ON LINEAR SAMPLED-DATA FEEDBACK SYSTEMS WITH FINITE PULSE WIDTH

by

E. I. Jury

Edwin P. F. Kan

Memorandum No. ERL-M216

4 October 1967

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force under AFOSR Grant AF-AFOSR-292-66/68-1463.

CONTENTS

I. INTRODUCTION

In the past decade, quite a number of papers have been written on the topic of sampled-data feedback systems with finite pulse width. It is the purpose of this report to give a summary, roughly in chronological order, of the methods presented by different researchers. In the appendix, an extension of Murphy's exact method [6] is proposed, and an example is worked out and compared with the results obtained by using two other methods.

The different methods fall into two categories:

(i) Exact methods

(ii) Approximate methods

The first category is further divided into two main branches:

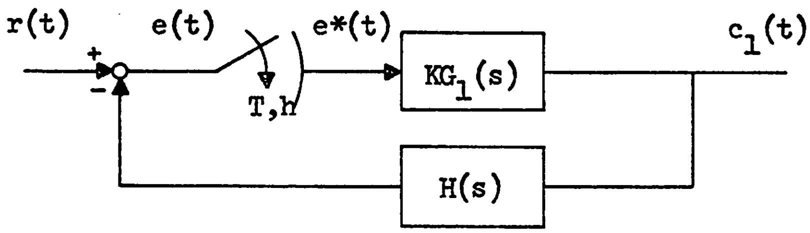
(a) Transform methods

(b) State space approach

The transform method has been studied by Farmanfarma [1, 2, 3], Murphy [6], Tsypkin [4], and Nishimura and Jury [5, 19, 20]. The state space approach was investigated by Kalman and Bertram [7], Gilbert [8], and Tou [9]. Bennett and Desoer [10, 11] have also written some papers on linear network analysis using essentially the same approach.

Approximate methods were developed by Tou [12], Kranc [13], and Murphy and Kennedy [14, 15], who used new transform methods, or the conventional transform methods after decomposing the finite-pulse-width sampler into a set of ideal (instantaneous) samplers.

II. DESCRIPTION OF THE SYSTEM



$r(t)$ = input signal

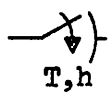
$e(t)$ = error signal

$e^*(t)$ = sampled signal

$c_1(t)$ = output signal

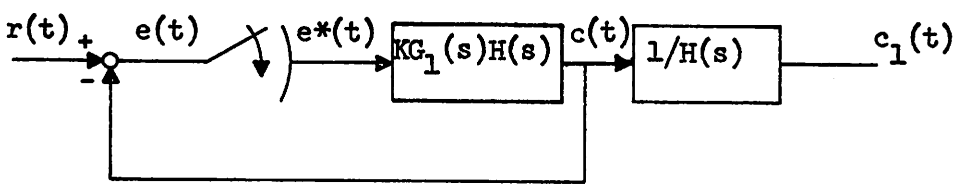
$KG_1(s)$ = forward transfer function

$H(s)$ = feedback transfer function

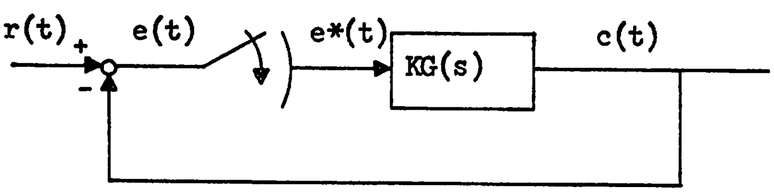


= sampler with period T and pulse width h

Transforming the above system,



So, essentially, the following system is studied:



III. EXACT ANALYSIS

A. Transform Methods

1. More or less a pioneer in sampled data systems with finite pulse width, Farmanfarma introduced a new transform, modified from the Laplace

transform, called the p-transform, where p stands for pulse width. Here, considering the sampler as a finite-pulse-train $u_T(t)$ modulator instead of an impulse-train $\delta_T(t)$ modulator, where

$$\delta_T(t) = \sum_{k=0}^{\infty} \delta(t - kT)$$

$$u_T(t) = \sum_{k=0}^{\infty} [u(t - kT) - u(t - \overline{kT + p})]$$

with the Laplace transform as

$$\Delta_T(s) = \frac{1}{1 - e^{-Ts}}$$

$$U_T(s) = \frac{1}{s} \frac{1 - e^{-ps}}{1 - e^{-Ts}}$$

The p-transform of a function f(t) will be

$$\mathcal{P}[f(t)] = \mathcal{L}[f(t)u_T(t)]$$

$$= F(s)*U_T(s)$$

$$= \frac{1}{2\pi j} \int F(s - \nu) \frac{1 - e^{-p\nu}}{\nu(1 - e^{-T\nu})} d\nu$$

While the z-transform of f(t) is

$$\mathcal{Z}[f(t)] = \mathcal{L}[f(t)\delta_T(t)]$$

$$= F(s)*\Delta_T(s)$$

$$= \frac{1}{2\pi j} \int F(s - \nu) \frac{1}{1 - e^{-T\nu}} d\nu$$

Using the new transform, the open loop linear sampled data system was analyzed in [1, 18] and extended to multiple-samplers and nonperiodic systems in [2, 18]. Closed loop analysis [3, 18] was studied in a direct way of attack by decomposing the sampler into an infinite set of identical noninstantaneous samplers of pulse duration p, each closing only once in different intervals. The response of the system in the first sampling interval is studied first by noting that this is a single loop, single sampler system. The continuous output is called $C_0(s)$. Then in the second interval, $C_0(s)$ is considered as an input other than R(s), so that the system is again a single loop, single sampler system. The total output is this output $C_1(s)$ superimposed on $C_0(s)$ with suitable delay. Similar procedures are taken for later intervals.

Due to the straightforwardness of this analysis no restriction is imposed on this method. Initial conditions can be thought of as an extra input to a system with zero initial conditions. KG(s) may have a discontinuous step response, in which case the response before and after the closing and opening of the samplers must be distinguished.

But, at the same time, it should be clear that the numerical calculation is extremely elaborate, as indicated by the example at the end of this paper.

2. In 1958, Tsypkin [4] presented an exact method to solve the same problem by introducing the convolution integral:

$$C(z, m)\Big|_{m=t/T} = \int_0^m G_0(z, m - \mu)E(z, \mu)d\mu$$

where C(z, m) and E(z, m) are the modified z-transforms of the output and error signals, and

$$G_0(z, m - \mu) = \mathfrak{Z}\left[G(s)e^{-\mu Ts}\right]$$

$$= \begin{cases} G(z, m - \mu) & \mu < m \\ e^{-\mu T}G(z, 1 + m - \mu) & m < \mu < 1 \end{cases}$$

So the problem of the finite-pulse-width sampler system can be solved through investigating the ordinary z-transform and the modified z-transform, and

$$E(z, m) = R(z, m) - C(z, m)$$

$$= R(z, m) - \int_0^m G_0(z, m - \mu)E(z, \mu)d\mu$$

$$= R(z, m) - LE(z, m)$$

where L is a Fredholm operator of the second type. Noting that the kernel of L is degenerate, a solution thus exists and the problem can be solved.

