SMALL SIGNAL BEHAVIOR OF NONLINEAR NETWORKS

by

C. A. Desoer and K. K. Wong

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ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

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Abstract - This paper develops two theorems concerning the small-signal behavior of nonlinear time-varying networks whose state equations are of the form $\dot{x} = f(x, u, t)$. The conclusions of the theorems are supported by experiments. The input is of the form $U(t) + u(t)$, where the bias, $U(t)$, is allowed to be time varying (typically, slowly varying) and $u(t)$ is the small signal. The bias induces a moving operating point $X(t)$. Given some simple assumptions concerning the linearized small-signal equivalent circuit it is shown that provided $u(t)$ is sufficiently small on $[0, \infty)$, the state trajectory about the operating point is bounded on $[0, \infty)$ and tends to zero as $u \rightarrow 0$. The method of proof also shows that this result applies to some distributed circuits. The second theorem shows that the push-pull connection reduces the distortion due to the nonlinearities of both resistors and energy storing elements. The third part of the paper describes numerical experiments that support the conclusions of the theory.

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The authors are with the Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, California.

I. Introduction

The purpose of this paper is to present new results on the small-signal behavior of nonlinear time-varying lumped networks. The need for such results arises from two developments: many of the new devices have a rather small region of linear operation so that the question naturally arises as to what happens if they are operated further into their nonlinear regions. The first part of this paper is a first step in this direction: it discusses the difference between the response of a nonlinear time-varying network $\mathcal{N}$ and that of its linearized small-signal equivalent circuit. Also the second part shows that this difference is further reduced by the push-pull configuration. The second development is the speed and convenience of digital computers: now engineers can calculate and plot easily and cheaply the responses of nonlinear networks. To back up such computation, theoretical results are necessary and this paper is a contribution in this direction.

In contrast to the somewhat restricted models of the classical literature [1] - [6] we take as our model the equations in the normal form: $\dot{x} = f(x, u, t)$. Note that $\dot{x}$ is not necessarily linearly dependent on u, as is usually assumed. We also allow a varying operating point. The reasons for these two departures are the following: fundamental studies of the equations of nonlinear time-varying lumped networks have shown that the usual model is not general enough [7] - [10]. Also recent developments in the study of constant resistance networks made of nonlinear time-varying elements raise the hope that one might envisage the design of electronically adjustable filters, equalizers and delay lines by cascading sections of constant resistance two-ports whose elements are nonlinear [11] and which would be operated in the small-signal mode with, in practice, slowly-varying operating points. These are the motivating thoughts for the first part of the paper. The first part of the paper also illustrates a number of fundamental techniques of nonlinear theory: Taylor expansions, manipulation of inequalities, a basic bootstrap technique, use of the Bellman-Gronwall inequality and iterations. The results of the first part generalize in several directions some previous perturbational analysis performed by the authors [12].

The second part gives a rigorous proof of the fact that by mounting two identical nonlinear time-varying dynamical networks in push-pull, the behavior of the push-pull connection is much closer to that of the linearized small-signal equivalent network. It should be stressed that in contrast to previous analyses, the networks mounted in push-pull may have an arbitrary number of nonlinear time-varying energy storing elements.

The last part of the paper describes in some detail one of the experiments carried out by the authors. Incidentally, it gives also a design method for electronically adjustable filters.

Notations

Following numerous authors, we define

R : set of all real numbers

R_+: set of all nonnegative real numbers

R^k (with k integer): space of k-tuples of real numbers.

C : class of continuous functions.

C^3: class of functions that have a continuous third derivative.

In the following we shall encounter scalars, vectors (in R^n or R^m) and elements of function spaces. The symbol $|\cdot|$ is used to denote the magnitude of a scalar and the norm of a vector in R^n or R^m. The developments that follow are valid independently of the choice of norm in R^n because all norms in R^n are equivalent. The symbol $||\cdot||$ is used as follows: let w map R_+ into R^n (or R^m), then by definition

$$||w|| = \sup_{t \geq 0} |w(t)|$$

II. Small-signal Behavior

It is well known that a large class of nonlinear time-varying networks can be described by equations in the normal form [7]-[10]. Thus we assume for the network $\mathcal{N}$ under consideration the state description

$$\dot{x} = f(x, \hat{u}, t) \tag{1}$$

where the n-vector x represents the state and the m-vector $\hat{u}$ represents the input (which consists of m independent sources). We assume throughout that the network is in the zero state at $t = 0$:

$$x(0) = 0 \tag{2}$$

Since our purpose is to discuss the small-signal behavior of such networks we take the input $\hat{u}$ to be the sum of a time-varying bias U and a perturbational signal u. Throughout the following we shall have $\|u\|$ small.

Basic Assumptions

In order to proceed with the analysis we must make a number of assumptions which we label F1, F2, ... The first one is

(F1) $$f(0, 0, t) = 0 \qquad \forall t \geq 0.$$

This will be the case, for instance, if all element characteristics go through the origin at all times.

The second assumption is that the bias is a bounded continuous function. For future reference we set

(U 1) $$\sup_{t \geq 0} |U(t)| = U_m < \infty$$

The operating point X(t) satisfies the equation

$$\dot{X} = f(X(t), U(t), t), \qquad X(0) = 0. \tag{3}$$

We assume that X is bounded and we set

(F2) $$\sup_{t \geq 0} |X(t)| = X_m < \infty$$

We restrict the behavior of f as a function of t as follows:

(F3) For each $x \in R^n$, $u \in R^m$, $f(x, u, \cdot) : R_+ \to R^n$ is a bounded and regulated [13, see sec. 7.6] function.

The next assumption requires more discussion and is related to the fact that the network must maintain a "good" behavior as time goes on. More specifically, for each <u>fixed</u> $t \geq 0$, f is a mapping of $R^n \times R^m$ into R^n. Call $D_1 f(X(t), U(t), t)$ the derivative of f with respect to x and evaluated at $(X(t), U(t), t)$. This derivative is a <u>linear</u> map of $R^n \to R^n$; it is represented by a matrix whose elements are $\left.\frac{\partial f_i}{\partial x_j}\right|_{(X(t), U(t), t)}$ with $i, j = 1, 2, \ldots, n$. Similarly for $D_2 f(X(t), U(t), t)$, the derivative of f with respect to u; its elements are $\left.\frac{\partial f_i}{\partial u_k}\right|_{(X(t), U(t), t)}$, $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$. By Taylor's expansion theorem we can write <u>for each</u> t

$$f(X(t) + \xi, U(t) + u, t) = f(X(t), U(t), t) + D_1 f(X(t), U(t), t) \cdot \xi + D_2 f(X(t), U(t), t) \cdot u + g(\xi, u, t) \tag{4}$$

Clearly g represents the second order terms, and its dependence on X(t) and U(t) has been absorbed in its dependence on t. The next step is to obtain bounds on g. In order not to get bogged down in notation we consider, for each fixed t, f as a mapping of $R^n \times R^m \to R^n$ and we denote by $f^{(2)}$ its second derivative. With these notations in mind we have [13, sec. 8.14.3]

$$g(\xi, u, t) = \left[\int_0^1 (1 - \alpha) f^{(2)} (X(t) + \alpha \xi, U(t) + \alpha u, t) \, d\alpha \right] \cdot (\xi + u)^{(2)} \tag{5}$$

where in the expression "$\xi + u$" we consider ξ to be a vector in $R^n \times R^m$ (with its last m components zero) and u also in $R^n \times R^m$ (with its first n components zero).

The representation of $f^{(2)}$ involves terms of the form

$$\frac{\partial^2 f_i}{\partial x_k \, \partial x_\ell}, \quad \frac{\partial^2 f_i}{\partial x_k \, \partial u_j}, \quad \frac{\partial^2 f_i}{\partial u_j \, \partial u_p} \tag{6}$$

where $i, k, \ell = 1, 2, \ldots, n$ and $j, p = 1, 2, \ldots, m$.

In order to justify the above Taylor expansion and obtain suitable bounds on g we impose the following requirements:

(F4) For the bias signal U(t) under consideration and for the resulting operating point X(t) and for <u>any</u> finite numbers ξ_m and u_m, all the second order partial derivatives of the form (6) are continuous in the domain $|X + \xi| \leq X_m + \xi_m$, $|U + u| \leq U_m + u_m$ and $t \geq 0$; furthermore, they are bounded: more precisely this means that for any finite numbers ξ_m and u_m, the following quantities are finite:

$$\sup \frac{\partial^2 f_i}{\partial x_k \, \partial x_\ell} \qquad i, k, \ell = 1, 2, \ldots, n ; \tag{7}$$

$$\sup \frac{\partial^2 f_i}{\partial x_k \, \partial u_j} \qquad i, k = 1, 2, \ldots, n ; \; j = 1, 2, \ldots, m ; \tag{8}$$

$$\sup \frac{\partial^2 f_i}{\partial u_j \, \partial u_p} \qquad i = 1, 2, \ldots, n ; \; j, p = 1, 2, \ldots, m ; \tag{9}$$

and where all the partial derivatives are evaluated at $(X(t) + \xi, \; U(t) + u, \; t)$, and all sup are taken over the set

$$|X(t) + \xi| \leq X_m + \xi_m , \quad |U(t) + u| \leq U_m + u_m , \quad t \geq 0 .$$

If this assumption F3 is satisfied, the bracket in the right-hand side of (5) is bounded and for some finite number S (which depends on $U(\cdot)$, $X(\cdot)$, ξ_m and u_m)

$$|g(\xi, u, t)| \leq S \, (|\xi| + |u|)^2 . \tag{10}$$

It is important to note that S is independent of t .

