A MULTI-PARAMETER SENSITIVITY MEASURE FOR LINEAR SYSTEMS

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ABSTRACT

A scalar sensitivity measure for linear, time-invariant, single-input, single-output, multiparameter systems has been defined in terms of the generalized root and gain sensitivities and the fractional perturbations of the parameters. The definition works equally well whether the transfer function of interest has simple or multiple poles and zeros. Based on this definition, a multi-parameter sensitivity index independent of the parameter perturbations has been introduced, taking the statistics of the random parameters into account. Application of this sensitivity index for optimal design is given in the companion paper.

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1. INTRODUCTION

The problems of analyzing the change of performance of a system due to perturbations in its parameters and the optimum synthesis of a system for minimum sensitivity have always been of considerable interest and importance to engineers. The first systematic approach to these problems was made by Bode[1] whose theory on return difference and sensitivity has been the basis for analysis and design of linear feedback systems. While in the old days one considered the effect of a single parameter in a feedback system, such as the gain of an amplifier; now one frequently deals with systems which contain multi-parameters that are subject to variations. For example in integrated circuits all circuit elements are sensitive to temperature and environmental changes. Therefore, it is crucial in the analysis and design of such systems to introduce the concept of multi-parameter sensitivity and to propose a sensitivity measure which is not only convenient for comparison of various systems but also useful for optimization.

In linear time-invariant lumped systems the conventional single-parameter sensitivity function is a rational function of the complex frequency variable, and is a measure of the relative change of the transfer function of interest with respect to the change of a particular parameter. Extensions of the scalar sensitivity function to multi-parameters have been made by various people. For example, a multi-parameter vector sensitivity function can be introduced[2,3] by using the gradient of the transfer function with respect to a parameter vector. However, from the designer's viewpoint the vector sensitivity function is not particularly attractive since it is a vector-valued function of complex frequency variable and hence it is difficult to use for comparison and for optimum design. Other extensions such as the use of sensitivity matrix formulation, while useful in some control problems, have limitations for general analysis and synthesis.[4]

In this paper we propose a general scalar sensitivity measure which is not a function of frequency. It is related simply to the conventional single-parameter gain and root sensitivities. Moreover, it is flexible in that simple weighting factors are incorporated in the definition for specific design purposes. An additional feature is that it can take into account the statistics of the multi-parameters. This is especially useful since, in general, the perturbations of various parameters are functions of some random variables with certain correlations among them.

In Section 2 we will review some important aspects of the sensitivity function and its relation to gain and root sensitivities. In Section 3 we will introduce the sensitivity measure and derive some useful properties. In Section 4 we will present the statistical considerations and define a sensitivity index. The application of the proposed measure is given in a companion paper.[5]

2. THE SENSITIVITY FUNCTION

Consider a linear time-invariant, single-input, single-output, lumped system. Let the transfer function be w:

$$w(s) = K \frac{\prod_{\ell=1}^{r} (s - z_\ell)}{\prod_{j=1}^{q} (s - p_j)} \tag{1}$$

where s is the complex frequency variable, z_ℓ and p_j are, respectively, the zeros and poles of the transfer function, and K represents the gain. Let x_k, $k = 1, 2, \ldots, n$ be the n parameters whose effects on the transfer function are of interest. Then the conventional scalar sensitivity function of the transfer function w with respect to the parameter x_k is denoted by $S_{x_k}^{w}$ and is defined as

$$S_{x_k}^{w}(s) \triangleq \frac{\partial \ln w(s)}{\partial \ln x_k} = x_k \frac{\partial \ln w(s)}{\partial x_k} = \frac{x_k}{w(s)} \frac{\partial w(s)}{\partial x_k} \tag{2}$$

For simplicity, we assume that x_k is not a function of the complex frequency variable s. Combining Eqs. (1) and (2), we obtain

$$S_{x_k}^{w}(s) = x_k \frac{\partial \ln K}{\partial x_k} + x_k \sum_{\ell=1}^{r} \frac{-\frac{\partial z_\ell}{\partial x_k}}{s - z_\ell} - x_k \sum_{j=1}^{q} \frac{-\frac{\partial p_j}{\partial x_k}}{s - p_j} \tag{3}$$