3. In 1960, Nishimura [5, 19, 20] introduced the double z-transform, which is a series summation (by applying the z-transform in the infinite power series form) of the "incremental responses." This incremental response $\Delta c_n(t)$ is defined as the response of the system due to control error which enters KG(s) during the (n + 1)st closure term, i.e., $nT \leqq \tau \leqq nT + h$,

$$\Delta C_n(s) = C_n(s) - R[c_n^{\,j}(0^-)]$$

where $C_n(s)$ = response of system to initial conditions and $r(\tau)$ for $nT \leqq \tau \leqq nT + h$

$R[c_n{}^j(0^-)]$ = response of system caused by the storage of the system before $\tau = nT$

$c_n{}^j(0^-)$ = the j^{th} derivative of the initial condition $c(\tau)$ at $\tau = nT^-$

and is further equated to

$$\Delta C_n(s) = W_{rn}(s) - R[c_n{}^j(0^+)]$$

where $W_{rn}(s)$ = response of system due to control error which originates from the input supplied to the system during $nT \leq \tau \leq nT + h$

$R[c_n{}^j(0^+)]$ = response of system due to control error which originates from the initial conditions of the system at $\tau = nT^+$

and so

$$C(z) = \sum_{n=0}^{\infty} z^{-n} \Delta C_n(z)$$

The response between sampling instants may also be obtained by using the modified z-transform

$$C(z, m) = \sum_{n=0}^{\infty} z^{-n} \Delta C_n(z, m)$$

This summation process is defined by Nishimura by the symbol $\mathfrak{Z}_d[\ ]$, i.e.,

$$C(z) = \mathfrak{Z}_d[\Delta C_n(z)]$$

and the overall procedure is called the "double z-transform."

One of the advantages of this method is that, because of the use of the z-transform and the modified z-transform, it is possible to eliminate the step of taking the inverse Laplace transform.* Also, the response is

---

*The labor involved in this step will be clear from the example in Appendix II.

given in a closed form so that the final-value theorem can be directly applied to find the steady-state condition, not only at the sampling instants but also between them. This method was also extended to a 2-sampler system and a simple open-loop multirate sampling system. Further extension to finite pulse-clamped systems, closed loop multirate systems, and multi-sampler systems are expected.

Due to the use of the z-transform, this method is restricted to periodic sampling systems.

4. Murphy [6] transformed the circuit from an error-sampled system to a feedback-sampled system (still finite-pulse width), then transformed the sampler into a set of ideal samplers followed by a transfer function. KG(s) in this case may contain a time transport operator $e^{\alpha s}$, the restriction of α being discussed in the appendix.

The reason for this decomposition is that during a portion* of the sampling interval, the input to the loop is zero so that the state variables, obtained by partial factorization of KG(s), vary as an exponential decay of the initial states. Thus, the same response could be implemented by an instantaneous sampler (to get the initial states) followed by a transfer function of the form $k/s + p$ (to describe the decay).

In Murphy's paper, KG(s) is assumed to possess only simple poles. The extension to multiple poles is presented in the appendix.

Murphy's method is restricted to periodic single loop systems with only one sampler of nonnegligible pulse width. In numerical computation, the labor involved in this method is considerably less than Farmanfarma's method, as indicated by the example in the appendix.

---

* This procedure will be clarified in Appendix I.

B. State Space Approach

1. Kalman and Bertram [7] presented a unified approach to the theory of sampling systems, including noninstantaneous sampling systems, using the state space approach. Here, the important fact that

$$\Phi(t,t_0) = \Phi(t,t_1)\Phi(t_1,t_0)$$

is made use of, where Φ is the transition matrix of the continuous dynamic element (CDE); and the total response is the product of transition matrices of the sampling element (SE), with or without a hold, the discrete dynamic element (DDE) and CDE, taken in order.

This method is widely applicable, ranging from the most basic conventional sampling systems, giving rise to stationary (i.e., time-invariant) difference equations, to random sampling systems, giving rise to nonstationary (time-varying) difference equations.

This method, and the state space approach in general, is the neatest and simplest exact method to solve the present kind of problem; the least amount of labor is involved compared with the other methods mentioned above. One drawback might be that the response at individual points in the sampling interval must be calculated individually, unlike the modified z-transform, which is a general expression of the response in the whole interval with m as a variable parameter.

2. Gilbert [8] also gave a general systematic approach to the analysis of linear periodic feedback systems by the use of the state vector representation; this is especially suitable to cases of periodically varying parameters such as gain constants, time constants, pulse duration, and periodic variation. He introduced the terms "interval number" and "sampling number," and presented a unified symbolic representation directly applicable

to differential analyzer setups. System response at multiples of the fundamental period, obtained by z-transform methods, is presented in a concise vector matrix notation.

Comments similar to those given above in section B.1 are applicable here.

3. Tou also wrote a section in his book [9] concerning the same kind of problem, using the state space approach.

4. In Desoer's paper [10] on a linear network containing a periodically operated switch he chose the voltages across the capacitors and the currents through the inductors as the state variables. Dividing the period T into two parts (i) when switch is open and (ii) when switch is closed, Desoer found a solution of the steady-state transmission as well as the transient response of the network. He also solved the problem of discontinuity in the state vector at switching instants by writing down the transition matrices. So essentially his approach is the same as those of the other researchers mentioned above.

Two years earlier Bennett [11] also wrote a paper on the same topic.

IV. APPROXIMATE ANALYSIS

1. Tou [12] introduced a new transform called the τ-transform, developed through the use of the delayed z-transform. This is actually the decomposition of the noninstantaneous sampler of duration τ into a set of n identical samplers each with duration Δ such that $n\Delta = \tau$. By adding the individual outputs from the samplers, derived through the delayed z-transform, the total output is obtained. Applying the limit when $\Delta \to 0$, and thus $n \to \infty$, the samplers become ideal samplers and so this total output tends to the output from the practical sampler. The whole procedure is known as the τ-transform.

Due to the complexity of this τ-transform, only exact open loop analysis was developed. For closed loop analysis, the outputs of the samplers are approximated by flat-topped pulses, leading to the equivalent ideal samplers followed by transfer functions $\frac{1 - e^{-\tau s}}{s}$ representing the noninstantaneous samplers, and thus conventional methods apply and the problem is solved.

2. Kranc [13] attacked the same problem by representing the finite-pulse-width sampler by a parallel set of n idealized switches followed by triangular holds. Then by solving a set of $n + 1$ linear simultaneous algebraic equations, the problem is solved.

Incidentally, the same method can be used to find an exact solution to multirate sampling systems.

3. Murphy [14] developed an approximate method for small h/T, in which each pulse appearing at the output of the sampler was replaced with a rectangular pulse of height equal to the height of the actual pulse midway between the leading and trailing edges of that pulse. For large h/T, Murphy showed that it is possible to transform the sampled data system into a continuous system with the sampler replaced by an attenuator h/T with good approximation.

4. Murphy and Kennedy [15] also proposed an approximate method by decomposing the noninstantaneous sampler into n samplers of pulse width τ, $n\tau = h$, and then approximated the resulting samplers by equivalent ideal samplers. Again, solving a set of linear simultaneous equations, the problem is solved.

5. For small pulse duration, Desoer [16] also wrote a paper on "A network containing a periodically operated switch solved by successive approximations."

6. In a corresponding letter to IEEE, Mr. C. Y. Lee [17] approximated the pulsed signal of width h by splitting it into 3 narrow pulses of width

h/3 each, then used Simpson's 3/8 rule to define the strength of this pulse so that

$$e^*(t) = \sum_{k=0}^{\infty} [u(t - kT) - u(t - \overline{kT + h})]$$

$$\cdot \left[\frac{1}{8}\, e(kT) + \frac{3}{8}\, e\left(kT + \frac{h}{3}\right) + \frac{3}{8}\, e\left(kT + \frac{2h}{3}\right) + \frac{1}{8}\, e(kT + h)\right]$$

Taking the Laplace transform and using the approximation

$$\sum_{k=0}^{\infty} e\left(kT + \frac{\alpha h}{3}\right) e^{-\left(kT + \frac{\alpha h}{3}\right)s} = \sum_{k=0}^{\infty} e(kT)e^{-kTs} = E^*(s) \qquad \alpha = 1,\ 2,\ 3$$

the following is then obtained

$$E_h^*(s) = E^*(s)\left(\frac{1}{8} + \frac{3}{8}\, e^{hs/3} + \frac{3}{8}\, e^{2hs/3} + \frac{1}{8}\, e^{hs}\right)\left(\frac{1 - e^{-hs}}{s}\right)$$

From this, the noninstantaneous sampler could be represented by 4 ideal samplers preceded and followed by the appropriate transfer functions.