Analysis

In the following we consider U to be a given bias signal (and, consequently, so is X, by (3)) and u to be arbitrary but small. The exact equations of the nonlinear network $\mathcal{N}$ are of the form

$$\dot{x}(t) = f(x(t),\ U(t) + u(t),\ t) \qquad x(0) = 0 \tag{11}$$

or, since we think of its state as being an operating point X(t) plus a small displacement $\xi(t)$,

$$\dot{X}(t) + \dot{\xi}(t) = f(X(t) + \xi(t),\ U(t) + u(t),\ t) \tag{12}$$

If we call

$$A(t) = D_1 f(X(t),\ U(t),\ t) \tag{13}$$

$$B(t) = D_2 f(X(t),\ U(t),\ t) \tag{14}$$

then, using (3) and (4), we may rewrite the exact equations of $\mathcal{N}$ in the form

$$\dot{\xi}(t) = A(t)\,\xi(t) + B(t)\,u(t) + g(\xi(t),\ u(t),\ t) \tag{15}$$

and

$$\xi(0) = 0 .$$

Since, for all t, the term $g(\xi, u, t)$ is of second order in $(\xi + u)$, the differential equation

$$\dot{\xi}_0(t) = A(t)\,\xi_0(t) + B(t)\,u(t) \qquad \xi_0(0) = 0 \tag{16}$$

represents the small-signal equivalent network (linearized about the operating point X(t)). We call this network $\mathcal{N}_0$. The results of this paper compare ξ and ξ_0 for small u.

Since the small signal equivalent network is a linear time-varying network, it is natural to introduce its state transition matrix $\Phi(t, t')$.

From (16), it follows that

$$\xi_0(t) = \int_0^t \Phi(t, t') B(t') u(t') dt' \qquad t \geq 0 . \tag{17}$$

Remark. The consequences of assumption (F4) were given in the form of the inequality (10). This inequality implies that g also satisfies the following condition:

For all $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|g(\xi, u, t)| \leq \varepsilon(|\xi| + |u|) \tag{18}$$

for all $\qquad (|\xi| + |u|) \leq 2\delta .$

Note that in the new inequality the right-hand side is linear in $(|\xi| + |u|)$: it is this feature that allows ε to be taken arbitrarily small.

The complete statement of our main result is:

Theorem I: Consider a nonlinear time-varying network whose state equations are of the form (11) and assume that (F1), (F2), (F3) and (U1) hold. Let $\xi(\cdot)$ and $\xi_0(\cdot)$ be defined by (15) and (16) respectively. Assume further that

(i) there exists a finite constant $M > 0$ such that for all $t \geq 0$,

$$\int_0^t |\Phi(t, t')| \, dt' \leq M$$

(ii) $\sup_{t \geq 0} |B(t)| = B_m < \infty$

(iii) for all $\varepsilon > 0$ there is a δ such that

$$(|\xi| + |u|) \leq 2\delta \quad \text{implies} \quad |g(\xi, u, t)| \leq \varepsilon(|\xi| + |u|)$$

Assumption (iii) allows us to pick ε as small as we wish; in the following we consider only those ε smaller than $1/M$.

Under these conditions, if

$$||u|| \leq \frac{1 - \varepsilon M}{M(B_m + \varepsilon)} \delta \tag{19}$$

then

(a) $||\xi_0|| \leq M B_m \, ||u|| \leq (1 - \varepsilon M) \, \delta$

(b) $||\xi|| \leq \delta$

(c) $\dfrac{||\xi - \xi_0||}{||u||} \leq \dfrac{\varepsilon M(1 + M B_m)}{1 - \varepsilon M}$.

The proof of the theorem is in Appendix I.

Remarks I. Assertions (a) and (b) mean that the responses of both $\mathcal{N}_0$ (the linearized small-signal equivalent network) and that of $\mathcal{N}$ (the given nonlinear network) are bounded on $[0, \infty)$. Furthermore, by taking $||u||$ small (hence δ small), the responses ξ_0 and ξ can be made as small as we wish.

II. Assertion (c) states that the ratio of $||\xi - \xi_0||$ (i.e., the "peak" value of the difference $\xi(t) - \xi_0(t)$) to $||u||$ (i.e., the "peak" value of the input) can be made as small as we wish by taking $||u||$ sufficiently small. Indeed to take $||u||$ small is equivalent to taking δ small, and the smaller δ is, the smaller the corresponding ε.

III. It is important to stress the fact that nowhere shall we use the group properties of the state transition matrix [14]. As a consequence the theorem also applies to the nonlinear integral equation equivalent to (15), namely,

$$\xi(t) = \xi_0(t) + \int_0^t \Phi(t, t')\, g(\xi(t'), u(t'), t')\, dt' \qquad t \geq 0 . \tag{20}$$

Therefore, the results above are applicable to some distributed circuits.

IV. At first sight one might believe that only the assumption

$$|\Phi(t, t')| \leq K \qquad \text{for all } t \geq t' \geq 0$$

(where K is a constant) is needed. However this is not the case as shown by the following scalar example:

$$\dot{\xi} = \xi^2 + u \qquad \xi(0) = 0 .$$

In this instance, $\Phi(t, t') = 1(t - t')$, where $1(t)$ is the unit step. Now if $u(t) = \varepsilon > 0$ on $[0, 1]$ and zero elsewhere, it is easy to see that the solution increases monotonically and has a finite escape time!

Two Special Cases

A. We consider a class of nonlinear time-varying networks whose state equations can be written in the form

$$\dot{x} = f(x, t) + B(t)\, u \qquad x(0) = 0 . \tag{21}$$

There is no easy way of describing the class of networks for which state equations can be written in the form (21). For example, this will be the case for nonlinear time-varying RLC networks for which a proper tree exists such that each fundamental loop defined by a link resistor includes no tree-branch resistor; furthermore, the location of the independent sources are restricted in the following way: if a current source is in parallel with a tree-branch resistor, that resistor is linear (although possibly time-varying), and if a voltage source is in series with a link resistor, that resistor is linear.

Manipulations similar to those of the previous case result in the following equations: for the nonlinear network, $\mathcal{N}$,

$$\dot{\xi}(t) = A(t)\,\xi(t) + B(t)\,u(t) + g(\xi(t), t) \qquad \xi(0) = 0 \tag{22}$$

and for the small-signal equivalent network, $\mathcal{N}_0$,

$$\dot{\xi}_0(t) = A(t)\, \xi_0(t) + B(t)\, u(t) \qquad \xi_0(0) = 0. \tag{23}$$

Because u does not enter the higher order term g in (22) the results of the main theorem may be sharpened.

Corollary I. Under the same assumptions as in Theorem I, (except for that g depends now only on ξ and t, hence assumption (iii) must be modified to read: $|\xi| \leq \delta$ implies $|g(\xi, t)| \leq \varepsilon\, |\xi|$) not only the conclusions (a) and (b) hold but also

$$\text{(c')} \quad \frac{||\xi - \xi_0||}{||\xi||} \leq \varepsilon M$$

$$\text{(d)} \quad \frac{||\xi - \xi_0||}{||\xi_0||} \leq \frac{\varepsilon M}{1 - \varepsilon M}$$

The proof of Corollary I is given in Appendix I.

In addition to the remarks following the previous theorem we wish to stress the interpretation of (c') and (d). Consider (d) for example. The left-hand side is

$$\frac{\sup |\xi(t) - \xi_0(t)|}{\sup |\xi_0(t)|} \qquad \text{(sup taken over } t \geq 0) \tag{24}$$

that is the ratio of the "peak" instantaneous deviation of $\xi(t)$ from $\xi_0(t)$ to the "peak" value of $\xi_0(t)$; (d) asserts that this ratio goes to zero as $\varepsilon \to 0$. Now if we consider a sequence of experiments where $||u||$ is taken successively smaller and going to zero, then ε can be taken to go to zero. Thus (d) asserts that by taking the input sufficiently small we can make arbitrarily small the ratio of the "peak" instantaneous deviation of the nonlinear system response from that of the small-signal linearized system to the "peak" value of the linearized system response. In short not only, $\xi \to \xi_0$ (uniformly in $t \geq 0$) as $||u|| \to 0$ but

$\frac{||\xi - \xi_0||}{||\xi_0||} \to 0$: the relative deviation goes to zero.

