $co\{y_j\}$ denote the convex hulls of these sets, respectively. For $n = 1, 2, \ldots,$ we shall say that the game $\mathcal{G}_n = \{co\{x_i\}, co\{y_i\}, \mathfrak{F}\}$ is a 1/n-approximation to the game $\mathcal{G}$ if

43 $$X \subset \bigcup_{i=1}^{k} \{x_i + U_x(n, 1/n)\}$$

and

44 $$Y \subset \bigcup_{j=1}^{\ell} \{y_j + U_y(n, 1/n)\}$$

where $U_x(n, 1/n) \subset X$, $U_y(n, 1/n) \subset Y$ are sets defined by (18).

45 <u>Remark:</u> Obviously, since X is compact, the covering $\bigcup_{x \in X} \{x + U_x(n, 1/n)\}$ contains a finite subcover of the form $\bigcup_{i=1}^{k} \{x_i + U_x(n, 1/n)\}$, with $x_i \in X$. A similar argument also holds for Y. Consequently 1/n-approximations to a Lipschitzian game always exist.

46 <u>Lemma:</u> Let $\mathcal{G}_n = \{co\{x_i\}, co\{y_j\}, \mathcal{J}\}$ be a 1/n-approximation to a Lipschitzian, convex-concave game $\mathcal{G} = \{X, Y, \mathcal{J}\}$. Then $\mathcal{G}_n$ has a solution.

<u>Proof:</u> The sets $co\{x_i\}$, $co\{y_j\}$ are closed, convex subsets of the compact sets X and Y, respectively. Hence they are compact. It now follows from Theorem (6) that the game $\mathcal{G}_n$ has a solution.

We now show that any sequence $\{\mathcal{G}_n\}$, with $n = 1, 2, 3, \ldots,$ of 1/n—approximations to a Lipschitzian convex-concave game $\mathcal{G}$ satisfies the assumptions of Theorem (10).

47 <u>Theorem:</u> For $n = 1, 2, 3, \ldots,$ let $\{\mathcal{G}_n\}$ be a sequence of 1/n-approximations to a Lipschitzian convex-concave game $\mathcal{G} = \{X, Y, \mathcal{J}\}$, where

48 $$\mathcal{G}_n = \{co\{x_{i_n}\},\ co\{y_{j_n}\},\ \mathfrak{F}\},\ \text{with } i_n = 1, 2, \ldots, k_n \text{ and } j_n = 1, 2, \ldots, \ell_n .$$

Let X_n denote $co\{x_{i_n}\}$ and Y_n denote $co\{y_{j_n}\}$ for $n=1, 2, 3 \ldots$. Then $\mathcal{G}_n = \{X_n, Y_n, \mathfrak{F}\}$ and

(i) For any $x \in X$, $y \in Y$, there exist sequences $\{x_n\}$, $\{y_n\}$, with $x_n \in X_n$, $y_n \in X_n$, $n=1, 2, 3, \ldots$, such that $x_n \to x$ and $y_n \to y$ and

(ii) $X = \overline{\bigcup_{n=1}^{\infty} X_n}$, $Y = \overline{\bigcup_{n=1}^{\infty} Y_n}$

Proof: Let x be any point in X. Then, by definition (42) of $\mathcal{G}_n$, for every $n=1, 2, 3, \ldots$, there exists an index $\alpha_n \in \{1, 2, \ldots, k_n\}$ such that the vertex x_{α_n} of X_n satisfies

49 $$x \in \{x_{\alpha_n} + U_x(n, 1/n)\}$$

i.e., $\{x - x_{\alpha_n}\} \in U_x(n, 1/n)$.

But, the neighborhoods $U_x(n, 1/n)$ form a countable base about the origin and hence the sequence x_{α_n}, $n=1, 2, 3, \ldots$, with $x_{\alpha_n} \in X_n$, converges to x. A similar argument holds for arbitrary points in Y. This completes the proof of (i).

Now, from the above, it follows that

50 $$X \subset \overline{\bigcup_{n=1}^{\infty} X_n}, \quad Y \subset \overline{\bigcup_{n=1}^{\infty} Y_n}$$

But, since $X_n \subset X$ and $Y_n \subset Y$ for $n=1,2,\ldots$, it follows that

51 $$X \supset \bigcup_{n=1}^{\infty} X_n, \quad Y \supset \bigcup_{n=1}^{\infty} Y_n .$$

Since X, Y are compact, we conclude that

52 $$X \supset \overline{\bigcup_{n=1}^{\infty} X_n} \text{ and } Y \supset \overline{\bigcup_{i=1}^{\infty} Y_n} .$$

But (50) and (52) imply that (ii) is true, which completes our proof.