In addition, Lee incorporated a nonlinearity in front of the sampler. The stability of the sampled data system was then studied by the describing-function method after the reduction into a multiloop ideal sampled data system.

Obviously the approximation is only good for small h/T; otherwise the delayed z-transform and the τ-transform should be used.

V. CONCLUSION

A summary of the work by past researchers on the topic of linear sampled data feedback systems with finite pulse width has been given above. It should be noted that all the methods, except those using the state space approach, no matter whether exact or approximate, aim at essentially the same goal: to propose a new transform method, or to reduce the non-instantaneous sampler into some configuration consisting of only ideal samplers, and then to employ time-transport factors to bring all the samplers into synchronism, so that the wealth of the z-transform may be utilized. In passing, the authors would like to indicate that the state space approach in some cases is often preferred to the other methods.

REFERENCES

1. G. Farmanfarma, Analysis of Linear Sampled-Data Systems with Finite Pulse Width: Open Loop, AIEE Transactions, Vol. 75, Pt. I (1956) (Jan. 1957 section), pp. 808-19.

2. G. Farmanfarma, Analysis of Multiple Sampler Systems with Finite Pulse Width, Open Loop, AIEE Transactions, Vol. 77, Pt. II (1958), pp. 20-28.

3. G. Farmanfarma, General Analysis and Stability Study of Finite Pulsed Feedback Systems, AIEE Transactions, Vol. 77, Pt. II (1958) pp. 148-62.

4. Ya. Z. Tsypkin, Theory of Impulse Systems (in Russian), Moscow, State Press for Physico-Mathematical Literature (1964).

5. T. Nishimura, Operational Analysis of Finite Pulsed Sampled Data Systems. ERL Report Series No. 60, Issue No. 279, May 1960.

6. G. J. Murphy, On the Exact Analysis of Sampled-Data Feedback Systems with Appreciable Pulse Width, Proceedings National Electronic Conference 1961, Vol. 17, pp. 214-221.

7. R. E. Kalman and J. E. Bertram, A Unified Approach to the Theory of Sampling Systems, Journal, Franklin Institute, Vol. 267, No. 5, (1959), pp. 405-436.

8. Edward O. Gilbert, A Method for the Symbolic Representation and Analysis of Linear Periodic Feedback Systems, Transactions IEEE, Pt. II (1960), pp. 512-523.

9. Julius T. Tou, Modern Control Theory, New York, McGraw Hill Co. Inc., (1964).

10. C. A. Desoer, Transmission Through a Linear Network Containing a Periodically Operated Switch, IRE Wescon Convention Records, Pt. 2 (1958), pp. 34-41.

11. C. A. Desoer, Steady State Transmission through Networks Containing Periodically Operated Switches, IRE Transactions, PGCT 2, 1 (1955); pp. 17-22.

12. Julius Tou, Analysis of Sampled-data Control Systems with Finite Sampling Duration, Proceedings NEC, Vol. 13 (1957), pp. 561-573.

13. G. M. Kranc, Additional Techniques for Sampled-Data Feedback Problems, IRE Wescon Convention Records (1957), Pt. 4, pp. 157-165.

14. G. J. Murphy, An Approximation to Simplify the Analysis of Closed Loop Sampled-Data Systems with Non-negligible Pulse Width, Proceedings NEC Vol. 14 (1958), pp. 65-74.

15. G. J. Murphy and H. B. Kennedy, Closed-Loop Analysis of Sampled-Data Systems with Appreciable Pulse Width, IEEE Transactions, Pt. II (1959), pp. 659-665.

16. C. A. Desoer, A Network Containing a Periodically Operated Switch Solved by Successive Approximations, Bell Systems Technical Journal, Vol. 36 (1957), pp. 1403-1428.

17. C. Y. Lee, Stability of Nonlinear Sampled-Data Systems with Finite Sampling Duration, IEEE Proceedings (1964), pp. 734.

18. E. I. Jury, Sampled-Data Control Systems, New York, John Wiley & Sons, Inc. (1958).

19. E. I. Jury and T. Nishimura, "Analysis of Finite Pulsed Systems with Periodically Varying Sampling Rate and Pulse Width," AIEE, CP 60-866, June, 1960.

20. T. Nishimura, "Operational Analysis of Finite Pulsed Sampled-Data Systems, IRE Trans. on Automatic Control, Vol. AC-6, Sept. 1961, No. 3. pp. 344-345.

21. O. A. Z. Leneman, "On Finite Pulse Width Sampling," <u>IEEE Trans. on Communication Technology</u>, Vol. COM-15, No. 5, to be published October 1967.

APPENDIX I

Discussion on Murphy's Exact Analysis [6]

1. Restriction of α

In Murphy's paper, equation (1) reads

$$G(s)H(s) = e^{\alpha s} \sum_{k=1}^{N} \frac{K_k}{s - p_k}$$

without defining the limits of α.

Figures 2 and 3 in the paper are repeated here:

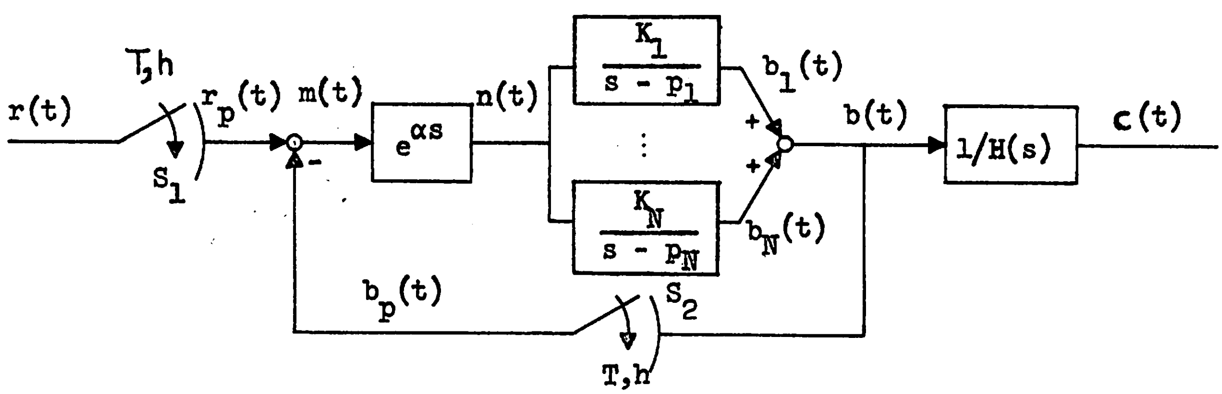


Figure 2

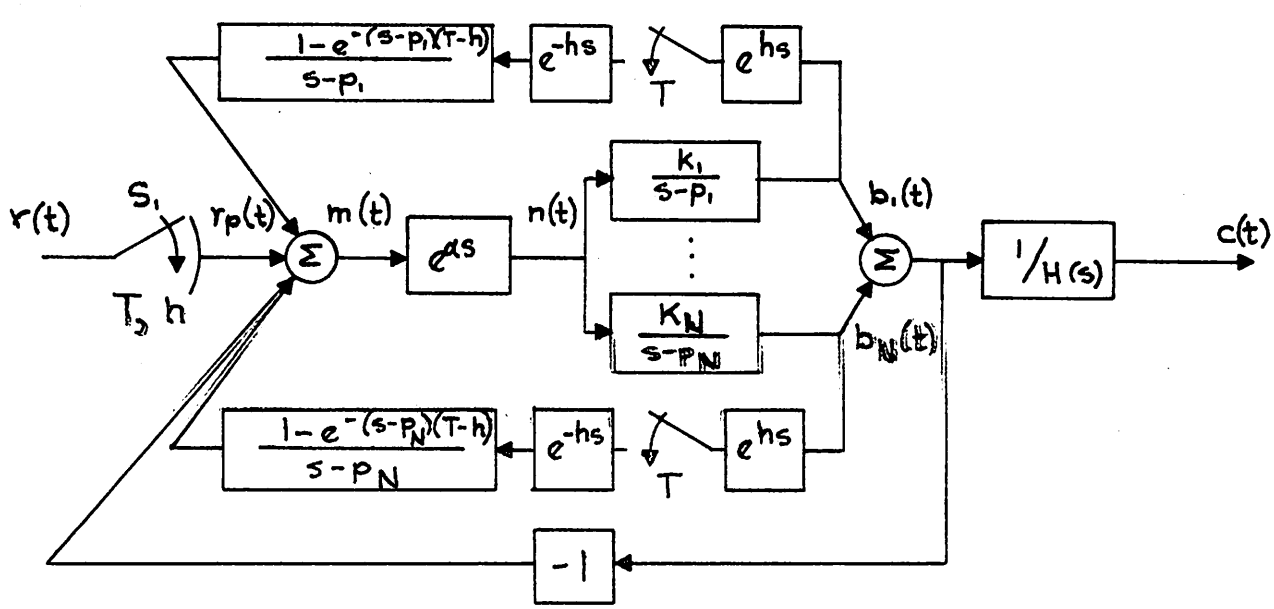


Figure 3

<u>Claim</u>: Murphy's analysis is only good for nonnegative α.

<u>Proof</u>: In equation (5):

$$x(t) = \begin{cases} 0 & nT \leq t \leq nT + h \\ \sum_{k=1}^{N} b_k(t) & nT + h \leq t < (n+1)T \end{cases} \tag{5}$$

$$= \begin{cases} 0 & nT \leq t < nT + h \\ \sum_{k=1}^{N} b_k(nT + h)e^{p_k(t-nT-h)} & nT + h \leq t < (n+1)T \end{cases} \tag{6}$$

The equality of (5) and (6) lies in the fact that $b_k(t)$ during $nT + h \leq t < (n + 1)T$ is an exponential decay of the initial condition $b_k(nT + h)$. But this is only valid when there is no input n(t) during this period of time. Obviously, during the same period, samplers S_1 and S_2 in Fig. 2 are open so that $m(t) = 0$. Now if α is nonnegative, n(t) will also be zero. Therefore, the equality between (5) and (6) is valid. But, if α is negative, n(t) will not be zero during a part of this period, e.g., if $|\alpha| < T$, n(t) will not be definitely zero till $t = nT + h + |\alpha|$. So the equality of (5) and (6) is invalidated and thus the whole analysis needs modification.

2. <u>Correction of equation (7)</u>

Eq. (6):

$$x(t) = \begin{cases} 0 & nT \leqq t < nT + h \\ \sum_{k=1}^{N} b_k(nT + h)e^{p_k(t-nT-h)} & nT + h \leqq t < (n+1)T \end{cases}$$

$$= \sum_{k=1}^{N} b_k(nT + h)e^{p_k(t-nT-h)} \left[u_{-1}(t - \overline{nT + h}) - u_{-1}(t - \overline{n + 1T})\right]$$

as compared to equation (7) in the paper

$$= \sum_{k=1}^{N} b_k(nT + h)\left[e^{p_k(t-nT-h)} u_{-1}(t - \overline{nT + h}) - e^{p_k(t-\overline{n+1T})} u_{-1}(t - \overline{n + 1T})e^{p_k(T-h)}\right]$$

∴ the transfer function in Fig. 3 is

$$\frac{1 - e^{-(T-h)s}\, e^{p_k(T-h)}}{s - p_k} = \frac{1 - e^{-(s-p_k)(T-h)}}{s - p_k}$$

as in the paper.

3. <u>Extension to multiple poles</u>

Consider a $G(s)H(s)$ with only one multiple pole and some other simple poles, i.e.,

$$G(s)H(s) = e^{\alpha s}\left[\sum_{j=1}^{N} \frac{K_j}{(s - p)^j} + \text{simple poles}\right]$$

Disregarding the simple poles (because the analysis is exactly the same as before) and assuming nonnegative α,

$$G(s)H(s) = e^{\alpha s} \sum_{k=1}^{N} \frac{K_k}{(s - p)^k}$$

Similar argument will give (see Fig. 2')

$$x(t) = \begin{cases} 0 & nT \leq t \leq nT + h \\ \sum_{k=1}^{N} b_k(t) & nT + h \leq t < (n + 1)T \end{cases}$$

$$= \begin{cases} 0 & nT \leq t < nT + h \\ \sum_{k=1}^{N} b_k(nT + h) \frac{(t - \overline{nT + h})^{k-1}}{(k - 1)!} e^{p(t-\overline{nT+h})} & nT + h \leq t < (n + 1)T \end{cases}$$

$$= \sum_{k=1}^{N} b_k(nT + h) \frac{\tau^{k-1}}{(k - 1)!} e^{p\tau}\left[u_{-1}(\tau) - u_{-1}(t - \overline{T - h})\right]$$

where $\tau = t - \overline{nT + h}$

The transfer function required will be

$$\mathcal{L}\left\{\frac{\tau^{k-1}}{(k - 1)!} e^{p\tau}\left[u_{-1}(\tau) - u_{-1}(\tau - \overline{T - h})\right]\right\}$$

$$= \mathcal{L}\left[\frac{\tau^{k-1}}{(k - 1)!} e^{p\tau}u_{-1}(\tau)\right] - \mathcal{L}\left[\frac{\tau^{k-1}}{(k - 1)!} e^{p\tau}u_{-1}(\tau - \Delta)\right]$$

where $\Delta = T - h$

The first term = $\dfrac{1}{(s-p)^k}$

The second term = $e^{-\Delta s}\,\mathcal{L}\left[\dfrac{(\tau+\Delta)^{k-1}}{(k-1)!}\,e^{p(\tau+\Delta)}\right]$

$$= e^{-(s-p)\Delta}\,\mathcal{L}\left[\sum_{j=0}^{k-1}\frac{1}{(k-1)!}\,{}_{k-1}C_j\,\tau^{k-1-j}\,\Delta^j e^{p\tau}\right]$$

where ${}_{k-1}C_j$ is a binomial coefficient

$$= e^{-(s-p)\Delta}\sum_{j=0}^{k-1}\frac{1}{(k-1)!}\,{}_{k-1}C_j\,\tau^{k-1-j}\,\Delta^j\,\frac{(k-1-j)!}{(s-p)^{k-j}}$$

$$= e^{-(s-p)\Delta}\sum_{j=0}^{k-1}\frac{\Delta^j}{j!(s-p)^{k-j}}$$

$\Rightarrow$ Transfer function = $\dfrac{1}{(s-p)^k} - e^{-(s-p)\Delta}\displaystyle\sum_{j=0}^{k-1}\frac{\Delta^j}{j!(s-p)^{k-j}}$

The equivalent block diagrams will be:

Fig. 2':