B. Going back to the general case, i.e., to networks described by (1), we can by sharpening the assumption on Φ obtain an additional conclusion on asymptotic behavior.

Corollary II. Let assumption (i) of Theorem I be sharpened to read

(i') there exist positive constants K and σ such that for all $t \geq t' \geq 0$

$$|\Phi(t, t')| \leq K e^{-\sigma(t-t')}$$

Let (ii) and (iii) hold.

We pick ε so that $\sigma - \varepsilon K = \sigma' > 0$ and $||u||$ sufficiently small so that $||\xi|| \leq \delta$.

Under these conditions, if $u(t) \to 0$ as $t \to \infty$, then $\xi_0(t) \to 0$ and $\xi(t) \to 0$.

The proof of the corollary is given in Appendix I.

III. Distortion Correction by Push-Pull

In the first part of this paper, we considered the state trajectory of a nonlinear network $\mathcal{N}$ and that of its linearized version, $\mathcal{N}_0$, about a moving operating point X. We gave conditions under which if the small-signal input u is sufficiently small, then the relative distance between the states is uniformly small on $[0, \infty)$. Is there a way of reducing this distance by appropriate design? In the following we shall prove rigorously that a push-pull connection will do just that. All the studies of the push-pull connection of nonlinear networks known to the authors assume a model consisting of a linear network driving a dependent source which has a memoryless nonlinear characteristic. Our analysis allows an arbitrary number of nonlinearities in energy storing elements as well as in resistors.

We formulate the problem as follows. Let $\mathcal{N}$ be a nonlinear network whose state and output description is of the form

$$\dot{x} = f(x, \hat{u}, t) \qquad x(0) = 0 \tag{25}$$

$$v = h(x, \hat{u}, t) \tag{26}$$

where $x(t) \in R^n$ and $\hat{u}(t) \in R^m$. For simplicity, we assume that the output v is scalar valued, i.e., $v(t) \in R$. The smoothness conditions on f and h will be specified later.

Suppose that $\mathcal{N}_1$ and $\mathcal{N}_2$ are two networks identical to $\mathcal{N}$ and that we connect them in the manner shown on Fig. 1. We denote this push-pull configuration as $\mathcal{P}$. The state description of $\mathcal{P}$ is then

$$\begin{cases} \dot{x}_1 = f(x_1, \hat{u}_1, t), \ x_1(0) = 0 & \dot{x}_2 = f(x_2, \hat{u}_2, t), \ x_2(0) = 0 \\ v_1 = h(x_1, \hat{u}_1, t) & v_2 = h(x_2, \hat{u}_2, t) \\ \hat{u}_1 = U + u & \hat{u}_2 = U - u \end{cases} \tag{27}$$

The output voltage is

$$v = v_1 - v_2 = h(x_1, \hat{u}_1, t) - h(x_2, \hat{u}_2, t) \tag{28}$$

Intuitively we expect that the symmetry of the push-pull configuration $\mathcal{P}$ and the antisymmetry of the small-signal sources would lead to cancellation of second order terms in $||u||$: in other words, the push-pull configuration is more linear. This intuition is made precise by the following theorem.

Theorem II. Consider the push-pull configuration $\mathcal{P}$ shown on Fig. 1 and described by Eqs. (27) and (28). Suppose that

(i) for each t and each $u \in R^m$, $f(\cdot, u, t)$ is Lipschitz,

(ii) for each t, $f(\cdot, \cdot, t) \in C^3$ and $h(\cdot, \cdot, t) \in C^3$ (with respect to both arguments),

(iii) for each $x \in R^n$ and each $u \in R^m$, $f(x, u, \cdot) \in C$ and $h(x, u, \cdot) \in C$.

The functions $X(\cdot)$ and $\xi_0(\cdot)$ are defined by

$$\dot{X} = f(X, U, t), \qquad X(0) = 0 \tag{29}$$

$$\dot{\xi}_0 = A(t)\,\xi_0 + B(t)\,u, \qquad \xi_0(0) = 0 \tag{30}$$

where, as before, $A(t) = D_1 f(X(t), U(t), t)$, $B(t) = D_2 f(X(t), U(t), t)$. It is assumed that for all $T > 0$, the matrices $A(\cdot)$ and $B(\cdot)$ are bounded on $[0, T]$.

Let h_i denote the derivative of h with respect to its ith argument and

$$z(t) = h_1(X, U, t)\,\xi_0(t) + h_2(X, U, t)\,u(t) \tag{31}$$

Under these conditions, for any $T > 0$, the output voltage of the push-pull configuration can be written in the form

$$v(t) = 2z(t) + O(\|u\|_T^3) \qquad 0 \le t \le T \tag{32}$$

where $\|u\|_T = \sup_{0 \le t \le T} |u(t)|$

The proof of this theorem is given in Appendix II.

<u>Comments</u>

(I) The function $z(\cdot)$ defined by (31) is the zero-state response of the linear network obtained by linearizing $\mathcal{N}_1$ about the operating point X. Whereas the response of $\mathcal{N}_1$ differs from the response of its linearized version by a term $O(\|u\|_T^2)$, the zero-state response of the push-pull configuration $\mathcal{P}$ differs from that of $\mathcal{N}_1$ by a term $O(\|u\|_T^3)$.

(II) It is a fact that the output v of the push-pull configuration is an odd function of u : the even part of the dependence of v_1 and v_2 on u cancels out.

IV. Experimental Results

Since by their very nature the results above give bounds whose scale factors are hard to calculate; experiments are required to find out how these theoretical claims turn out in practice. Several experiments were carried out: some with sinusoidal inputs, some with square wave inputs, some with fixed bias and some with varying bias. In all cases the experiments confirmed the expectations of the theory. We shall report in detail on one experiment. In doing so we shall tie together a number of the ideas developed in the theory, we shall verify that the claims do apply to networks which have nonlinear energy storing elements and we shall exhibit a design procedure for nonlinear networks to be operated in the small signal mode.

Design

We propose to design a Butterworth low-pass filter whose 3-db cutoff frequency can be adjusted electronically by varying the bias. The circuit is shown in Fig. 2. A linear time-invariant Butterworth low-pass filter with ω_0 as 3 db cutoff frequency has the following state and output equations:

$$\begin{bmatrix} \dot{q} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = \omega_0 \begin{bmatrix} 0 & \frac{2}{3} & -2 \\ -\frac{3}{4} & 0 & 0 \\ \frac{3}{4} & 0 & -2 \end{bmatrix} \begin{bmatrix} q \\ \phi_1 \\ \phi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot e \tag{35}$$

$$v_0 = 2\,\omega_0\,\phi_2 \tag{36}$$

where the capacitor is $\frac{4}{3\,\omega_0}$ F and the inductances are $\frac{3}{2\,\omega_0}$ H and $\frac{1}{2\,\omega_0}$ H, respectively. The design requirement is that, about any constant operating point created by the dc bias E, the small-signal response must be described by equations of the form (35) and (36). We assume that the nonlinear elements have characteristics that are monotonically increasing, differentiable functions that go through zero and that map the real line onto the real line in a one-to-one fashion. We call them

$$q = \psi(v) \tag{37}$$

$$\phi_1 = \psi_1(i_1) \tag{38}$$

$$\phi_2 = \psi_2(i_2) \tag{39}$$

The state equations of the nonlinear network are

$$\dot{q} = \psi_1^{-1}(\phi_1) - \psi_2^{-1}(\phi_2) \tag{40a}$$

$$\dot{\phi}_1 = -\,\psi^{-1}(q) + E + e \tag{40b}$$

$$\dot{\phi}_2 = \psi^{-1}(q) - \psi_2^{-1}(\phi_2) \tag{40c}$$

It can be shown that, for any constant E, the resulting operating point is asymptotically stable in the large [12]. The operating point can be calculated from Eqs. (40a-c) by setting $\dot{q} = \dot{\phi}_1 = \dot{\phi}_2 = 0$. Using obvious notations, the operating point (V, I_1, I_2) is characterized by

$$V = I_1 = I_2 = E \tag{41}$$

About each operating point specified by E, the linearized small-signal equivalent network has the following equations

$$\dot{\xi} = A\,\xi + b\,e \tag{42}$$

where

$$\xi = (q - \psi(E),\ \phi_1 - \psi_1(E),\ \phi_2 - \psi_2(E))',$$

$$A = \begin{bmatrix} 0 & \dfrac{1}{\psi_1'(E)} & \dfrac{1}{\psi_2'(E)} \\ -\dfrac{1}{\psi'(E)} & 0 & 0 \\ -\dfrac{1}{\psi'(E)} & 0 & \dfrac{1}{\psi_2'(E)} \end{bmatrix} \tag{43}$$

and

$$b = (0,\ 1,\ 0)'.$$

Comparison of (42), (43) with (35) shows that ω_0 will depend on E and that $\psi'(E)$, $\psi_1'(E)$ and $\psi_2'(E)$ will be related. For convenience we introduce a new function $h(\cdot)$ defined by

$$\omega_0(E) = 1/h'(E) \qquad \text{for all } E. \tag{44}$$

The comparison and the requirement that the characteristic go through the origin give

$$\psi(\cdot) = \frac{4}{3}h(\cdot)\ ; \qquad \psi_1(\cdot) = \frac{3}{2}h(\cdot)\ ; \qquad \psi_2(\cdot) = \frac{1}{2}h(\cdot)\ . \tag{45}$$

When the three nonlinear characteristics are thus related, then, for all values of the bias E, the linearized small-signal equivalent network about the corresponding operating point is a Butterworth filter whose 3 db cutoff is $1/h'(E)$.

