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OPTIMUM SYNTHESIS OF A CLASS OF MULTIPLE LOOP FEEDBACK SYSTEMS

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ABSTRACT

Realization of arbitrary transfer functions by a special class of multiple loop feedback configuration has been investigated. The nth order system consists of n nominally identical single-pole active stages with arbitrary but constant interconnections and feedbacks. The constraints on the sensitivity functions with respect to the active stages of such a system have been obtained, and conditions have been derived for the minimum of the multiparameter sensitivity index, defined in the companion paper. For second order systems, the optimal design has been analytically obtained. For higher order systems, a general optimization scheme, employing steepest descent from an initial design, has been outlined. The optimal design of a fourth order stagger-tuned bandpass filter has been presented as illustration.

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1. INTRODUCTION

With the advent of integrated circuits the philosophy of electronic circuit and system design has changed drastically. First, active circuit elements which used to be an order of magnitude more expansive than passive components are often cheaper and easier to obtain. Thus with the exception of possible stability considerations the number of active elements employed is in general of no concern to designers. Secondly, integrated circuit elements are more sensitive to temperature and enviromental variations. The designer is forced to consider the possible change of performance of a system due to perturbations in its parameters.[1] Therefore certain multi-parameter sensitivity measure must be introduced in order to compare circuits and to optimize the design. Thirdly, because of complicated fabrication techniques which is usually involved in the manufacturing of integrated circuits it is desirable to design circuits with dominant symmetry such as circuits employing many identical components. For these reasons we propose to investigate a special class of multiple loop feedback systems. The feedback system is to realize an arbitrary transfer function by means of n identical active stages and is to be optimized in the sense that a multi-parameter sensitivity index is minimized.

Even though in the discussion above we give our motivation to the proposed problem in terms of integrated circuits, the multiple-loop feedback configuration has obvious application in the optimum synthesis of linear systems. As a matter of fact the problem is related to the familiar problem of minimal realization of a transfer function by means of state variables.[2, 3]

In Section 2 we will describe the proposed multiple loop feedback configuration and its properties. In Section 3 we will illustrate the use of the multiple loop configuration to the synthesis of a second order transfer function. The optimum sensitivity measure is derived in Section 4. In Section 5 we will indicate the method of synthesis of high order systems by means of computer and give an illustration design.

2. A MULTIPLE LOOP FEEDBACK MODEL

Consider a single-input, single output, multiple-loop feedback system which is represented by the vector signal flow graph representation of Fig. 1.[4,5] In the figure u and y are, respectively, the scalar input and output of the system related to each other by the transfer function w, according to the equation

$$y = wu \tag{1}$$

$\underline{X}$ is an $n \times n$ diagonal matrix

$$\underline{X} = \text{diag}\,(x_1, x_2, \ldots, x_n) \tag{2}$$

The n x_k's are the transfer functions of the n active stages and are assumed to be nominally identical. Typically, x_k may correspond to a transistor having a single-pole transfer function of the form

$$x_k = \frac{\gamma_k}{s+1} \tag{3}$$

where γ_k represents the gain of the stage and the bandwidth is normalized to unity. We further assume that n γ_k's are the only parameters which are sensitive to change. We will optimize the transfer function with respect to a sensitivity index which is a function of the n γ_k's. Note that the sensitivity function of the transfer function w with respect to γ_k is the same as that with respect to x_k. For convenience, we will use

x_k's as the sensitive parameters from here on.

The other elements in the signal flow graph are all assumed to be constants and are insensitive. Thus d is a scalar which represents a direct transmission from the input to the output. $\underline{c}$ and $\underline{b}$ are constant n-vectors. $\underline{A}$ is a constant $n \times n$ matrix which may represent a general frequency independent feedback connection. In terms of the elements of the vector signal flow graph the transfer function w is given by

$$w = d + \underline{c}^T \underline{W}(\underline{X}) \underline{b} \tag{4}$$

where

$$\underline{W}(\underline{X}) = (\underline{X}^{-1} - \underline{A})^{-1} \tag{5}$$

If x_k is assumed to have single pole as in Eq. (3), then with n x_k's, the general configuration can realize a transfer function w with n poles. The synthesis problem can be described as the determination of the quadruple $\{\underline{A}, \underline{b}, \underline{c}, d\}$ from the given transfer function w and the given diagonal matrix $\underline{X}$. The optimum synthesis problem can be described as finding a particular quadruple which not only realizes the given transfer function w but also minimizes a specified sensitivity index.