Thus the sensitivity function has been put in the form of partial fraction expansion. For convenience, we rewrite (3) as follows:

$$S_{x_k}^{w}(s) = \sum_{i=1}^{m} \frac{h_{ik}}{s - s_i} + h_{m+1,k} \tag{4}$$

Note that the sensitivity function of w with respect to any x_k is a rational function whose poles, s_i, $i = 1, 2, \ldots, m$, are the poles and zeros of w. The residue of $S_{x_k}^{w}$ at the pole s_i, denoted by h_{ik} is given by

$$h_{ik} = \pm x_k \frac{\partial s_i}{\partial x_k} \tag{5}$$

The + sign is used if s_i is a pole of w, and the - sign is used if s_i is a zero of w. Equation (5) indicates that the residue at a pole s_i is simply the root sensitivity[6-8] or its negative. The constant term in Eq. (4) denoted by $h_{m+1,k}$ is from (3)

$$h_{m+1,k} = x_k \frac{\partial \ln K}{\partial x_k} = S_{x_k}^{K} \tag{6}$$

and is recognized as the gain sensitivity.

It should be pointed out that when some of the poles or zeros of the transfer function w are repeated, the root sensitivity in the conventional sense, as in Eq. (5), is not uniquely defined, since a multiple root is a branch point of the root locus. Consequently, Eq. (3) is no longer valid. However, the sensitivity function $S_{x_k}^{w}$ still has the partial fraction expansion form given by Eq. (4), where its poles s_i, $i = 1, 2, \ldots, m$ are now the m <u>distinct</u> poles and zeros of w. This can be shown by considering, for example, the polynomial containing multiple roots at $s = z$ of order ν:

$$P(s) \triangleq (s - z)^{\nu} \tag{7}$$

Let a first order perturbation be introduced so that the roots of the polynomial are moved from $s = z$ to $s = z + \delta z_{\ell}$, $\ell = 1, 2, \ldots, \nu$. Then

$$\delta P(s) = \prod_{\ell=1}^{\nu} (s - z - \delta z_{\ell}) - (s - z)^{\nu} \tag{8}$$

represents the perturbation of the given polynomial $P(s)$. To a first order approximation

$$\frac{\delta P(\delta)}{P(\delta)} = \frac{- \sum_{\ell=1}^{\nu} \delta z_{\ell}}{s - z} \tag{9}$$

It is then straightforward to obtain Eq. (4) directly from Eq. (2), and it is easily seen that the sensitivity function $S_{x_k}^{w}$ still has simple poles. Thus the partial fraction expansion form of Eq. (4) is valid regardless whether the given transfer function w has repeated poles and zeros or not. This suggests a logical extension of the definition of root sensitivity for multiple roots, say s_i, to be the residue h_{ik} of the sensitivity function in Eq. (4) at the root location s_i*.

Let us summarize the above properties as follows:

(i) The sensitivity function $S_{x_k}^{w}$ which is defined as the sensitivity of the transfer function w with respect to the parameter x_k is a rational function of the complex frequency variable s. It contains m simple poles at s_i which are the <u>distinct</u> poles and zeros of the transfer function w.

(ii) The sensitivity function $S_{x_k}^{w}$ usually has the same degree in the numerator and the denominator polynomials. The behavior at infinity as given by $h_{m+1,k}$ is the gain sensitivity with respect to x_k.

(iii) The residue h_{ik} at a pole s_i which is a pole of the transfer function is the pole-sensitivity with respect to x_k.

(iv) The residue at a pole which is a zero of the transfer function is the negative of the zero-sensitivity.