The corresponding push-pull configuration is shown in Fig. 3. The state of the network $(q, \phi_1, \phi_2, \hat{q}, \hat{\phi}_1, \hat{\phi}_2)$ is related to the inductor currents and capacitor charges by

$$\begin{aligned} v &= h^{-1}(\tfrac{3}{4}q) & \hat{v} &= h^{-1}(\tfrac{3}{4}\hat{q}) \\ i_1 &= h^{-1}(\tfrac{2}{3}\phi_1) & \hat{i}_1 &= h^{-1}(\tfrac{2}{3}\hat{\phi}_1) \\ i_2 &= h^{-1}(2\,\phi_2) & \hat{i}_2 &= h^{-1}(2\,\hat{\phi}_2) \end{aligned} \tag{46}$$

The output is given by

$$v_p = h^{-1}(2\,\phi_2) - h^{-1}(2\,\hat{\phi}_2)\ . \tag{47}$$

Results

We choose

$$h(E) = E + \tanh E$$

consequently, the small-signal cutoff frequency is

$$\omega_0(E) = 1 + \operatorname{sech}^2 E.$$

Thus as E increases from 0 to ∞, ω_0 decreases from 2 to 1 rad/s. The responses shown in Figs. 4, 5 and 6 correspond to a square wave input of amplitude A with period 2 sec., and to a bias of E = 1V. (Hence $\omega_0(1) = 1.7708$ rad/s.). The amplitude A varies from figure to figure: for Fig. 4, A = 1.00; for Fig. 5, A = 0.316; for Fig. 6, A = 0.100. The curves are the output voltages of the network and are labelled as follows:

- v_N for the nonlinear network
- v_L for the linearized small-signal equivalent network
- $v_{N'}$ for the "pull" part of the push-pull connection.

In each figure, we subtracted the contribution of the bias. These curves show clearly how $v_N - v_L$ decreases as the amplitude A of the input decreases. The lack of symmetry of v_N in Fig. 4 can easily be related to the curvature of the characteristics.

Figures 7, 8 and 9 (with A = 1.00, A = 0.316 and A = 0.100, respectively) show the differences between the nonlinear network response and the push-pull response and that of the linearized small-signal equivalent network:

$$\Delta_N = v_N - v_L$$

$$\Delta_P = (v_N - v_{N'}) - 2 v_L$$

What is particularly notable is not only the absolute decrease of the distortion (both Δ_N and Δ_P) as the input amplitude A decreases, but also the relative decrease of the ratio $\frac{\Delta_P}{\Delta_N}$. These dramatic decreases are calculated in Table I. A final point: this calculation requires high precision as shown by the ratio $||v_L|| / ||\Delta_P||$ which is 10^3 (for A = 0.100) and noise is not visible in the curve of Δ_P ! Such computation could not be done on an analog computer.

Table I

A	1.00	0.316	0.100
$\|\|v_L\|\|$	1.161	0.367	0.116
$\|\|\Delta_N\|\|$	$1.96\ 10^{-1}$	$1.98\ 10^{-2}$	$1.93\ 10^{-3}$
$\|\|\Delta_P\|\|$	$5.1\ 10^{-2}$	$2.36\ 10^{-3}$	$1.0\ 10^{-4}$
$\frac{\|\|\Delta_N\|\|}{\|\|\Delta_P\|\|}$	3.8	8.4	19

Conclusion

This paper has shown that given a nonlinear time-varying lumped network subjected to a time-varying bias, it is possible to predict that its response to small signals would remain bounded only on the basis of properties of the linearized small-signal equivalent network. Furthermore, it showed that the nonlinear distortion can be further reduced by the use of the push-pull connection. Finally, it should be stressed that the theorems above are not restricted to nonlinear networks but apply also to differential nonlinear time-varying systems.

Acknowledgments

The authors express their gratitude to Don Simcox whose programming skill coaxed the computer into calculating and plotting the curves of Figs. 4-9.

Appendix I

Proof of Theorem I.

For the proof ε is fixed and assumed smaller than $1/M$. Also $\frac{1-\varepsilon M}{M(B_m+\varepsilon)}$ may be assumed to be < 1; indeed it can be always made so by taking B_m larger. This proof is based on a bootstrap technique: more precisely, for the purpose of the proof, we assume temporarily that for the ε under consideration and for all $(\xi, u, t) \in R^n \times R^m \times R_+$

$$|g(\xi, u, t)| \leq \varepsilon(|\xi| + |u|). \tag{A.1}$$

Taking the norm of (20) and using the properties of the norm and (A.1), we obtain

$$|\xi(t)| \leq |\xi_0(t)| + \varepsilon \int_0^t |\Phi(t, t')| \, (|\xi(t')| + u(t')|) \, dt' \tag{A.2}$$

Let us majorize the right-hand side by replacing $|\xi(t')|$ and $|u(t')|$ by $||\xi||$ and $||u||$ respectively. Finally, with assumption (i) we obtain

$$|\xi(t)| \leq ||\xi_0|| + \varepsilon M(||\xi|| + ||u||) \qquad t \geq 0 . \tag{A.3}$$

Note that at this point $||\xi||$ may perfectly well be infinity. Since inequality (A.3) holds for all t and the right-hand side is independent of t, we may write

$$||\xi|| \leq ||\xi_0|| + \varepsilon M(||\xi|| + ||u||) \tag{A.4}$$

and, using similar steps on (17) we have

$$||\xi_0|| \leq M\, B_m\, ||u|| \tag{A.5}$$

Using (A.5) into (A.4), and transposing the term $\varepsilon M||\xi||$, we obtain

$$(1 - \varepsilon M)\, ||\xi|| \leq M(B_m + \varepsilon)\, ||u|| \tag{A.6}$$

Since $\varepsilon M < 1$ and since $||u|| < \infty$, it follows that $||\xi|| < \infty$. In fact using (19), we obtain

$$||\xi|| \leq \delta \tag{A.7}$$

Therefore $|\xi(t)| \leq \delta$ for all $t \geq 0$; similarly, from (19) $|u(t)| \leq \delta$ for all $t \geq 0$. Therefore, the inequality (A.1) need only hold for $|\xi| \leq \delta$, $|u| \leq \delta$ and $t \geq 0$, but this is guaranteed to be so by (iii) which asserts that $(|\xi| + |u|) \leq 2\delta$ implies $|g(\xi, u, t)| \leq \varepsilon(|\xi| + |u|)$. Therefore (A.5) and (A.7) hold under the assumptions of the theorem, hence (a) and (b) have been proven.

From (20) we obtain successively

$$|\xi(t) - \xi_0(t)| \leq \varepsilon \int_0^t |\Phi(t, t')|\, (|u(t')| + |\xi(t')|)\, dt' \tag{A.8}$$

$$\leq \varepsilon M(||\xi|| + ||u||) \tag{A.9}$$

$$\leq \varepsilon M(||\xi - \xi_0|| + ||\xi_0|| + ||u||) \tag{A.10}$$

hence

$$||\xi - \xi_0|| \leq \frac{1}{1 - \varepsilon M}\, \varepsilon M(||\xi_0|| + ||u||) \tag{A.11}$$

$$\leq \frac{\varepsilon M(1 + MB_m)}{1 - \varepsilon M} \; ||u|| \tag{A.12}$$

From which (c) follows.

Q.E.D.

Alternate proof [12, see Appendix]. Conclusion (b) can also be established as follows: first obtain (A.5); then solve (20) by iteration. Observe that if $||\xi_k||$ (the norm of the kth iterate) is $\leq \delta$ then $||\xi_{k+1}|| \leq \delta$. Now with (F4), g satisfies a Lipschitz condition in ξ, hence the sequence ξ_k converges to the unique solution ξ of (20), and $||\xi|| \leq \delta$.

Proof of Corollary I.

The derivation is the same as that of the theorem except that assumption (iii) reads: for all $\varepsilon > 0$ there is a $\delta > 0$ such that for $|\xi| \leq \delta$ we have $|g(\xi, t)| \leq \varepsilon \; |\xi|$. Hence instead of (A.4) we obtain

$$||\xi|| \leq ||\xi_0|| + \varepsilon M \; ||\xi|| \tag{A.13}$$

from which (c') follows immediately. Assertion (d) is obtained by using the triangle inequality in (A.13) to obtain

$$||\xi|| \leq ||\xi_0|| + \varepsilon M [\, ||\xi - \xi_0|| + ||\xi_0||\,] \tag{A.14}$$

from which (d) follows.