The advantage of the signal flow graph model is in its generality. Consider a given quadruple $\{\underline{A}, \underline{b}, \underline{c}, d\}$ which realizes a specified transfer function w with a given $\underline{X}$. Then if we let $\underline{T}$ be any nonsingular real $n \times n$ matrix, the quadruple

$$\{\tilde{\underline{A}}, \tilde{\underline{b}}, \tilde{\underline{c}}, \tilde{d}\} \triangleq \{\underline{T}^{-1}\underline{A}\underline{T}, \underline{T}^{-1}\underline{b}, \underline{T}^{T}\underline{c}, d\} \tag{6}$$

also realizes the same transfer function w. This is easily proven since

$$\tilde{\underline{W}} = (\underline{X}^{-1} - \tilde{\underline{A}})^{-1} = \underline{T}^{-1}\underline{W}\underline{T} \tag{7}$$

Substituting Eqs. (6) and (7) in (4), we can easily see that $\tilde{w} = w$. Since $\underline{T}$ is arbitrary, we can then optimize the transfer function synthesis by starting with any quadruple realization of w and then minimizing a sensitivity index with respect the elements of the transformation matrix $\underline{T}$. It can be shown that all possible quadruple realizations may be obtained by such a transformation, i.e., given any two quadruples $\{\underline{A}, \underline{b}, \underline{c}, d\}$ and $\{\tilde{\underline{A}}, \tilde{\underline{b}}, \tilde{\underline{c}}, \tilde{d}\}$ realizing the same transfer function w according to Eqs. (4) and (5), a real nonsingular transformation matrix $\underline{T}$ can always be found so as to satisfy Fig. (6). The problem is similar to that of finding all equivalent minimal realization of a given transfer function as proposed by Kalman.[2] In Kalman's problem the matrix $\underline{X}$ is $\frac{1}{s}\underline{1}$.

3. A SECOND-ORDER EXAMPLE

Before going into the optimum synthesis of a general transfer function, we first consider a second-order example. Let the transfer function be

$$w(s) = \frac{2k\,s}{s^2 + 2\xi s + \omega_0^2} \tag{8}$$

and let

$$x_1 = x_2 = \frac{1}{s+1} \tag{9}$$

Thus the two identical stages correspond to unity-gain single-pole transfer functions with unity bandwidth. The synthesis problem is to determine any quadruple $\{\underline{A}, \underline{b}, \underline{c}, d\}$ which realizes the specified $w(s)$. The scalar signal flow graph is shown in Fig. 2, where the elements $a_{11} \triangleq f_{11} + 1$, $a_{22} \triangleq f_{22} + 1$, a_{12}, a_{21}, b_1, b_2, c_1 and c_2 are to be determined. It is obvious that since $w(\infty) = 0$, the direct transmission path d which is in the general signal flow graph of Fig. 1 is not needed.

The matrix $\underline{W}$ is given by Eq. (5)

$$\underline{W} = \begin{pmatrix} s+1-a_{11} & -a_{12} \\ -a_{21} & s+1-a_{22} \end{pmatrix}^{-1} = \begin{pmatrix} s-f_{11} & -a_{12} \\ -a_{21} & s-f_{22} \end{pmatrix}^{-1} \tag{10}$$

(cont'd.)

$$= \frac{1}{\Delta(s)} \begin{pmatrix} s - f_{22} & a_{12} \\ a_{21} & s - f_{11} \end{pmatrix} \tag{10}$$

where

$$\Delta(s) = s^2 - (f_{11} + f_{22}) s + f_{11} f_{22} - a_{12} a_{21} \tag{11}$$

and is to be identified with the denominator polynomial of w in (8). From Eq. (4), we can express the transfer function as

$$w(s) = \underline{c}^T \underline{W} \, \underline{b} = \frac{1}{\Delta(s)} (c_1 \; c_2) \begin{pmatrix} s - f_{22} & a_{12} \\ a_{21} & s - f_{11} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{12}$$

Thus the numerator polynomial of w is given by

$$N(s) = (c_1 \; c_2) \begin{pmatrix} s - f_{22} & a_{12} \\ a_{21} & s - f_{11} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$= s(b_1 c_1 + b_2 c_2) - b_1 c_1 f_{22} - b_2 c_2 f_{11} + b_1 c_2 a_{21} + b_2 c_1 a_{12} \tag{13}$$

Identifying $\Delta(s)$ of (11) and $N(s)$ of (13) with the given transfer function $w(s)$ of (8), we obtain

$$f_{11} + f_{22} = -2\xi \tag{14a}$$

$$f_{11} f_{22} - a_{12} a_{21} = \omega_0^2 \tag{14b}$$

$$b_1 c_1 + b_2 c_2 = 2k \tag{14c}$$

$$b_1 c_1 f_{22} + b_2 c_2 f_{11} - b_1 c_2 a_{21} - b_2 c_1 a_{12} = 0 \tag{14d}$$