(v) The over-all sensitivity property of the transfer function w with respect to the n parameters x_k, $k=1, 2, \ldots, n$ is characterized by n sensitivity functions $S_{x_k}^{w}$, $k=1, 2, \ldots, n$. Since the poles of $S_{x_k}^{w}$ are s_i, $i=1, 2, \ldots, m$, which are specified by the transfer function w, we can

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* This is different from the definition used by Horowitz.[7] However, with our present formulation, we do not need to distinguish between simple and multiple roots.

say that the over-all sensitivity property of the complete system is characterized by an $(m+1) \times n$ matrix

$$\underline{H} = [h_{ik}] \tag{10}$$

where h_{ik} is the residue of the sensitivity function $S_{x_k}^{w}$ at the pole s_i for $i = 1, 2, \ldots, m$, and is the constant term in the partial fraction expansion of $S_{x_k}^{w}$ when $i = m+1$ [see Eq. (4)].

3. THE SENSITIVITY MEASURE

In the previous section we have given expression of the sensitivity function for the transfer function with respect to the parameter x_k. Since there are n parameters, $x_1, x_2, \ldots, x_n$, an incremental transfer function δw can be written in terms of incremental changes δx_k, $k = 1, 2, \ldots, n$. Using the first order terms of Taylor's expansion, we have

$$\delta w(s) = \sum_{k=1}^{n} \frac{\partial w(s)}{\partial x_k} \delta x_k \tag{11}$$

We can also express the incremental transfer function in terms of the n sensitivity functions by means of Eq. (2):

$$\delta w(s) = \sum_{k=1}^{n} w(s)\, S_{x_k}^{w}(s) \frac{\delta x_k}{x_k} \tag{12}$$

Let us introduce the following notations:

$$f(s) \triangleq \frac{\delta w}{w}(s) \quad \text{and} \quad \epsilon_k \triangleq \frac{\delta x_k}{x_k} \tag{13}$$

Then Eq. (12) becomes

$$f(s) = \sum_{k=1}^{n} S_{x_k}^{w}(s)\,\epsilon_k \tag{14}$$

and we call ϵ_k's the fractional parameter perturbations. The function f measures the fractional change in the transfer function w. Knowing the parameter perturbations and the n sensitivity functions we can calculate the function f. From Eq. (14) we see that the function f has the same poles as the sensitivity functions. They are poles and zeros of the transfer function. The partial fraction expansion of f is

$$f(s) = \sum_{i=1}^{m} \frac{f_i}{s - s_i} + f_{m+1} \tag{15}$$

The scalar sensitivity measure $\mathcal{M}$ is now defined as follows:

$$\mathcal{M}^2 \triangleq \sum_{i=1}^{m+1} \left[\alpha(s_i)(\mathrm{Re}\, f_i)^2 + \beta(s_i)(\mathrm{Im}\, f_i)^2\right] \tag{16}$$

where α and β are positive weighting functions which may be chosen arbitrarily by the designer to assign desired weights to the different poles and zeros of the transfer function. For notational convenience, let $\alpha_i \triangleq \alpha(s_i)$ and $\beta_i \triangleq \beta(s_i)$. Furthermore we choose $\alpha_{m+1} = 1$ and $\beta_{m+1} = 0$ since f_{m+1} is always real. In matrix notation, we denote

$$\underline{A} \triangleq \mathrm{diag}\,(\alpha_1, \alpha_2, \ldots, \alpha_m, 1) \tag{17a}$$

$$\underline{B} \triangleq \mathrm{diag}\,(\beta_1, \beta_2, \ldots, \beta_m, 0) \tag{18b}$$

as the diagonal weighting matrices, and

$$\underline{f} \triangleq (f_1, f_2, \ldots, f_{m+1})^T \tag{17c}$$

Then

$$\mathcal{M}^2 = (\mathrm{Re}\,\underline{f}^T)\,\underline{A}\,(\mathrm{Re}\,\underline{f}) + (\mathrm{Im}\,\underline{f}^T)\,\underline{B}\,(\mathrm{Im}\,\underline{f}) \tag{18}$$