Q.E.D.

Proof of Corollary II.

Define

$$\psi(t) \triangleq |B(t)\, u(t)| \tag{A.15}$$

$$\phi(t) \triangleq K \int_0^t e^{-\sigma(t-t')}\, \psi(t')\, dt' \tag{A.16}$$

Using these definitions and assumptions (i') and (iii) in (20), we obtain

$$|\xi(t)| \leq K \int_0^t e^{-\sigma(t-t')}\, \psi(t')\, dt' + \varepsilon K \int_0^t e^{-\sigma(t-t')}\, |\xi(t')|\, dt'$$

or

$$e^{\sigma t}\, |\xi(t)| \leq e^{\sigma t}\, \phi(t) + \varepsilon K \int_0^t e^{\sigma t'} |\xi(t')|\, dt'$$

Using the Bellman Gronwall inequality [3]

$$|\xi(t)| \leq \phi(t) + \varepsilon K \int_0^t e^{-\sigma'(t-t')}\, dt' \tag{A.17}$$

where $\sigma' = \sigma - \varepsilon K > 0$.

Now ψ defined in (A.15) is bounded on $[0, \infty)$ and $\psi(t) \to 0$ as $t \to \infty$: this is a consequence of (ii) and the assumption $u(t) \to 0$ as $t \to \infty$. It can be checked from (A.16) that ϕ is bounded on $[0, \infty)$ and $\phi(t) \to 0$ as $t \to \infty$. Repeating this implication in the second term of (A.17), we conclude that the right-hand side of (A.17) is bounded and $\to 0$ as $t \to \infty$.

Q.E.D.

Appendix II

Proof of Theorem II.

Conceptual clarity requires us to view the vector-valued functions $x_1(\cdot)$, $x_2(\cdot)$, $\hat{u}_1(\cdot)$, $\hat{u}_2(\cdot)$, ... as points in linear function spaces: $x_1(\cdot)$, $x_2(\cdot)$ are points in the Banach space $\mathcal{B}_n$ of functions mapping $[0, T]$ into R^n with $\sup_{0 \leq t \leq T} |x(t)|$ as norm. The same holds for $u_1(\cdot)$ and $u_2(\cdot)$ except that they are continuous functions mapping $[0, T]$ into R^m, hence they are points in the corresponding Banach space which we call $\mathcal{B}_m$.

In view of the assumptions (i), (ii) and (iii) on f, the differential equation $\dot{x} = f(x, \hat{u}, t)$, $x(0) = 0$ has a unique continuous solution on $[0, T]$ for each $\hat{u} \in \mathcal{B}_m$. This solution is defined implicitly by the integral equation

$$x(t) - \int_0^t f(x(t'), \hat{u}(t'), t')\, dt' = 0 \qquad 0 \leq t \leq T.$$

From the function space point of view, this equation is of the form

$$F(u, x) = 0 \tag{A.20}$$

where $F : \mathcal{B}_m \times \mathcal{B}_n \rightarrow \mathcal{B}_n$. This equation defines implicitly a map $\Psi : \mathcal{B}_m \rightarrow \mathcal{B}_n$ such that $F(u, \Psi(u)) = 0$. Thus

$$x_1 = \Psi(\hat{u}_1) \tag{A.21}$$

$$x_2 = \Psi(\hat{u}_2) \tag{A.22}$$

We assert that $\Psi \in C^3$ on a sufficiently small neighborhood of U. We shall obtain this conclusion by showing that the assumption of the implicit function theorem are satisfied [13, see 10.2.1 and 10.2.3]. First by (ii), $F \in C^3$. Next consider $D_2 F(U, X)$, the Fréchet derivative of F with respect to its second argument, evaluated at the point (X, U): it is a linear map of $\mathcal{B}_n$ into $\mathcal{B}_n$ given by

$$(D_2 F(U, X))\ \delta x \Big|_t = \delta x(t) - \int_0^t D_1 f(X(t'), U(t'), t')\ \delta x(t')\ dt' \quad \text{(A.23)}$$

(Note that the kernel is A(t'), as in (30).) Furthermore, the linear map $D_2 F(X, U)$ is one-to-one, onto and has a continuous inverse: indeed for any $\zeta \in \mathcal{B}_n$, the linear integral equation

$$\delta x(t) - \int_0^t A(t')\ \delta x(t')\ dt' = \zeta(t) \qquad 0 \leq t \leq T \qquad \text{(A.24)}$$

can be solved by successive approximations. It is well known that, for each $\zeta \in \mathcal{B}_n$, it has one and only one solution and that the solution δx depends continuously on ζ since $A(\cdot)$ is bounded on [0, T]. Thus $F \in C^3$ and $D_2 F(U, X)$ is a homeomorphism of $\mathcal{B}_n$ onto $\mathcal{B}_n$, consequently, by the implicit function theorem, $\Psi \in C^3$.

In terms of Ψ we have

$$X + x = \Psi(U + u)$$

$$\xi_0 = \Psi'(U) \cdot u \qquad \text{(A.25)}$$

and

$$v = h[\Psi(U + u), U + u] - h[\Psi(U - u), U - u]$$

or, using obvious notations

$$v = k(U + u) - k(U - u). \tag{A.26}$$

Note that $k(\cdot) \in C^3$ since both h and $\Psi \in C^3$ [13, see 8.12.10].

By Taylor's theorem [13, see 8.14.3], we obtain

$$v = 2k'(U) \cdot u + O(\|u\|_T^3) \tag{A.27}$$

Using the rule for differentiating a composition of two functions

$$k'(U) \cdot u = h_1[\Psi(U), U] \cdot \Psi'(U) \cdot u + h_2[\Psi(U), U] \cdot u$$

or

$$k'(U) \cdot u = h_1(X, U) \cdot \xi_0 + h_2(X, U) \cdot u \tag{A.28}$$

Thus we see that $k'(U) \cdot u$ is the zero-state response of the linearized network, i.e., z. Thus from (A.27) and (A.28), Eq. (32) follows.

Q.E.D.

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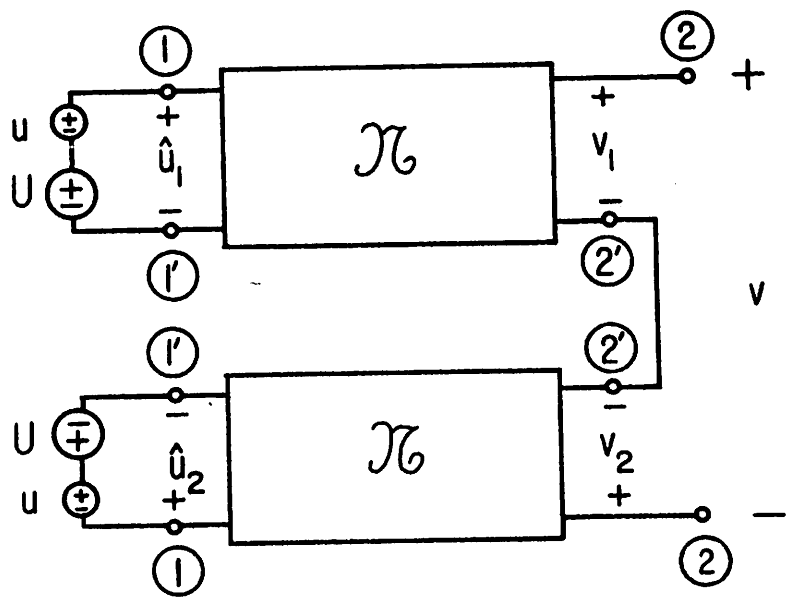

Fig. 1. Push-pull connection.

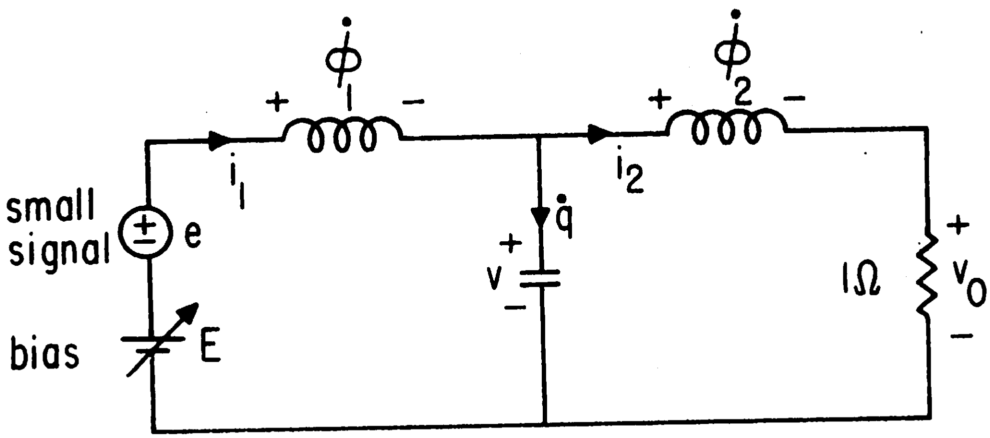

Fig. 2. Nonlinear network under study.