Any solution of the above four equations represents a realization of the given transfer function w. One set of solutions is given below

$$f_{11} = f_{22} = -\xi \tag{15a}$$

$$a_{21} = k(\pm\omega_0 - \xi), \quad a_{12} = \frac{1}{k}(\mp\omega_0 - \xi) \tag{15b}$$

$$b_1 = c_2 = 1 \tag{15c}$$

$$b_2 = c_1 = k \tag{15d}$$

It will be seen in the following section that these elements as given by Eq. (15) also satisfy the minimum sensitivity condition and therefore represents an optimum synthesis.

A lumped electronic circuit realization of the system is shown in Fig. 3b with $k=1$, $\xi = .2$ and a normalized $\omega_0 = 1$ which corresponds to an actual frequency of 231 kHz. For simplicity, the two local feedback circuits as given by $x_1 = x_2 = \frac{1}{s+1}$ and $a_{11} = a_{22} = 1 - \xi = .8$ are realized by the transfer function $\frac{1}{s+.2}$ without feedback as shown in Fig. 3a. The required time constant is realized by an external RC combination with R = 5kΩ and C = 690pF as shown in Fig. 3b. The measured and the designed magnitude curves are shown in Fig. 4. The discrepancy between the two curves can be eliminated by adjusting the feedback resistance R_f.

4. OPTIMUM SENSITIVITY MEASURE

In this section we first derive some constraints of the sensitivity functions and then use the sensitivity index introduced in the companion paper[6] to obtain the conditions for minimum sensitivity index.

The conventional scalar sensitivity of the transfer function w with respect to the parameter x_k is

$$S_{x_k}^{w} = \frac{x_k}{w} \frac{\partial w}{\partial x_k} \tag{16}$$

In terms of the elements of the vector signal flow graph, we obtain from Eqs. (4) and (5)

$$S_{x_k}^{w} = \frac{x_k}{w} \underline{c}^T \frac{\partial \underline{W}}{\partial x_k} \underline{b}$$

$$= \frac{x_k}{w} \underline{c}^T \underline{W}\,\underline{X}^{-1} \underline{\Lambda}_k \underline{X}^{-1} \underline{W}\,\underline{b}$$

$$= \frac{1}{x_k w} \sum_i \sum_j c_i w_{ik} w_{kj} b_j \tag{17}$$

where $\underline{\Lambda}_k$ is a diagonal matrix with unity in the kth term and zeros elsewhere, w_{ik} is the ikth element of $\underline{W}$, c_i and b_j are respectively the i th and the j th elements of $\underline{c}$ and $\underline{b}$. We can express the sensitivity function $S_{x_k}^{w}$ more conveniently by introducing the matrix

$$\underline{D} = \frac{1}{w}\, \underline{W}\, \underline{b}\, \underline{c}^T\, \underline{W} \tag{18}$$

The $\underline{k}$th diagonal element of $\underline{D}$ is

$$d_{kk} = \frac{1}{w} \sum_i \sum_j c_i w_{ik} w_{kj} b_j \tag{19}$$

Thus Eq. (17) becomes

$$S^w_{x_k} = \frac{d_{kk}}{x_k} \tag{20}$$

and can be calculated from the matrix $\underline{D}$. The trace of $\underline{D}$ can be written, from (18) and (20),

$$\operatorname{tr} \underline{D} = \sum_{k=1}^{n} d_{kk} = \frac{1}{w}\, \underline{c}^T \underline{W}^2 \underline{b} = \sum_{k=1}^{n} x_k S^w_{x_k} \tag{21}$$

It is interesting to note that Eq. (21) constitutes a basic invariant on the sensitivity functions. Consider the transformation of the quadruple $\{\underline{A}, \underline{b}, \underline{c}, d\}$ which corresponds to a realization of the given w by an arbitrary nonsingular matrix $\underline{T}$. We have shown that the resulting realization $\{\tilde{\underline{A}}, \tilde{\underline{b}}, \tilde{\underline{c}}, \tilde{d}\}$ yields the same transfer function w. The matrix $\tilde{\underline{D}}$ after the transformation becomes

$$\tilde{\underline{D}} = \frac{1}{w}\, \tilde{\underline{W}}\, \tilde{\underline{b}}\, \tilde{\underline{c}}^T\, \tilde{\underline{W}} = \underline{T}^{-1}\, \underline{D}\, \underline{T} \tag{22}$$

Since the trace is invariant with respect to a similarity transformation, we obtain

$$\mathrm{tr}\ \underline{\tilde{D}} = \mathrm{tr}\ \underline{D} = \sum_{k=1}^{n} x_k S_{x_k}^{w} \tag{23}$$

In our configuration, x_k is assumed to be the same for all k, we therefore obtain the following constraint:

$$\sum_{k=1}^{n} S_{x_k}^{w} \quad \text{is invariant} \tag{24}$$

Eq. (26) represents a basic invariant for equivalent realizations of the transfer function w.