53 Theorem: For $n=1,2,3,\ldots$, let $\{\mathcal{G}_n\}$, with $\mathcal{G}_n = \{co\{x_{i_n}\}, co\{y_{j_n}\}, \mathfrak{F}\}$, $i_n = 1,2,\ldots,k_n$, $j_n = 1,2,\ldots,\ell_n$, be a sequence of $1/n$-approximations to a Lipschitzian convex-concave game $\mathcal{G} = \{X, Y, \mathfrak{F}\}$. Let $(\bar{x}_n, \bar{y}_n)$ be a solution to $\mathcal{G}_n$ and let $\{(\bar{x}_\alpha, \bar{y}_\alpha)\}$ be any subsequence of $\{(\bar{x}_n, \bar{y}_n)\}$, converging to a pair $(\bar{x}, \bar{y})$. Then $(\bar{x}, \bar{y})$ is a solution to $\mathcal{G}$.

53 a Corollary: If the game $\mathcal{G} = \{X, Y, \mathfrak{F}\}$ has a unique solution $(\bar{x}, \bar{y})$, then any sequence of solutions $\{\bar{x}_n, \bar{y}_n\}$ to $1/n$-approximations $\mathcal{G}_n$ to $\mathcal{G}$ converges to $(\bar{x}, \bar{y})$.

Proof: This theorem is an immediate consequence of theorems (10) and (47).

V. CONSTRUCTION OF APPROXIMATIONS TO GAMES

We consider in this section a special class of Lipschitzian convex-concave games for which the sets X, Y are of the form $\Lambda(L, M)$, as defined in (24). As we shall now show, it is not difficult to construct finite dimensional approximations to the sets $\Lambda(L, M)$. We begin by assuming that the functions $f \in \Lambda(L, M)$ are real valued. The extension of our results to vector valued functions is trivial.

54 <u>Theorem:</u> Let Λ be the space of real valued Lipschitzian function defined in (16) and let $\Lambda(L, M)$ be as in (24). Then for any positive integer n there exists a finite set of functions f_i, $i = 1, 2, \ldots, \ell$ such that

55
$$\Lambda(L, M) \subset \bigcup_{i=1}^{\ell} \{f_i + U(n, 1/n)\},$$

with the cardinality ℓ of the set $\{f_i\}$ satisfying

56
$$\ell \leq 2^N \left(2\left[\frac{nL}{2}\right] - 1 \right),$$

where for all real x, $[x]$ denotes the smallest integer larger than x and $N = [n^2 M]$.

57 <u>Remark:</u> The cardinality number ℓ defines the dimension of the approximation $\mathrm{co}\{f_i\}$ to the set $\Lambda(L, M)$.

<u>Proof:</u> We shall characterize the functions f_i, $i = 1, 2, \ldots, \ell$, as paths

in a graph. First we note that if $f \in \Lambda(L, M)$, then

$$(\forall t)_{[t_0, \infty)} ||f(t)|| \leq L + Mt.$$

Hence the set $\Lambda(L, M)$ consists of functions whose graphs never leave the shaded area of Fig. 1. To construct the graph for characterizing the f_i (see Fig. 2), we draw half lines parallel to the upper and lower boundaries of Fig. 1. These half lines originate from points with abscissa t_0 and ordinate $-[nL/2]+1$, $-[nL/2]+2$, ..., -2, -1, 0, 1, 2, ..., $[nL/2]-2$, $[nL/2]-1$. We define a <u>forward path</u> as any path in the graph of Fig. 2, connecting an extreme left node, such as A_i to an extreme right node such as B_j, with the path always travelling from left to right. In Fig. 2, we show the case $\ell = 9$, $N=10$. With each path we now associate a function of the form

58 $$f(t) = \frac{2r}{n} + \alpha_0 \, s(Mt - Mt_0) + \sum_{k=1}^{N-1} (\alpha_k - \alpha_{k-1}) \, s\left(Mt - Mt_0 - \frac{k}{n}\right)$$

where, $r \in \{-[nL/2]+1, -[nL/2]+2, \ldots, -2, -1, 0, +1, +2, \ldots, [nL/2]-2, [nL/2]-1\}$, $s: R \to R$ is the ramp function, i.e.,

59 $$s(t) = \begin{cases} 0 & \text{for} \quad t \leq 0 \\ t & \text{for} \quad t > 0 \end{cases}$$

and $\alpha_k \in \{-1, +1\}$ for $k = 0, 1, 2, \ldots, N-1$. Note that $f(t)$ as defined in (58), is Lipschitzian and belongs to $\Lambda(L, M)$. We shall denote by $\mathcal{L}$ the sets of all functions obtained from (58) when $r, \alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ range through all their permissible values.