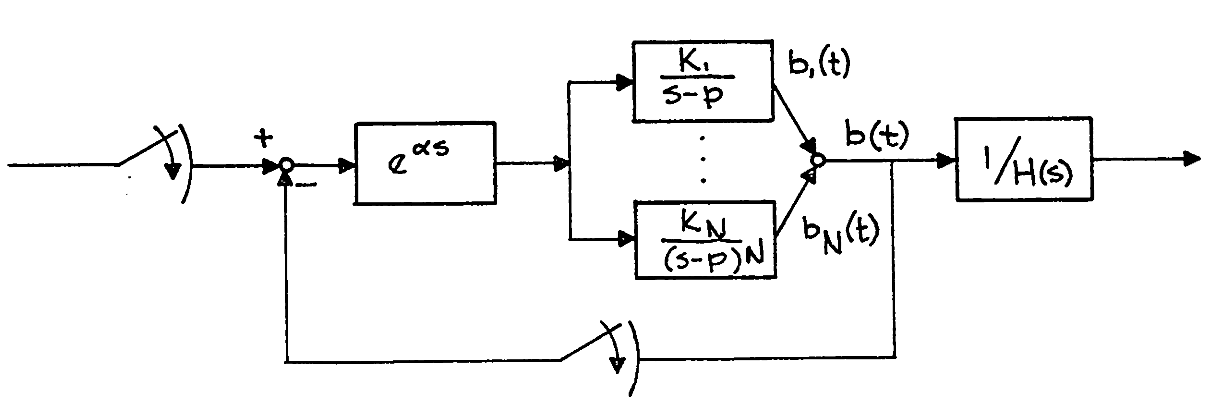
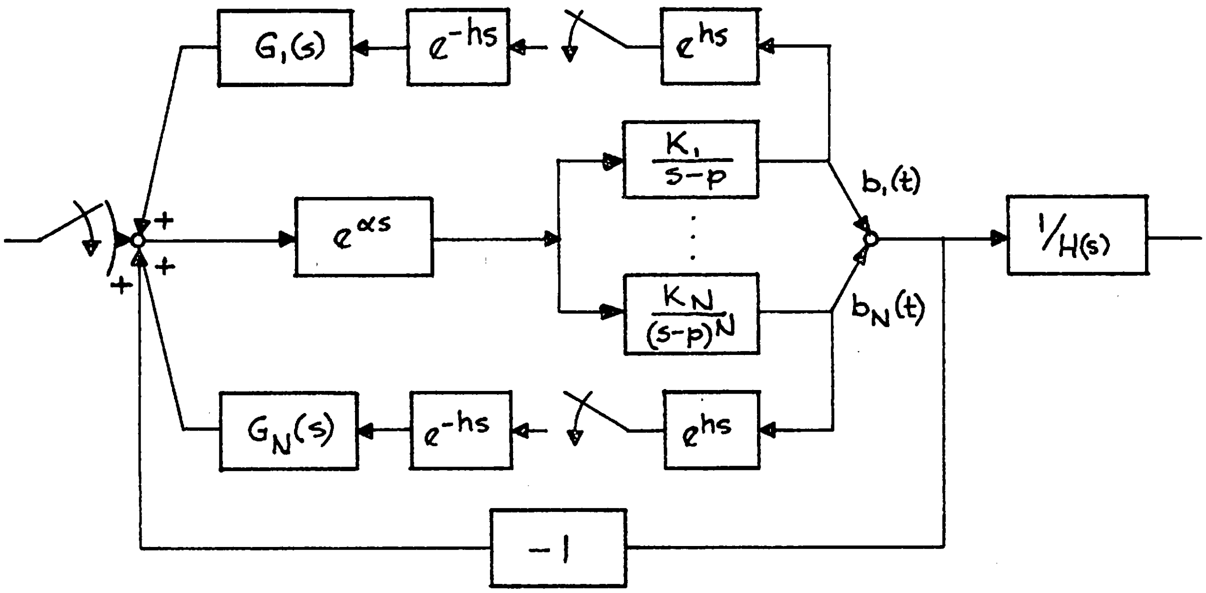


Fig. 3':



$$G_k(s) = \frac{1}{(s-p)^k} - e^{-(s-p)\Delta} \sum_{j=0}^{k-1} \frac{\Delta^j}{j!(s-p)^{k-j}}$$

So, the analysis in the paper is again applicable because Fig. 3' has the same configuration as Fig. 3 in the paper.

4. <u>Extension to negative α with $T - h > |\alpha| > h$</u>

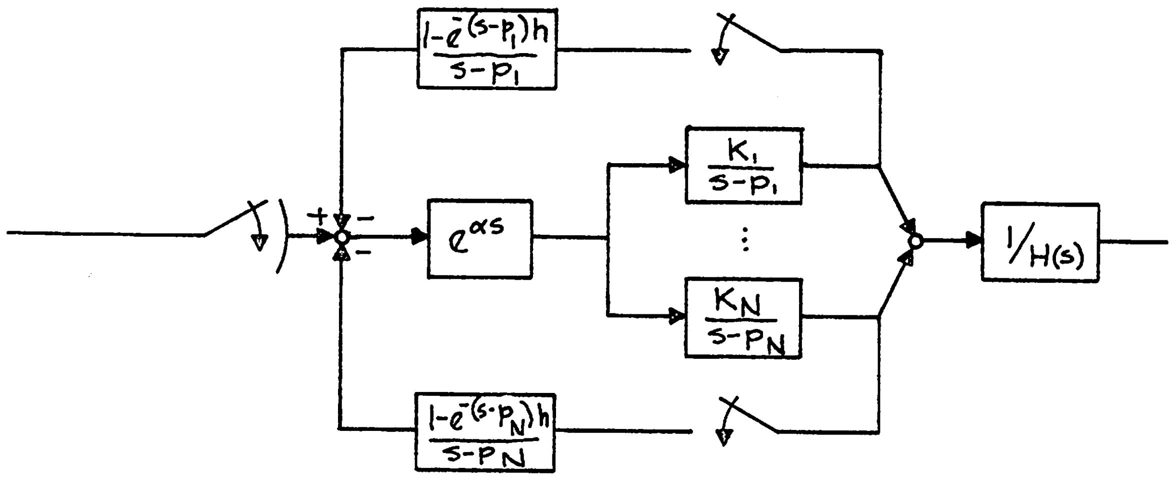
Referring to Fig. 2 again,

$$b_p(t) = \begin{cases} \sum_{k=1}^{N} b_k(t) & nT \leq t \leq nT + h \\ 0 & nT + h \leq t < (n+1)T \end{cases}$$

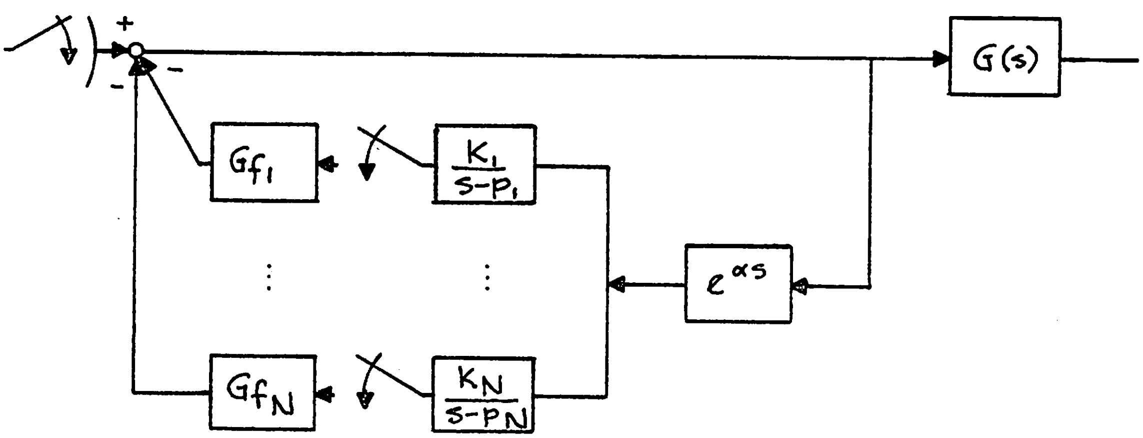
$$= \begin{cases} \sum_{k=1}^{N} b_k(nT)e^{p_k(t-nT)} & nT \leq t < nT + h \\ 0 & nT + h \leq t < (n+1)T \end{cases}$$

$$= \sum_{k=1}^{N} b_k(nT)e^{p_k(t-nT)}\left[u_{-1}(t - nT) - u_{-1}(t - \overline{nT + h})\right]$$

So, the equivalent Fig. 3 is:



The equivalent Fig. 4 is:



$$G_{f_k} = \frac{1 - e^{-(s-p_k)h}}{s - p_k}$$

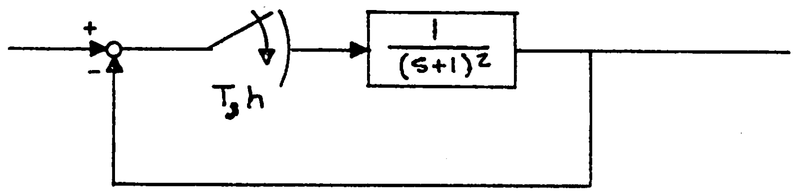
This block diagram is just a special case of Fig. 4 in the paper. So, Murphy's analysis can again be applied.