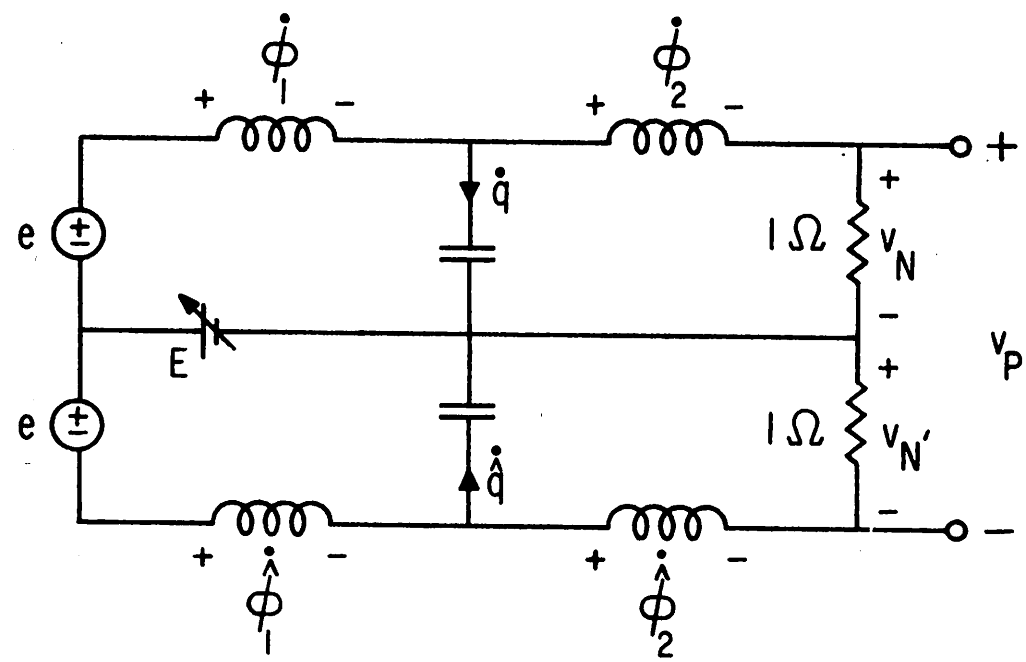

Fig. 3. Push-pull connection under study.

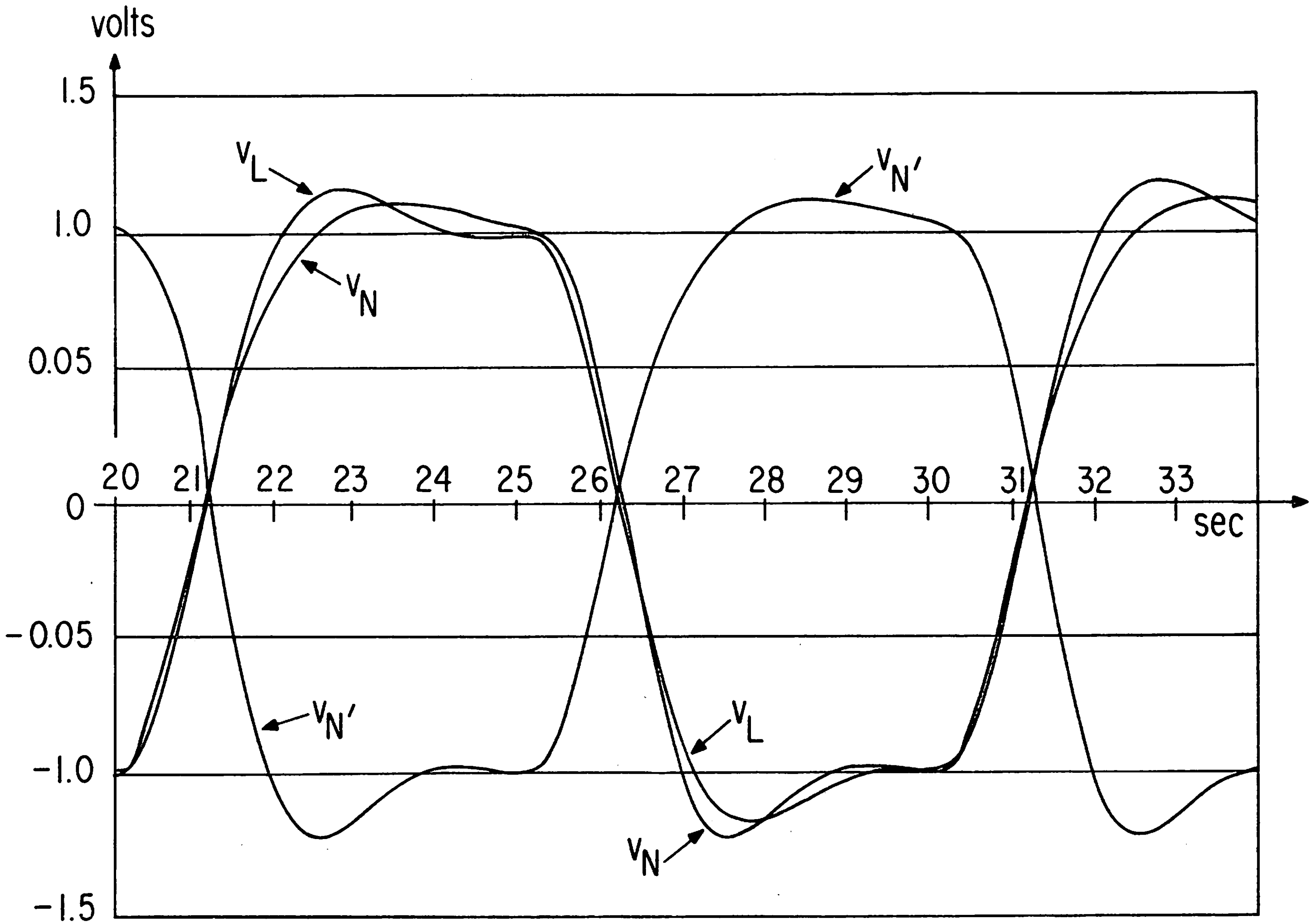

Fig. 4. Zero-state responses to a square wave of amplitude $A = 1V$ and period $2\,s$; v_L is the response of the linearized equivalent network, v_N that of the nonlinear network of Fig. 2, and $v_{N'}$ that of the lower part of the push-pull configuration of Fig. 3. (In v_N and $v_{N'}$ the constant contribution of the bias of 1V has been removed.)