We first show that the cardinality of $\mathcal{L}$ is ℓ. Obviously, r can assume any one of $2\left[\frac{nL}{2}\right] - 1$ possible values. Similarly, each of the N α_i's can assume 2 different values. It is immediately clear that no two different (N+1)-tuples $(r, \alpha_0, \ldots, \alpha_{N-1})$ define the same function $f(t)$. Hence the cardinality of $\mathcal{L}$ is indeed ℓ.

Next we prove that (55) holds. The reader will find it helpful again to refer to Fig. 2. Note that (55) is equivalent to the statement that for every $f \in \Lambda(L, M)$ there exists a $f_i \in \mathcal{L}$ such that

60 $$f \in (\{f_i\} + U(n, \frac{1}{n}))$$

Equivalently, we only have to establish that

61 $$(\forall f)_{\Lambda(L,M)} (\exists f_i)_{\mathcal{L}} \text{ s.t. } (\forall t)_{\left[t_0, t_0 + \frac{N}{Mn}\right]} \|f(t) - f_i(t)\| \leq \frac{1}{n}$$

and $f_i(t)$ is as in (58) for some (N+1)-tuple $(r, \alpha_0, \alpha_1, \ldots, \alpha_{N-1})$.

The proof of (61) proceeds by induction on N†. For N=1, (61) is

---

† Rather than proceding as below, we could have proved (61) by establishing a one-to-one correspondence between Lipschitzian functions and monotonic increasing, upper semicontinuous functions and then used known results in measure theory. However, we prefer to give a direct demonstration.

clearly true for $f_i = \frac{2r}{n} + \alpha_0 S(Mt - Mt_0)$ where r is such that $||f(t_0) - \frac{2r}{n}|| \leq \frac{1}{n}$ and $\alpha_0 \epsilon \{+1, -1\}$. We now assume that (61) is true for $N-1$ and will prove that it is true for N. By the induction hypothesis, we have been able to find an $(N-1)$-tuple $(r, \alpha_0, \alpha_1, \ldots, \alpha_{N-2})$ such that

62 $$(\forall t)_{\left[t_0, t_0 + \frac{N-1}{nM}\right]} ||f(t) - g_i(t)|| \leq \frac{1}{n}$$

where

63 $$g_i(t) = \frac{2r}{n} + \alpha_0 s(Mt - Mt_0) + \sum_{k=1}^{N-2} (\alpha_k - \alpha_{k-1}) s(Mt - \frac{k}{n} - Mt_0)$$

Let us now define a function f_i by

64 $$f_i(t) = g_i(t) + (\alpha_{N-1} - \alpha_{N-2}) s(Mt - \frac{(N-1)}{n} - Mt_0)$$

Obviously $(\forall t)_{\left[t_0, t_0 + \frac{N-1}{nM}\right]} f_i(t) = g_i(t)$, i.e., from (62),

65 $$(\forall t)_{\left[t_0, t_0 + \frac{N-1}{nM}\right]} ||f(t) - f_i(t)|| \leq \frac{1}{n}$$

We now proceed to determine α_{N-1} such that

66 $$(\forall t)_{\left[t_0 + \frac{N-1}{nM}, t_0 + \frac{N}{nM}\right]} ||f(t) - f_i(t)|| \leq \frac{1}{n}$$

From (64), we can rewrite $f_i(t)$ in the interval $\Delta_N \triangleq \left[t_0 + \frac{N-1}{nM}, t_0 + \frac{N}{nM} \right]$ as

67 $$f_i(t) = g_i\left(\frac{N-1}{nM}\right) + \alpha_{N-1}\left(Mt - \frac{N-1}{n} - Mt_0 \right)$$

Note that for $\alpha_{N-1} = 1$ the graph of the function $f_i(t)$ goes upwards in the above time interval and for $\alpha_{N-1} = -1$ it goes downwards. To determine α_{N-1} we note that if $(\forall\, t)_{\Delta_N} \left|\left| f(t) - g_i\left(\frac{N-1}{nM}\right) - \left(Mt - Mt_0 - \frac{N-1}{n} \right) \right|\right| \leq \frac{1}{n}$ we should choose $\alpha_{N-1} = 1$. On the other hand, if this relation does not hold, it means that there exists a $t_1 \in \Delta_N$ such that

68 $$\left|\left| f(t_1) - g_i\left(\frac{N-1}{nM}\right) - \left(Mt_1 - Mt_0 - \frac{(N-1)}{n} \right) \right|\right| > \frac{1}{n}$$

and we should choose $\alpha_{N-1} = -1$. Indeed, in this case, if we could find a $t_2 \in \Delta_N$ such that

69 $$\left|\left| f(t_2) - g_i\left(\frac{N-1}{nM}\right) + (Mt_2 - Mt_0 - \frac{(N-1)}{n}) \right|\right| > \frac{1}{n}$$

we would conclude from (68) and (69), together with the continuity of f, that $||f(t_1) - f(t_2)|| > M||t_1 - t_2||$, which contradicts the assumption that $f \in \Lambda(L, M)$.