We can now pursue to the problem of optimum synthesis. We will use the sensitivity index introduced in the comapanion paper for optimization. Briefly, given the partial fraction expansion of the sensitivity function

$$S_{x_k}^{w} = \sum_{i=1}^{m} \frac{h_{ik}}{s - s_i} + h_{m+1,k} \tag{25}$$

where the s_i's are the distinct poles and zeros of the transfer function w, the sensitivity index is

$$\mathscr{L} = \sum_{k=1}^{n} \sigma_k^2 \left[\sum_{i=1}^{m+1} \alpha_i (\mathrm{Re}\, h_{ik})^2 + \beta_i (\mathrm{Im}\ h_{ik})^2 \right] \tag{26}$$

where the fractional perturbation parameters $\epsilon_k \triangleq \frac{\delta x_k}{x_k}$ $(k=1, 2, \cdots, n)$ are assumed to be mutually independent random variables having standard deviations σ_k and α_i and β_i are weighting factors. Eqs. (24) and (25) imply that at each pole s_i and at $s = \infty$,

$$\sum_{k=1}^{n} h_{ik} = H_i, \quad i = 1, 2, \ldots, m+1 \tag{27}$$

where H_i's are constants. The optimization problem is then to minimize $\mathcal{S}$ in (26) subject to the $m+1$ constraints of (27). It is easily shown by means of the Lagrange multiplier's rule that the optimum solution for h_{ik} is given by:

$$h_{ik} = \frac{1}{\sigma_k^2} \frac{H_i}{\sum_{k=1}^{n} 1/\sigma_k^2} \quad \text{for} \quad i = 1, 2, \ldots, m+1 \tag{28}$$

The optimum sensitivity index becomes

$$\mathcal{S}\big|_{min} = \frac{1}{\sum_{k=1}^{n} 1/\sigma_k^2} \sum_{i=1}^{m+1} \left[\alpha_i (\mathrm{Re}\, H_i)^2 + \beta_i (\mathrm{Im}\, H_i)^2\right] \tag{29}$$

Condition (28) can be stated alternately in terms of the n sensitivity functions. Combining (28) and (25), we obtain the following result for optimum synthesis:

$$\sigma_1^2 S_{x_1}^w = \sigma_2^2 S_{x_2}^w = \dots = \sigma_n^2 S_{x_m}^w \tag{30}$$

If, as in our problem, all σ_k's are the same, then the optimum synthesis amounts to the synthesis which yields identical sensitivity functions for the n parameters x_k.

In the second-order example of the previous section, it is easily shown that to set $S_{x_1}^w = S_{x_2}^w$, we need

$$f_{11} = f_{22}$$

and

$$b_1 c_1 = b_2 c_2$$

The particular synthesis given satisfies the above equations, hence it represents the optimum synthesis.

5. OPTIMUM SYNTHESIS OF HIGHER-ORDER SYSTEMS

The optimum synthesis of high-order system cannot be done in general as in the second-order system by equating coefficients. Instead, we must depend on the method of finding an arbitrary initial design first and optimize the sensitivity index $\mathcal{S}$ with respect to the elements of the transformation matrix. The initial design can be, for example, a tandem or a parallel connection of second-order systems. The sensitivity index is calculated for the initial design and is to be minimized by employing any of the standard techniques for steepest descent along the gradient of $\mathcal{S}$ in the n^2-dimensional Euclidean space spanned by the elements of the transformation matrix $\underline{T}$. It is a straightforward task to program a computer to calculate the gradient as well as the Hessian matrix (in case zero gradient is encountered). A point where the gradient is zero is the location of a minimum, a maximum or a point of inflexion of $\mathcal{S}$. In the first case, comparison of the value of the sensitivity index $\mathcal{S}$ at the point with $\mathcal{S}|_{min}$ as given by Eq. (29) will provide the information whether the minimum is global or only local. In the other two cases, to continue steepest descent, we have to pick the eigenvector corresponding to the lowest eigenvalue of the Hessian matrix to obtain the transformation matrix. The details of the program will be omitted.