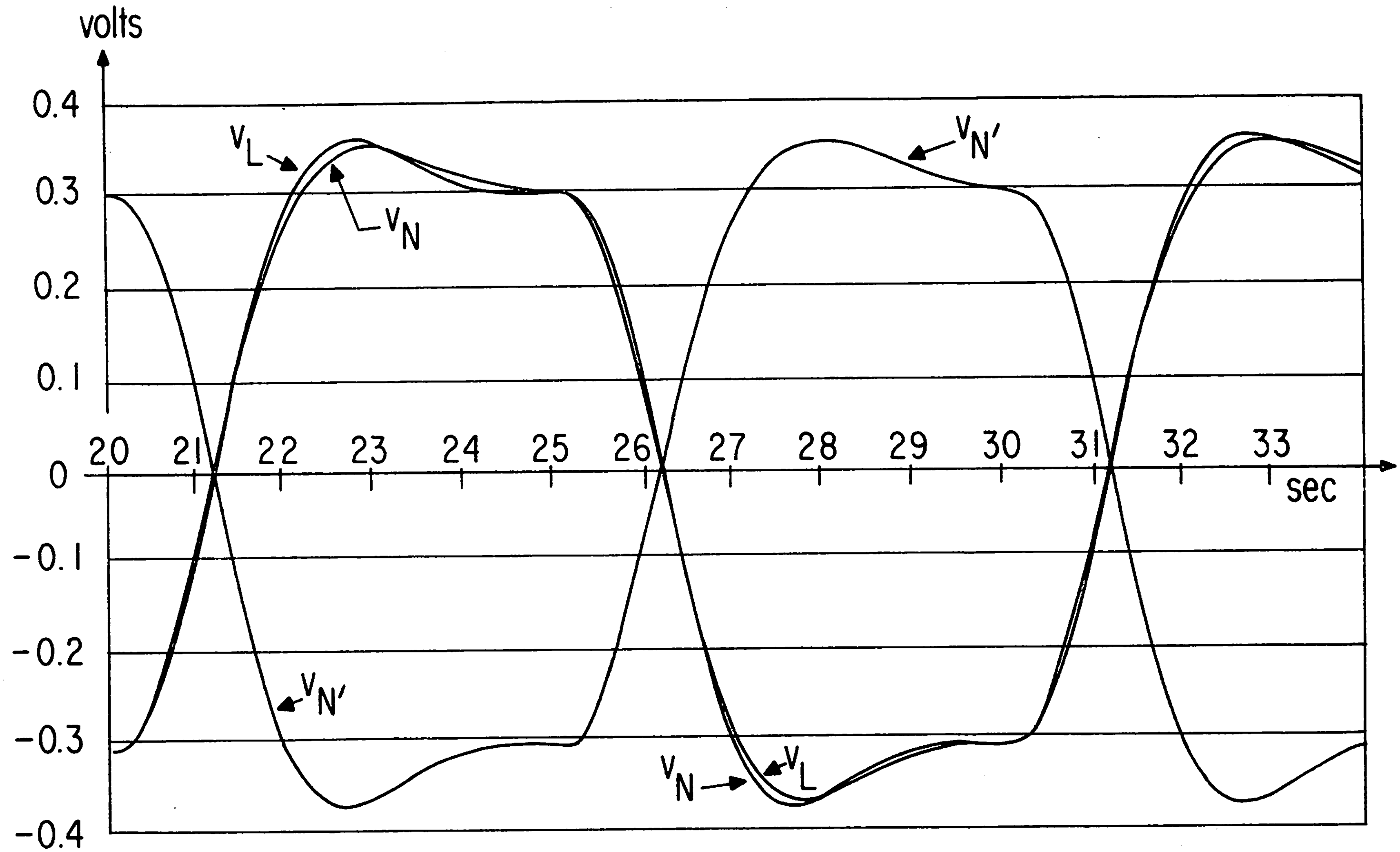

Fig. 5. Zero-state responses to a square wave of amplitude A = 0.316 V and period 2 s.

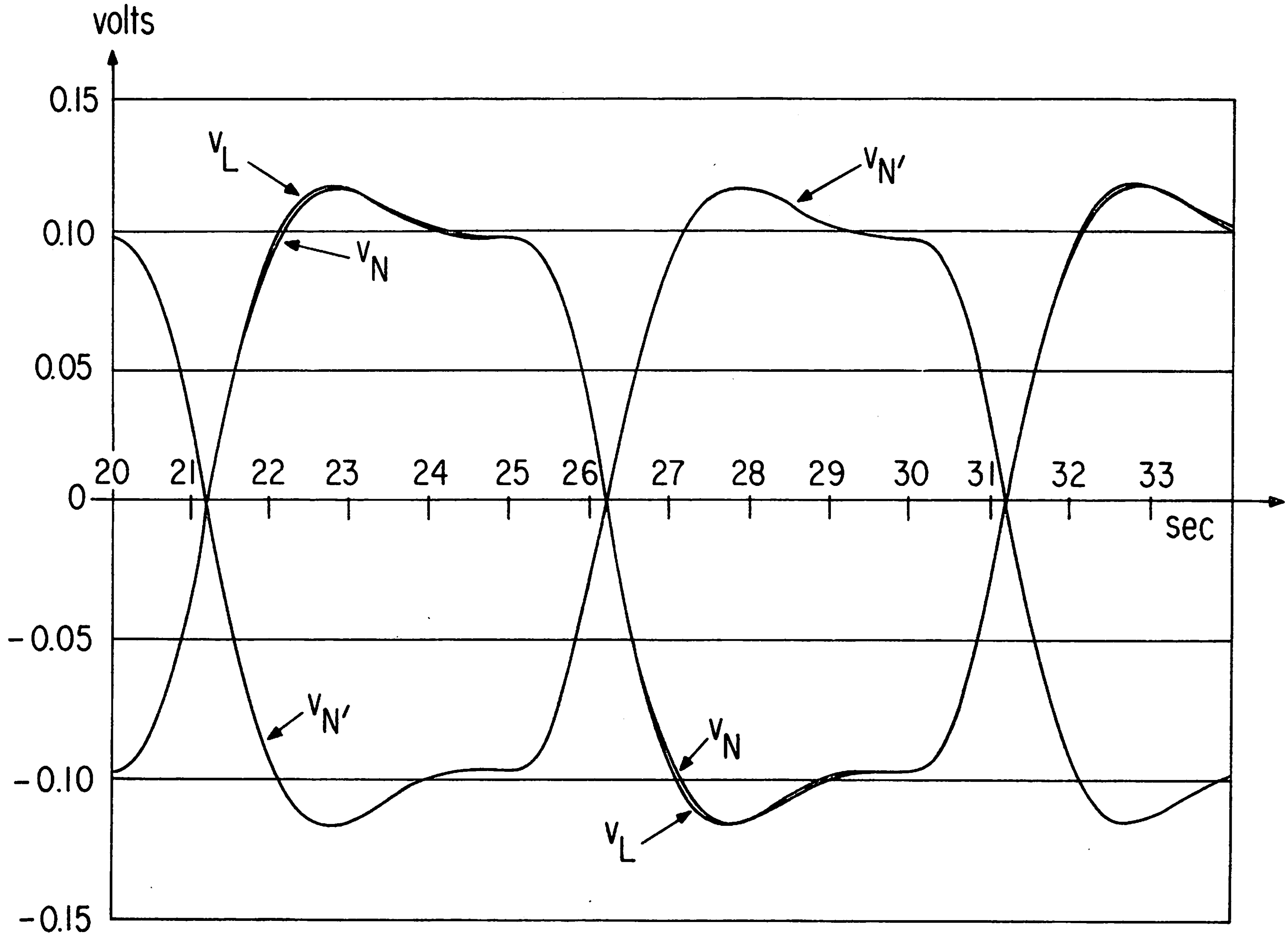

Fig. 6. Zero-state responses to a square wave of amplitude $A = 0.100$ V and period 2 s.

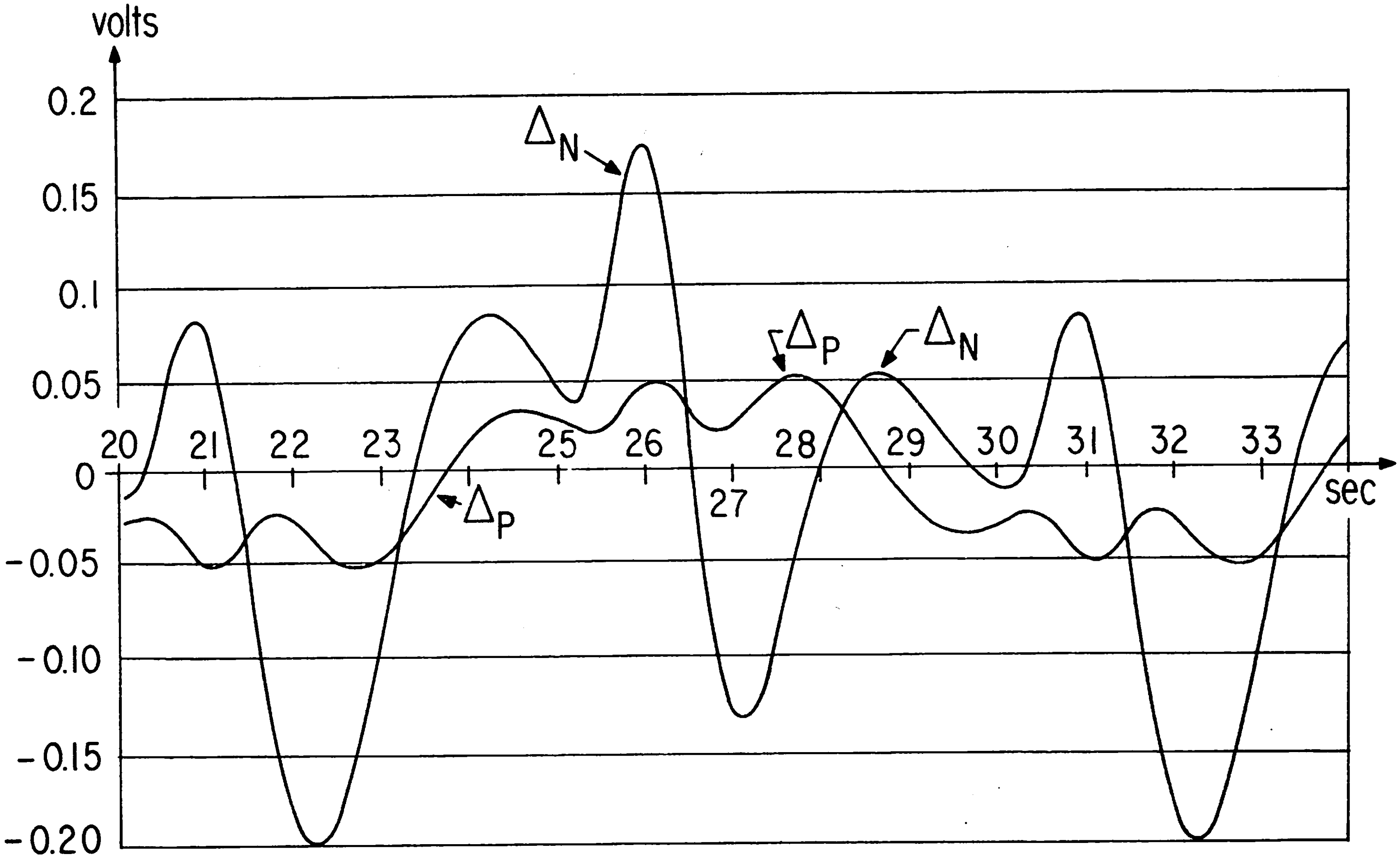

Fig. 7. Distortion corresponding to Fig. 4: Δ_N and Δ_P are defined as: $\Delta_N = v_N - v_L$, $\Delta_P = v_N - v_{N'} - 2v_L$. The present curves correspond to $A = 1\,V$.