70 <u>Remark:</u> The extension of Theorem (54) to vector valued functions $f = (f^1, f^2, \ldots, f^n)$ is obviously trivial under the norms

$$||f(t)|| = \sup_{i \in \{1, 2, \ldots, n\}} f^i(t), \quad \text{or} \quad ||f(t)|| = \sum_{i=1}^{n} |f^i(t)| .$$

With a very small amount of effort, the extension can also be carried out for other norms.

71 Remark: Instead of playing games in spaces of Lipschitzian functions f defined on the semi-infinite interval $[t_0, \infty)$, we could have played in spaces of Lipschitzian functions defined on a given finite interval $[t_0, T]$, as is the case in fixed time differential games. When the Lipschitzian functions we consider are defined only on a finite interval $[t_0, T]$, we find that the set of neighborhoods $\{U(T - t_0, 1/n),\ n = 1, 2, \ldots\}$ form a countable base for this Lipschitzian function space, with $U(T, 1/n)$ defined as in (18). Referring to Fig. 2, we now see that the number of functions f_i necessary for the construction of a $1/n$-approximation to the set $\Lambda(L, M)$, with the interval of definition changed to $[t_0, T]$ in (24), grows considerably slower with n than in the case where the interval of definition for the funtions is $[t_0, \infty)$. This is due to the fact that the index N, which was equal to $[Mn^2]$ in (54), now becomes $N = nM(T - t_0)$ and hence is approximately proportional to n and not to n^2.

CONCLUSION:

The underlying philosphy behind the use of penalty functions and decomposition methods in optimization problems is to substitute a sequence of relatively easy problems for a very difficult one. In this paper, we have applied this type of thinking to games defined on infinite dimensional spaces.

We have established a set of properties with which finite dimensional approximations must be endowed, and we have shown that for certain classes of differential games and games played on convex, compact subsets of Lipschitzian function spaces such approximations always exist. We have also shown how to construct finite dimensional approximations for games played on the subsets $\Lambda(L, M)$ of Lipschitzian function spaces, and have obtained an upper bound ℓ, on the minimum dimension of an approximation (of fineness $1/n$). We suspect that the bound ℓ is actually a least upper bound.

Although we have not done it in this paper, it is reasonably easy to show that the construction used to obtain finite dimensional approximations to games played on the subsets $\Lambda(L, M)$ of Lipschitzian function spaces can readily be modified to obtain finite dimensional approximations for games on subsets consisting of upper (lower) semi-continuous functions which are bounded from above and from below.

Thus, the decomposition techniques discussed in this paper can be applied to a broad class of problems and we hope that they will lead to new and interesting computational results.

ACKNOWLEDGMENT. We wish to thank Mr. P. Rossi, of IBM France, for his very helpful comments and his discussion of our results.

REFERENCES

[1] Edwards, R. E., "Functional Analysis: Theory and Applications," Holt, Rinehart and Winston, New York, 1965.

[2] Kelley, J. L., Namioca, I, Linear Topological Spaces, Van Nostrand, New Jersey, 1963.

[3] Kelley, J. L., "General Topology," Van Nostrand, New Jersey, 1955.

[4] Varaiya, P. P., "On the Trajectories of a Differential System," Proc. Conference on the Mathematical Theory of Control, Los Angeles, 1967.

[5] Jacob, J-P, Polak, E., "On a Class of Pursuit-Evasion Problems," IBM Report, RJ 421, Feb. 1967.