The following example indicates the optimum synthesis of a fourth-order staggered-tuned bandpass filter. The specified transfer function has two pairs of complex conjugate poles at $s = -1 \pm j \cdot 9$ and

$s = -1 \pm j1.1$. There are no finite zeros. The initial realization is obtained by cascading two optimal second-order systems, each of which realizes one of the two complex pole pairs. The realization is given by

$$\underline{A} = \begin{bmatrix} -.1 & -.81 & 0 & 0 \\ 1 & -.1 & 0 & 0 \\ 0 & 1.0 & -.1 & -1.21 \\ 0 & 0 & 1 & -.1 \end{bmatrix}$$

$$\underline{b} = (1, 0, 0, 0)^T$$

$$\underline{c} = (0, 0, 0, 1)^T$$

The flow graph representation is shown in Fig. 5. The sensitivity index with weighting factors

$$\alpha_i = -\frac{1}{\text{Re } s_i} \quad \text{and} \quad \beta_i = 0$$

and a normalized σ_k is chosen such that $\mathcal{L}_{min} = 1$ in Eq. (29). For the initial design, the sensitivity index has a value of 2. After five iterations in the minimization process, a sensitivity index of 1.001 has been achieved and the resulting realization is given by

$$\tilde{\underline{A}} = \begin{bmatrix} -.1 & -1.004 & 0 & .04 \\ 1.0 & -.1 & 0 & 0 \\ 0 & 1.0 & -.1 & -1.016 \\ 0 & 0 & 1 & -.1 \end{bmatrix}$$

$$\tilde{\underline{b}} = \underline{b} \quad \text{and} \quad \tilde{\underline{c}} = \underline{c}$$

The signal flow graph realization is shown in Fig. 6. The result indicates that for minimum sensitivity to random perturbations in the nominally identical stages, a 4th-order bandpass amplifier has to consist of two identical optimal 2nd-order systems in tandem with an overall feedback loop.

6. CONCLUSIONS

In this paper we have presented a special class of multiple loop feedback configuration which can realize arbitrary transfer functions. An optimum synthesis is achieved in the sense that a multiparameter sensitivity index is minimized. The basic result for minimum sensitivity synthesis of our proposed configuration is that the conventional sensitivity functions with respect to all parameters are the same. For transfer functions of order two, the optimum synthesis can be obtained analytically. For higher order transfer functions, a computer program has been written to obtain the optimum synthesis by the gradient technique. We believe that extensions of the proposed multiple loop configuration to more general ones would be of considerable interest.

We are grateful to Y. F. Zai who designed and measured the circuit in Fig. 3b.

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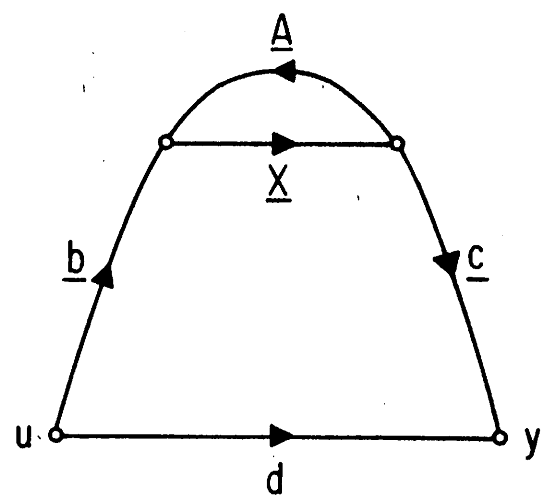

Fig. 1. A general multiple loop feedback configuration which is represented by a vector signal flow graph. u is the scalar input, y is the scalar output, and $\underline{X}$ is an $n \times n$ diagonal matrix whose individual elements represent n active stages.

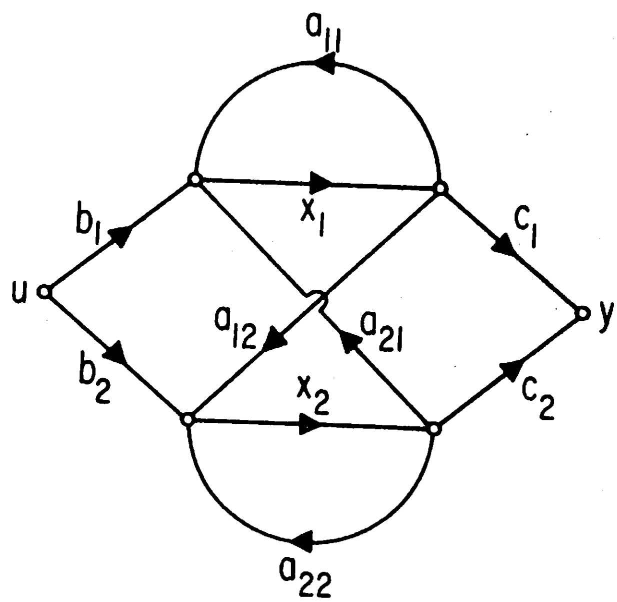

Fig. 2. A scalar signal flow graph which realizes a second-order transfer function.