The sensitivity measure can be expressed in terms of the fractional parameter perturbations ϵ_k, $k=1, 2, \ldots, n$ and the residues h_{ik}'s of the sensitivity functions. Comparing Eq. (14) with (15) and using Eq. (4), we obtain

$$f_i = \sum_{k=1}^{n} h_{ik}\,\epsilon_k, \qquad i = 1, 2, \ldots, m+1 \tag{19}$$

Thus (16) can be written as

$$\mathcal{M}^2 = \sum_{i=1}^{m+1} \left[\alpha_i \sum_{k=1}^{n} (\mathrm{Re}\, h_{ik}\, \epsilon_k)^2 + \beta_i \sum_{k=1}^{n} (\mathrm{Im}\, h_{ik}\, \epsilon_k)^2 \right] \quad (20)$$

In matrix notation, Eq. (15) is

$$\underline{f} = \underline{H}\,\underline{\epsilon} \quad (21a)$$

where

$$\underline{\epsilon} \triangleq (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)^T \quad (21b)$$

is the fractional parameter perturbation vector and

$$\underline{H} \triangleq [h_{ik}] \quad (21c)$$

Eq. (19) can now be written as

$$\mathcal{M}^2 = \underline{\epsilon}^T \left[(\mathrm{Re}\, \underline{H}^T)\, \underline{A}\, (\mathrm{Re}\, \underline{H}) + (\mathrm{Im}\, H^T)\, B\, (\mathrm{Im}\, H) \right] \underline{\epsilon} \quad (22)$$

Or, more conveniently, by defining a sensitivity matrix

$$\underline{S} \triangleq (\mathrm{Re}\, \underline{H}^T)\, \underline{A} (\mathrm{Re}\, \underline{H}) + (\mathrm{Im}\, \underline{H}^T)\, \underline{B} (\mathrm{Im}\, \underline{H}) \quad (23)$$

we can express the scalar sensitivity measure as

$$\mathcal{M}(\underline{\epsilon}) = \sqrt{\underline{\epsilon}^T \underline{S}\, \underline{\epsilon}} \quad (24)$$

Thus it can be easily calculated from the sensitivity matrix $\underline{S}$ and the fractional parameter perturbation vector $\underline{\epsilon}$.

The sensitivity measure $\mathcal{M}$ defined in Eq. (16) represents a norm in the n-dimensional parameter perturbation vector space. This fact can be easily established. First, it is obvious that $\mathcal{M} \geq 0$, and $\mathcal{M} = 0$ only when $\underline{\epsilon} = 0$. Second, for an arbitrary real constant k, it is a fact that $\mathcal{M}(k\underline{\epsilon}) = k\mathcal{M}(\underline{\epsilon})$. Third, it can be shown that $\mathcal{M}$ satisfies the triangular inequality, that is, $\mathcal{M}(\underline{\epsilon} + \underline{\delta}) \leq \mathcal{M}(\underline{\epsilon}) + \mathcal{M}(\underline{\delta})$.

4. STATISTICAL CONSIDERATIONS

The fractional parameter perturbations ϵ_k, $k = 1, 2, \ldots, n$ are n random variables with zero mean and may be correlated with one another. We are at liberty to pick any stochastic index of the spread in the value of our sensitivity measure $\mathcal{M}$. The simplest choice is the variance of the sensitivity measure, i.e., the expectation of $\mathcal{M}^2$, and we will call it the sensitivity index $\mathcal{S}$:

$$\mathcal{S} = E\{\mathcal{M}^2\} \tag{25}$$

The sensitivity index can be expressed directly in terms of the correlation coefficients among the ϵ_k's. Let $\underline{R}$ be the $n \times n$ covariance matrix of the n random variables $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, that is,

$$E\{\epsilon_k \epsilon_\ell\} = r_{k\ell}$$

and

$$E\{\underline{\epsilon}\,\underline{\epsilon}^T\} = \underline{R} = [r_{k\ell}] \tag{26}$$

From Eq. (24), we have

$$\mathscr{A} = E\{\mathscr{M}^2\} = E\{\epsilon^T \underline{S}\,\epsilon\} = \mathrm{tr}[\underline{S}\,\underline{R}] \tag{27}$$

where tr designates the trace of a matrix. Thus the sensitivity index can be computed easily from the sensitivity matrix $\underline{S}$ as given by Eq. (23) and the covariance matrix $\underline{R}$.