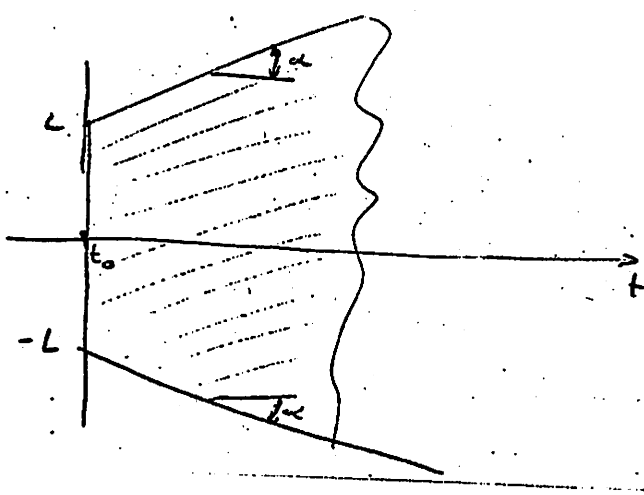

Fig. 1. The Bounds on Functions in $\Lambda(L,M)$.

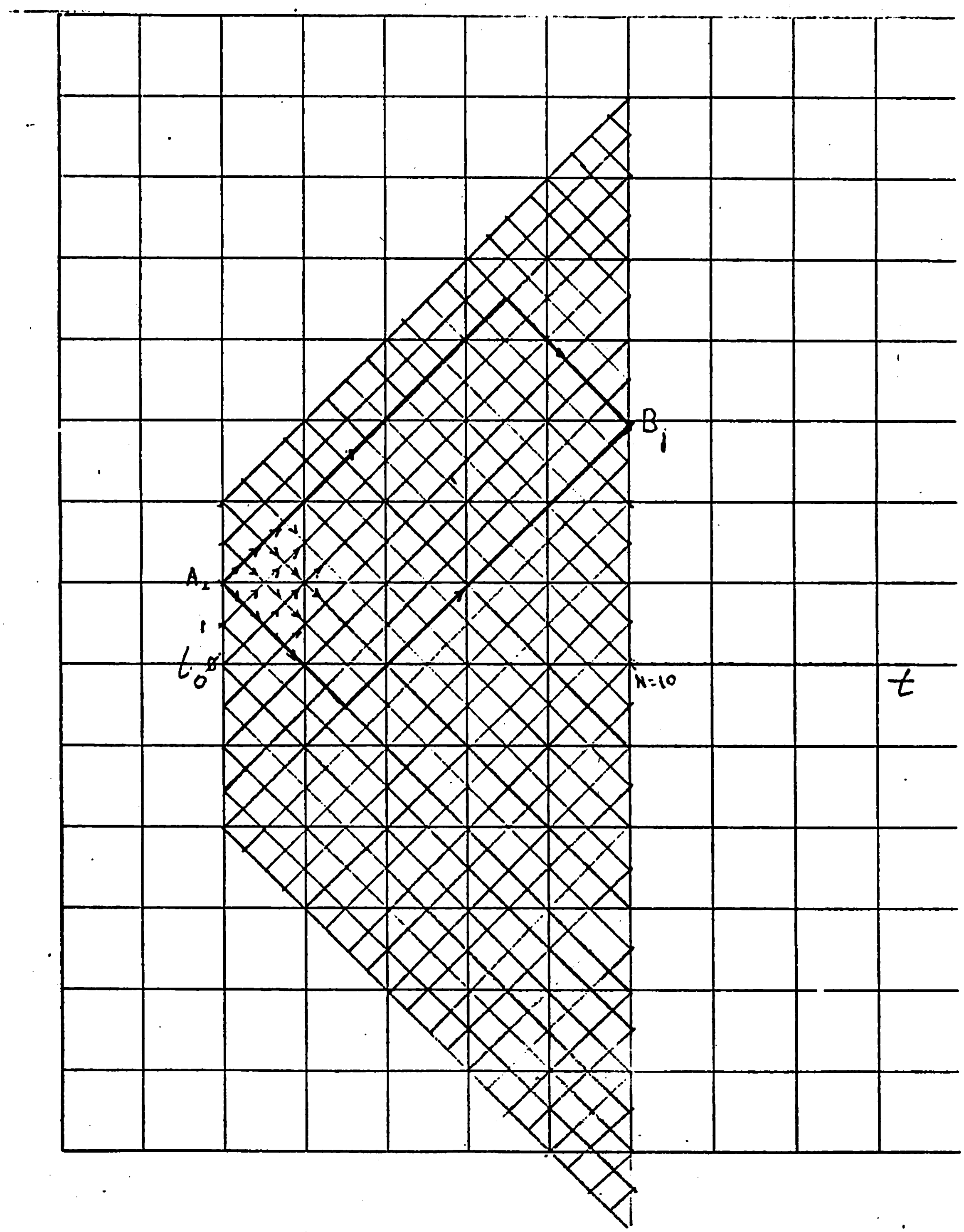

Fig. 2. Construction of the Functions f_i.