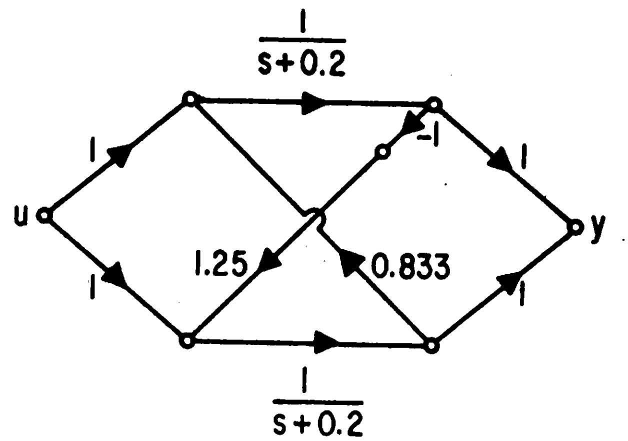

Fig. 3a. The signal flow graph representation of the circuit in Fig. 3b.

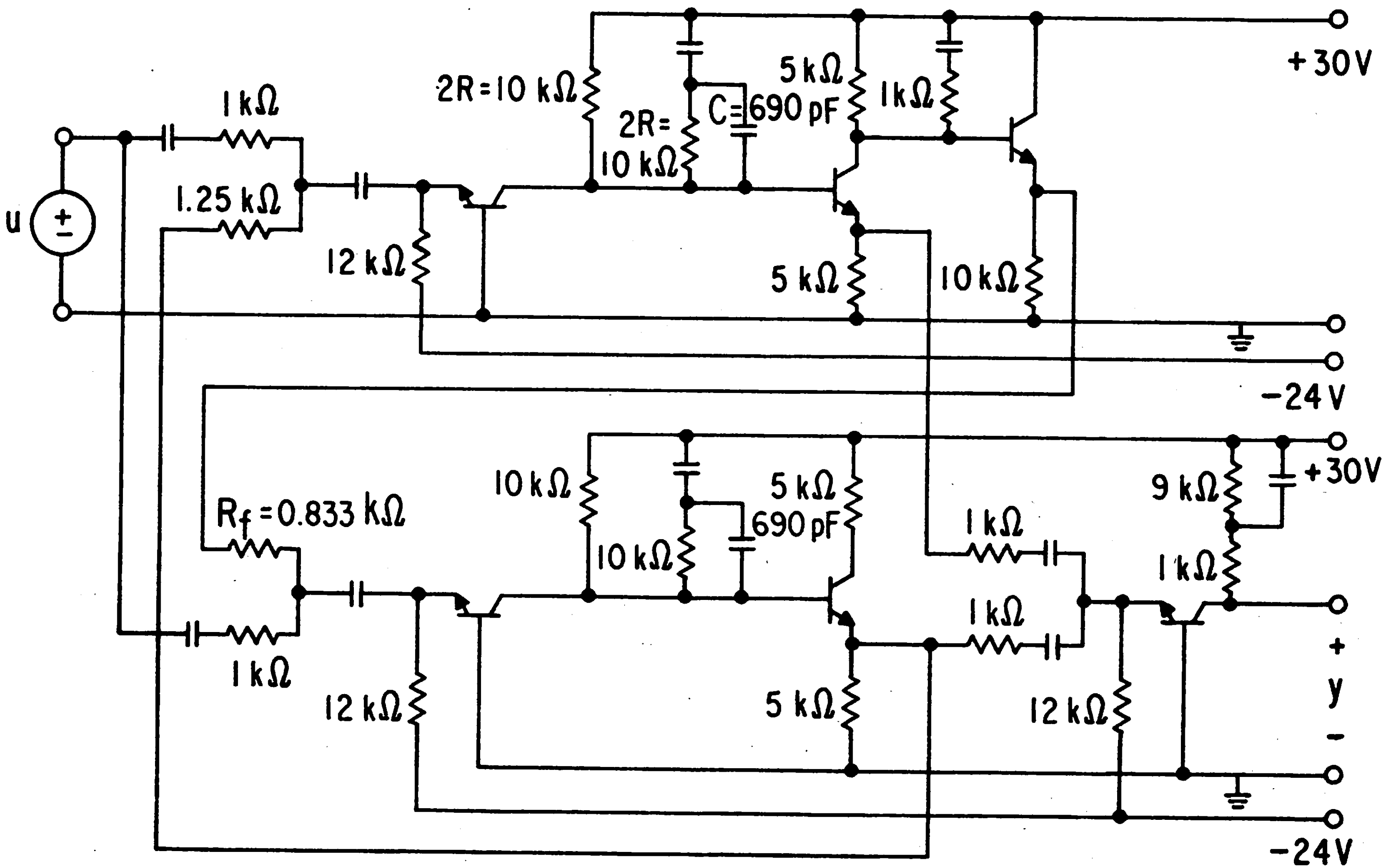

Fig. 3b. A transistor circuit realization of $w(s) = \frac{y}{u} = \frac{25}{s^2 + .4s + 1}$

Actual center frequency is 231 kHz.

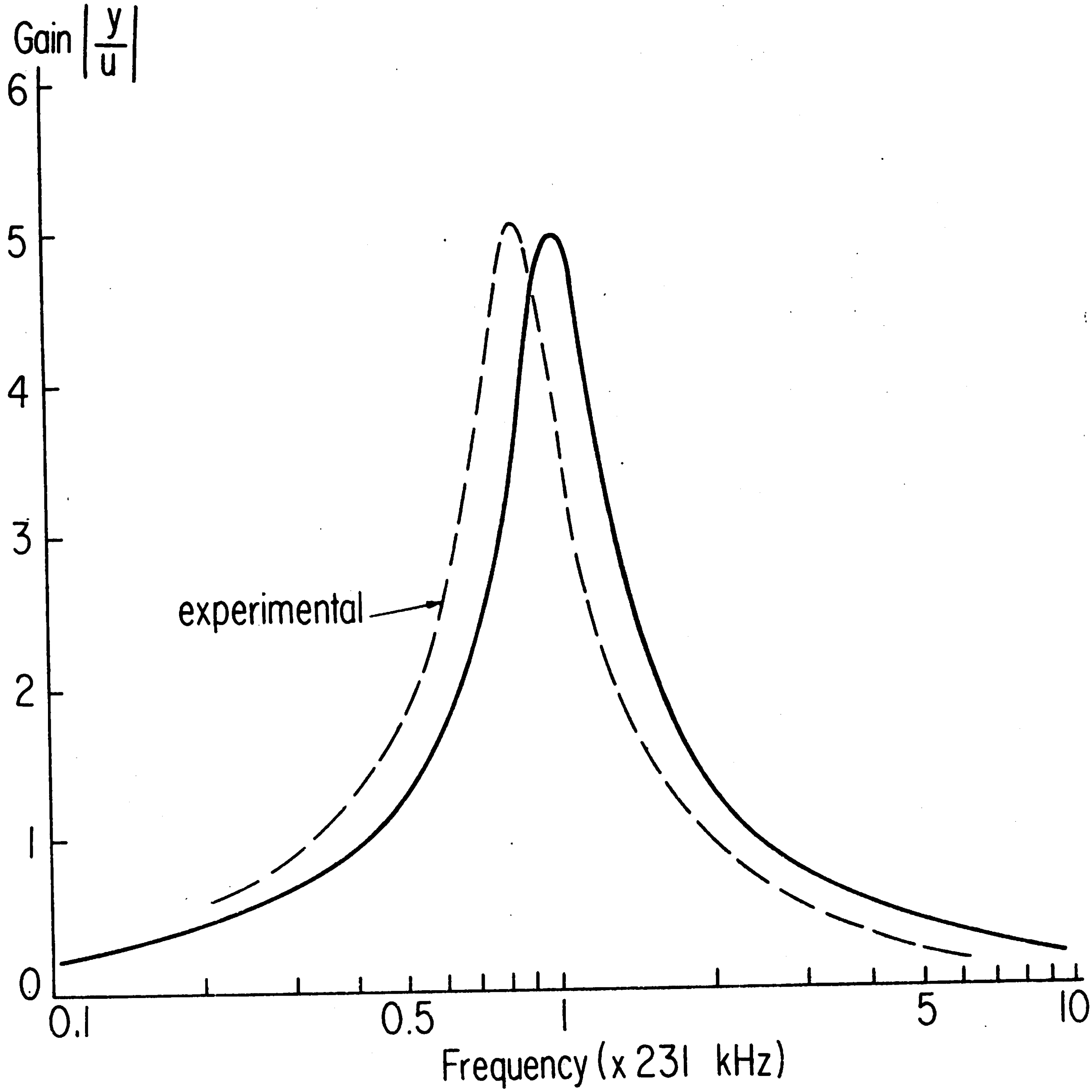

Fig. 4. **Measured and designed frequence responses for the circuit in Fig. 3 b.**