This analysis is obviously also applicable to the case when $T - h > |\alpha| - mT > h$, $m = 0, 1, 2, \ldots$, and can also be extended to the case with multiple poles.

The authors believe that, for α outside this range, Murphy's method cannot be applied, even with extension and modification of the block diagrams, because in neither portion of the period is the input n(t) completely zero so that $b_k(t)$ would not be an exponential decay completely in either portion, and transfer function representation is impossible.

APPENDIX II

An Example



Step input

$$KG(s) = \frac{1}{(s + 1)^2}$$

$$H(s) = 1$$

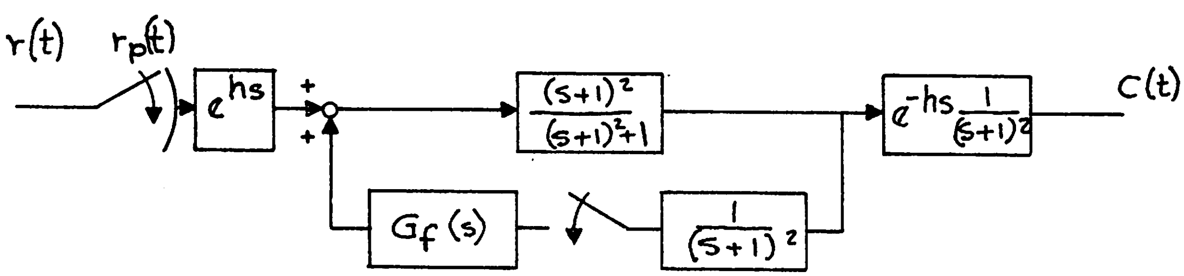
$$\alpha = 0$$

$$T = 1$$

$$h = 0.25$$

1. By Murphy's Method

As extended in Appendix I.3 the following equivalent block diagram is obtained:



$$\text{where } G_f(s) = \frac{1}{(s + 1)^2} - e^{-(s+1)(T-h)}\left[\frac{1}{(s + 1)^2} + \frac{(T - h)}{(s + 1)}\right]$$

Now

$$C(s) = R_p(s)\,\frac{1}{(s + 1)^2 + 1} + \frac{e^{-hs}}{(s + 1)^2 + 1}\,G_f(s)\,\mathcal{J}[e^{hs}C(s)] \qquad (1)$$

Multiplying both sides by e^{hs}:

$$e^{hs}C(s) = R_p(s)\frac{e^{hs}}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}G_f(s)\mathcal{Z}[e^{hs}C(s)]$$

Taking z-transform:

$$\mathcal{Z}[e^{hs}C(s)] = \mathcal{Z}\left[R_p(s)\frac{e^{hs}}{(s+1)^2+1}\right] + \mathcal{Z}\left[\frac{G_f(s)}{(s+1)^2+1}\right]\mathcal{Z}[e^{hs}C(s)]$$

Transposing:

$$\mathcal{Z}[e^{hs}C(s)] = \frac{\mathcal{Z}\left[R_p(s)\frac{e^{hs}}{(s+1)^2+1}\right]}{1-\mathcal{Z}\left[\frac{1}{(s+1)^2+1}G_f(s)\right]} \tag{2}$$

Substituting (2) in (1):

$$C(s) = R_p(s)\frac{1}{(s+1)^2+1} + \frac{e^{-hs}}{(s+1)^2+1}G_f(s)\frac{\mathcal{Z}\left[R_p(s)\frac{e^{hs}}{(s+1)^2+1}\right]}{1-\mathcal{Z}\left[\frac{1}{(s+1)^2+1}G_f(s)\right]} \tag{3}$$

Taking z-transform:

$$\boxed{C(z) = \mathcal{Z}\left[\frac{R_p(s)}{(s+1)^2+1}\right] + \mathcal{Z}\left[\frac{e^{-hs}}{(s+1)^2+1}G_f(s)\right]\frac{\mathcal{Z}\left[R_p(s)\frac{e^{hs}}{(s+1)^2+1}\right]}{1-\mathcal{Z}\left[\frac{1}{(s+1)^2+1}G_f(s)\right]}} \tag{4}$$

<u>Numerical calculations</u>

$$R_p(s) = \frac{1}{s}\,\frac{1 - e^{-hs}}{1 - e^{-Ts}} \qquad T = 1,\ h = 0.25,\ \frac{h}{T} = 0.25$$

$$\mathcal{Z}\left[\frac{R_p(s)}{(s + 1)^2 + 1}\right] = \frac{1}{1 - z^{-1}}\left(\frac{0.081z + 0.005}{z^2 - 0.397z + 0.135}\right)$$

$$\mathcal{Z}\left[R_p(s)\,\frac{e^{hs}}{(s + 1)^2 + 1}\right] = \frac{1}{1 - z^{-1}}\left(\frac{0.026z^2 + 0.056z}{z^2 - 0.397z + 0.135}\right)$$

$$G_f(s) = \frac{1}{(s + 1)^2}\left[1 - e^{-(s+1)(T-h)}\right] - \frac{e^{-(s+1)(T-h)}}{(s + 1)}\,(T - h)$$

$$\mathcal{Z}\left[\frac{1}{(s + 1)^2 + 1}\,G_f(s)\right] = \frac{0.048z + 0.018}{z^2 - 0.397z + 0.135}$$

$$\frac{1}{1 - \mathcal{Z}\left[\frac{1}{(s + 1)^2 + 1}\,G_f(s)\right]} = \frac{z^2 - 0.397z + 0.135}{z^2 - 0.445z + 0.117}$$

$$\mathcal{Z}\left[\frac{e^{-hs}}{(s + 1)^2 + 1}\,G_f(s)\right] = \frac{0.032z + 0.033}{z^2 - 0.397z + 0.135}$$

From (4):

$$C(z) = \frac{1}{1 - z^{-1}} \frac{0.08z + 0.005}{z^2 - 0.397z + 0.135}$$

$$+ \frac{0.032z + 0.033}{z^2 - 1.397z + 0.135} \cdot \frac{1}{1 - z^{-1}} \frac{0.026z^2 + 0.056z}{z^2 - 0.397z + 0.135} \cdot \frac{z^2 - 0.397z + 0.135}{z^2 - 0.445z + 0.117}$$

$$= \frac{0.0893z^4 - 0.0045z^3 + 0.0257z^2 + 0.0006z}{z^5 - 1.843z^4 + 1.272z^3 - 0.535z^2 + 0.122z - 0.016}$$