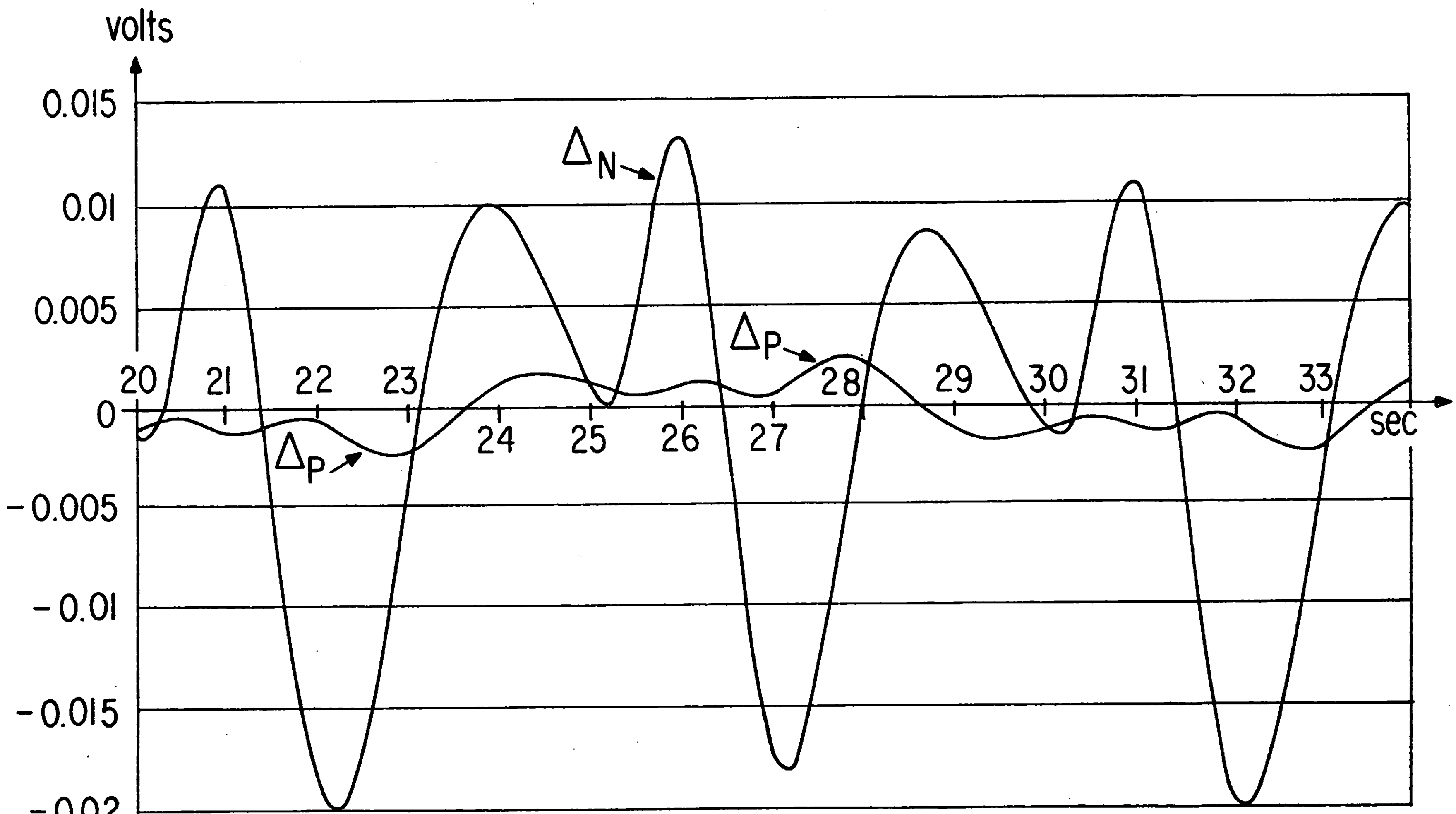

Fig. 8. Distortion corresponding to Fig. 5: A = 0.316 V.

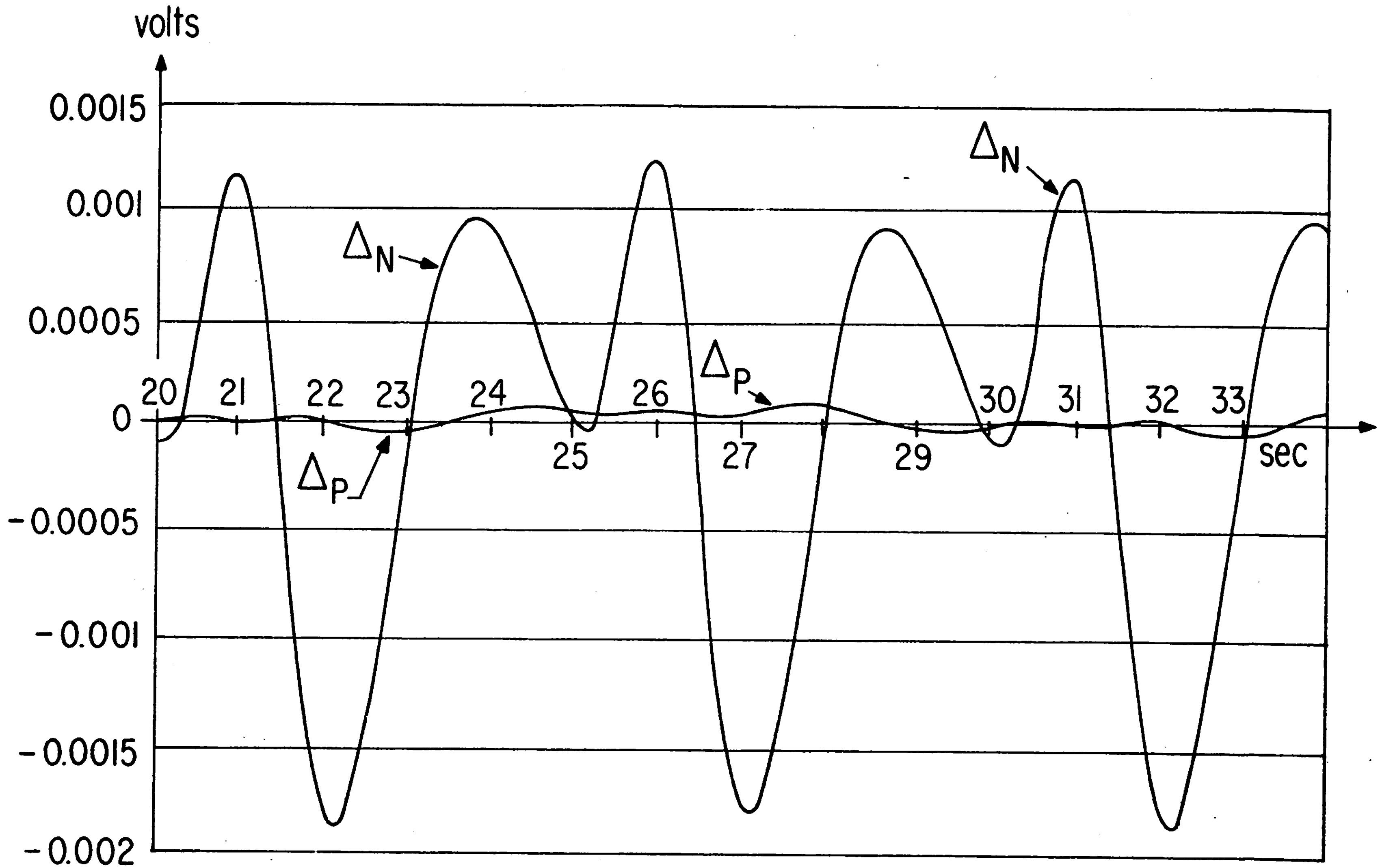

Fig. 9. Distortion corresponding to Fig. 6: A = 0.100 V.