We now consider the two extreme cases of the correlation among the perturbations of different parameters. One extreme corresponds to complete mutual independence of the random variables ϵ_k. This is typical in the synthesis of lumped passive circuits when the main source of perturbations is the random manufacturing tolerance on element values. At the other extreme lies the totally degenerate case where all ϵ_k depend on one random variable only. This is typical in the design of integrated circuits, where all components are subject to the systematic error due to change in temperature or some similar environmental condition.

When the random variables ϵ_k, $k = 1, 2, \ldots, n$ are mutually independent, the covariance matrix $\underline{R}$ is diagonal with the diagonal element $r_{kk} = \sigma_k^2$, the variance of ϵ_k. Hence, from (27), we obtain

$$\mathscr{A} = E\{\mathscr{M}^2\} = \mathrm{tr}[\underline{S}\,\underline{R}] = \sum_{k=1}^{n} s_{kk}\,\sigma_k^2 \tag{28}$$

where s_{kk} is the $\underline{k}$th diagonal element of the matrix $\underline{S}$. In terms of the element of matrix $\underline{H}$, we can write

$$\mathcal{S} = E\{\mathcal{M}^2\} = \sum_{k=1}^{n} \sigma_k^2 \left[\sum_{i=1}^{m+1} \alpha_i(\mathrm{Re}\, h_{ik})^2 + \beta_i(\mathrm{Im}\, h_{ik})^2 \right] \qquad (29)$$

When $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are all deterministically related to only one random variable, the rank of the covariance matrix $\underline{R}$ is unity, so that its elements are given by

$$r_{k\ell} = \sigma_k \sigma_\ell, \qquad k, \ell = 1, 2, \ldots, n \qquad (30)$$

where σ_k^2 is the variance of ϵ_k. Then it follows from Eqs. (23) and (27)

$$\mathcal{S} = E\{\mathcal{M}^2\} = \underline{\sigma}^T \underline{S}\, \underline{\sigma} \qquad (31)$$

where $\underline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)^T$ is the standard deviation vector.

5. CONCLUSION

In this paper we have introduced a scalar sensitivity measure for a linear, time-invariant, single input, single-output, multi-parameter system. The sensitivity measure is dependent upon the parameter perturbations and is related to conventional gain and root sensitivities. The advantage of this sensitivity measure is that it is a scalar and it is frequency independent. Based on this, a sensitivity index independent of the parameter perturbations has been defined by taking into account the statistics of random parameters. In a companion paper the optimal synthesis of a transfer function by means of a special signal flow graph configuration which minimizes the sensitivity index is described.[5]

REFERENCES

1. H. W. Bode, Network Analysis and Feedback Amplifier Design, Van Nostrand Co., Inc., New York, 1945.
2. A. J. Goldstein and F. F. Kuo, "Multiparameter Sensitivity," IRE Trans. on Circuit Theory, vol. CT-8, pp. 177-178; June 1961.
3. S. L. Hakimi and J. B. Cruz, Jr., "Measures of Sensitivity for Linear Systems with Large Multiple Parameter Variations," IRE Wescon Conv. Record, 1960.
4. J. B. Cruz, Jr., and W. W. Perkins, "A New Approach to the Sensitivity Problem in Multivariable Feedback System Design," IEEE Trans. on Automatic Control, vol. AC-9, pp. 216-223; July 1964.
5. R. N. Biswas and E. S. Kuh, "Optimum Synthesis of a Class of Multiple Loop Feedback Systems," to appear.
6. E. S. Kuh and R. A. Rohrer, Theory of Linear Active Networks, Holden-Day, Inc., San Francisco, 1967.
7. I. M. Horowitz, Synthesis of Feedback Systems, Academic Press, New York, 1963.
8. D. A. Calahan, Modern Network Synthesis, Hayden Book Co., Inc., New York, 1964.