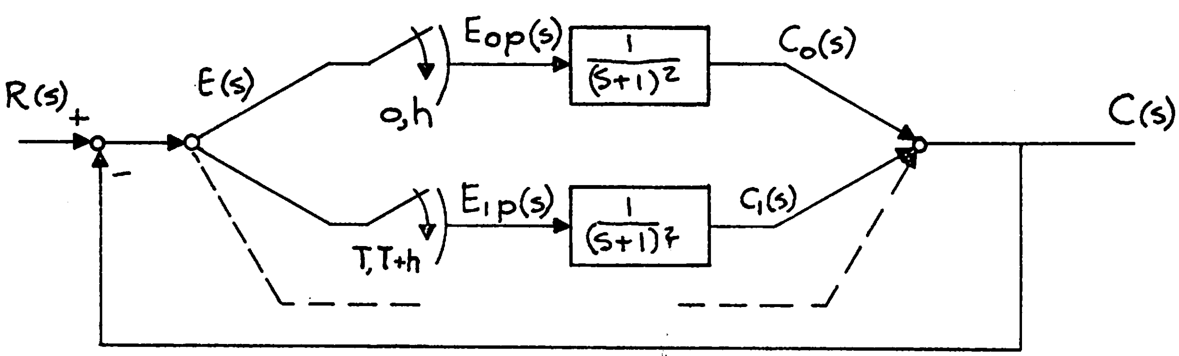
$$= 0.0893z^{-1} + 0.161z^{-2} + 0.209z^{-3} + 0.228z^{-4} + \cdots$$

i.e.,

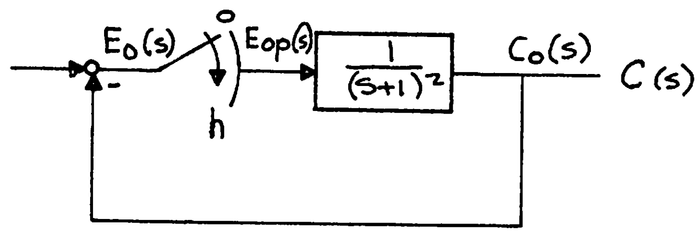
$$\left.\begin{aligned} c(0) &= 0 \\ c(1) &= 0.0893 \\ c(2) &= 0.161 \\ c(3) &= 0.209 \\ c(4) &= 0.228 \\ &\vdots \end{aligned}\right\}$$

2. <u>By Farmanfarma's Method</u>

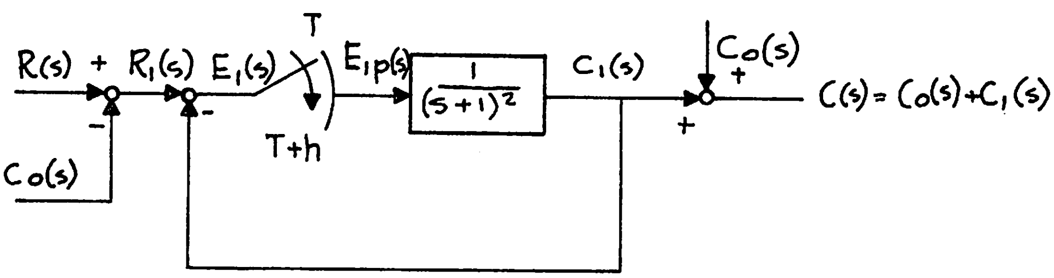
Equivalent block diagram



(i) For interval $0 \leqq t \leqq T$



(ii) $T \leqq t \leqq 2T$



From equation (55) in Farmanfarma's paper [3]

$$C_0(s) = \mathcal{P}_0^{\,h}\left[\frac{R(s)}{1 + KG(s)}\right] KG(s) \tag{5}$$

$$C_1(s) = e^{-Ts}\,\mathcal{P}_0^{\,h}\left\{\frac{e^{Ts}\,\mathcal{P}_T^{\,\infty}[R(s) - C_0(s)]}{1 + KG(s)}\right\} KG(s) \tag{6}$$

$\vdots$

$$C_n(s) = e^{-nTs}\,\mathcal{P}_0^{\,h}\left\{\frac{e^{nTs}\,\mathcal{P}_{nT}^{\,\infty}[R(s) - C_0(s) - C_1(s) - \cdots - C_{n-1}(s)]}{1 + KG(s)}\right\} KG(s)$$

Numerical calculations

$$\mathcal{P}_0^h\left[\frac{R(s)}{1 - pKG(s)}\right] = \mathcal{P}_0^h\left[\frac{1}{s}\frac{1}{1 + \frac{1}{(s+1)^2}}\right] = \mathcal{P}_0^h\left\{\frac{(s+1)^2}{s[(s+1)^2+1]}\right\}$$

Now:

$$\frac{(s+1)^2}{s[(s+1)^2+1]} = \frac{1}{2}\frac{1}{s} + \frac{\alpha_2}{4j}\frac{1}{(s+\alpha_1)} - \frac{\alpha_1}{4j}\frac{1}{(s+\alpha_2)}$$

where

$$\left.\begin{aligned}\alpha_1 &= 1 - j\\ \alpha_2 &= 1 + j\end{aligned}\right\}\ \text{roots of } (s+1)^2 + 1$$

$\therefore$

$$\mathcal{P}_0^h\left\{\frac{(s+1)^2}{s[(s+1)^2+1]}\right\} = \frac{1}{2}\mathcal{P}_0^h\left(\frac{1}{s}\right) + \frac{\alpha_2}{4j}\mathcal{P}_0^h\left(\frac{1}{s+\alpha_1}\right) - \frac{\alpha_1}{4j}\mathcal{P}_0^h\left(\frac{1}{s+\alpha_2}\right)$$

$$= \frac{1}{2}\frac{1}{s}\left(1 - e^{-hs}\right) + \frac{\alpha_2}{4j}\left[1 - e^{-h(s+\alpha_1)}\right]\frac{1}{s+\alpha_1} - \frac{\alpha_1}{4j}\left[1 - e^{-h(s+\alpha_2)}\right]\frac{1}{s+\alpha_2}$$

So

$$C(s) = \frac{1}{2}\frac{1}{s}\left(1 - e^{-hs}\right)\frac{1}{s(s+1)^2}$$

$$+ \frac{\alpha_2}{4j}\left(1 - e^{-h\alpha_1}e^{-hs}\right)\frac{1}{(s+\alpha_1)(s+1)^2} - \frac{\alpha_1}{4j}\left(1 - e^{-h\alpha_2}e^{-hs}\right)\frac{1}{(s+\alpha_2)(s+1)^2}$$

Now

$$\frac{1}{(s+\alpha_1)(s+1)^2} = -\frac{1}{s+\alpha_1} + \frac{1}{s+1} + \frac{j}{(s+1)^2}$$

$$\frac{1}{(s+\alpha_2)(s+1)^2} = -\frac{1}{s+\alpha_2} + \frac{1}{s+1} - \frac{j}{(s+1)^2}$$

$$\frac{1}{s(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

$$\therefore \quad C_0(s) = \frac{1}{2}\left(1 - e^{-hs}\right)\left[\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right]$$

$$+ \frac{\alpha_2}{4j}\left(1 - e^{-h\alpha_1}e^{-hs}\right)\left[-\frac{1}{s+\alpha_1} + \frac{1}{s+1} + j\frac{1}{(s+1)^2}\right]$$

$$- \frac{\alpha_1}{4j}\left(1 - e^{-h\alpha_2}e^{-hs}\right)\left[-\frac{1}{s+\alpha_2} + \frac{1}{s+1} - j\frac{1}{(s+1)^2}\right]$$

Taking inverse Laplace transform:

$$c(1) = \frac{1}{2} - \frac{1}{2}e^{-1} - \frac{1}{2}e^{-1} - \frac{1}{2} + \frac{1}{2}e^{-0.75} + \frac{1}{2}0.75e^{-0.75}$$

$$+ \frac{\alpha_2}{4j}\left(-e^{-\alpha_1} + e^{-\alpha_1} + e^{-1} - e^{-0.75}e^{-0.25\alpha_1} + je^{-1} - je^{-0.75}0.75e^{-0.25\alpha_1}\right)$$

$$- \frac{\alpha_1}{4j}\left(e^{-\alpha_2} - e^{-\alpha_2} + e^{-1} - e^{-0.75}e^{-0.25\alpha_2} - je^{-1} + j0.75e^{-0.75}e^{-0.25\alpha_2}\right)$$