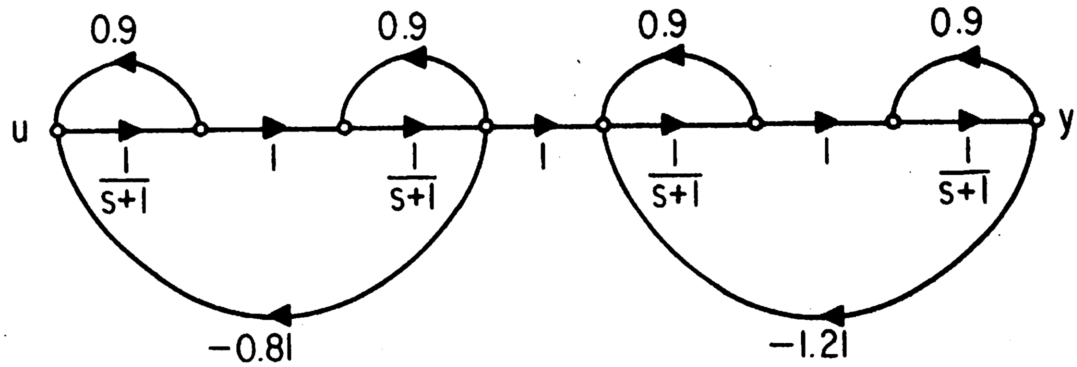

Fig. 5. Initial design of a 4th order system. $w(s) = \dfrac{K}{(s^2 + 2s + 1.81)(s^2 + 2s + 2.21)}$

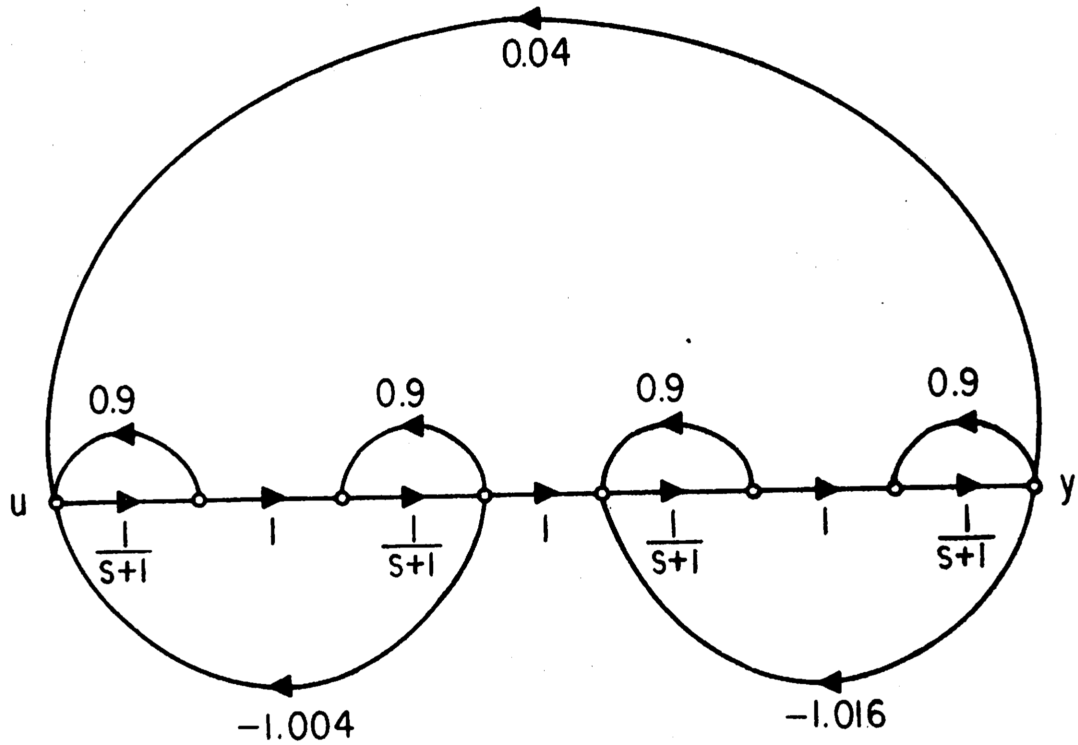

Fig. 6. Optimum system with the same transfer function as in Fig. 5.