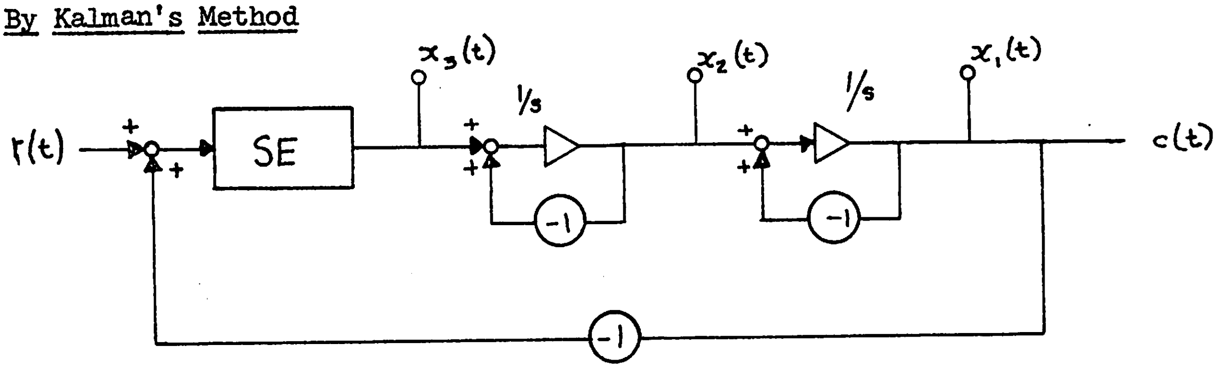
$$= 0.09$$

by observing $\mathcal{L}^{-1}[C_1(s)]$ at $t = T$ is zero because the plant has continuous step response.

Due to the extremely large number of terms in $\mathcal{L}^{-1}[C_1(s)]$, $c(2) = \mathcal{L}^{-1}[C_0(s) + C_1(s)]\big|_{t=2}$ is not calculated.

$$\therefore \quad \left.\begin{array}{l} c(0) = 0 \\ c(1) = 0.09 \\ \vdots \end{array}\right\}$$

3. <u>By Kalman's Method</u>



For $kT < t \leq kT + h$

$$\begin{cases} x_3(t) = r(t) - x_1(t) \\ \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = -x_2 + x_3 = -x_2 - x_1 + r(t) \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

Let $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$,

$$\Rightarrow \qquad e^{At} = e^{-t} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

as

$$\underline{x}(kT + \tau) = e^{A\tau}\underline{x}(kT) + \int_0^{\tau} e^{A(\tau-\nu)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(\nu)d\nu$$

$$\begin{bmatrix} x_1(kT+\tau) \\ x_2(kT+\tau) \\ x_3(kT+\tau) \end{bmatrix} = \begin{bmatrix} e^{-\tau}\cos\tau & e^{-\tau}\sin\tau & 0 \\ -e^{-\tau}\sin\tau & e^{-\tau}\cos\tau & 0 \\ -e^{-\tau}\cos\tau & -e^{-\tau}\sin\tau & 0 \end{bmatrix} \begin{bmatrix} x_1(kT) \\ x_2(kT) \\ x_3(kT) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - \frac{1}{2}e^{-\tau}(\sin\tau + \cos\tau) \\ \frac{1}{2} + \frac{1}{2}e^{-\tau}(\sin\tau - \cos\tau) \\ \frac{1}{2} + \frac{1}{2}e^{-\tau}(\sin\tau + \cos\tau) \end{bmatrix}$$

$$= \overline{\Phi}(\tau)\underline{x}(kT) + \underline{s}_1(\tau) \qquad (7)$$

For $kT + h \leq t \leq (k + 1)T$

$$x_3(t) = 0$$

$$\dot{x}_1 = -x_1 + x_2$$

$$\dot{x}_2 = -x_2$$

$$\Rightarrow \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let $B = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

$$e^{Bt} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} x_1(kT+h+\tau) \\ x_2(kT+h+\tau) \\ x_3(kT+h+\tau) \end{bmatrix} = \begin{bmatrix} e^{-\tau} & te^{-\tau} & 0 \\ 0 & e^{-\tau} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(kT+h) \\ x_2(kT+h) \\ x_3(kT+h) \end{bmatrix}$$

$$= \Phi(\tau)\underline{x}(kT+h) \qquad (8)$$

Combining (7) and (8):

$$\underline{x}(\overline{k+1}T) = \Phi(T-h)\underline{x}(kT+h)$$

$$= \Phi(T-h)\overline{\Phi}(h)\underline{x}(kT) + \Phi(T-h)\underline{s}_1(h) \qquad (9)$$

<u>Numerical</u> <u>calculations</u>

$$\Phi(T-h) = \begin{bmatrix} e^{-0.75} & 0.75e^{-0.75} & 0 \\ 0 & e^{-0.75} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.472 & 0.354 & 0 \\ 0 & 0.472 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{s}_1(h) = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}e^{-0.25}(\sin 0.25 + \cos 0.25) \\ \frac{1}{2} + \frac{1}{2}e^{-0.25}(\sin 0.25 - \cos 0.25) \\ \frac{1}{2} + \frac{1}{2}e^{-0.25}(\sin 0.25 + \cos 0.25) \end{bmatrix} = \begin{bmatrix} 0.027 \\ 0.220 \\ 0.993 \end{bmatrix}$$

$$\bar{\Phi}(h) = \begin{bmatrix} e^{-0.25}\cos 0.25 & e^{-0.25}\sin 0.25 & 0 \\ -e^{-0.25}\sin 0.25 & e^{-0.25}\cos 0.25 & 0 \\ -e^{-0.25}\cos 0.25 & -e^{-0.25}\sin 0.25 & 0 \end{bmatrix} = \begin{bmatrix} 0.755 & 0.192 & 0 \\ -0.192 & 0.755 & 0 \\ -0.755 & -0.192 & 0 \end{bmatrix}$$

$x_1(0) = 0 \qquad x_2(0) = 0$

$$\underline{x}(1^-) = \Phi(T - h)\underline{s}_1(h) = \begin{bmatrix} 0.091 \\ 0.104 \\ 0 \end{bmatrix} \qquad \underline{x}(1^+) = \begin{bmatrix} 0.091 \\ 0.104 \\ 1 \end{bmatrix}$$

$$\underline{x}(2^-) = \Phi(T - h)\bar{\Phi}(h)\underline{x}(1^+) + \Phi(T - h)\underline{s}_1(h)$$

$$= \begin{bmatrix} 0.154 \\ 0.140 \\ 0 \end{bmatrix} \qquad \underline{x}(2^+) = \begin{bmatrix} 0.154 \\ 0.140 \\ 1 \end{bmatrix}$$

$$\underline{x}(3^-) = \begin{bmatrix} 0.185 \\ 0.140 \\ 0 \end{bmatrix} \qquad \underline{x}(3^+) = \begin{bmatrix} 0.185 \\ 0.140 \\ 1 \end{bmatrix}$$

$$\underline{x}(4^-) = \begin{bmatrix} 0.194 \\ 0.137 \\ 0 \end{bmatrix} \qquad \underline{x}(4^+) = \begin{bmatrix} 0.194 \\ 0.140 \\ 1 \end{bmatrix}$$

i.e.,

$$\left.\begin{array}{l} c(0) = 0 \\ c(1) = 0.091 \\ c(2) = 0.154 \\ c(3) = 0.185 \\ c(4) = 0.194 \\ \vdots \end{array}\right\}$$

4. Comparison and comments on the results obtained by the three different methods

Murphy's method	Farmanfarma's	Kalman's
c(0) = 0	c(0) = 0	c(0) = 0
c(1) = 0.089	c(1) = 0.09	c(1) = 0.091
c(2) = 0.161	.	c(2) = 0.154
c(3) = 0.209	.	c(3) = 0.185
.	.	.
.		.
.		.

The result obtained by Kalman's method is most accurate because of least numerical calculation. Murphy's result is not as accurate because the numerical calculation involves too many differences between small numbers which lead to errors in final results. From the example it should be clear that the state space approach is the neatest and simplest.