SYNTHESIS OF IMPULSE RESPONSE MATRICES BY INTERNALLY STABLE AND PASSIVE REALIZATIONS

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SYNTHESIS OF IMPULSE RESPONSE MATRICES BY INTERNALLY STABLE AND PASSIVE REALIZATIONS*

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Abstract

A class of realizations (termed uniform realizations) for time-variable impulse response matrices is defined which plays a role similar to that of minimal (completely controllable and observable) realizations for time-invariant systems; members of the class have bounded coefficients and are uniformly asymptotically stable if the impulse response matrix represents a bounded-input bounded-output (BIBO) stable system. The necessary and sufficient conditions for an impulse response matrix to be uniformly realizable are derived together with an explicit realization procedure. Conditions for a system to be realizable as a passive network are also obtained and it is shown that any BIBO stable, uniformly realizable impulse response matrix may be synthesized as the transfer response of a passive network composed of constant positive inductors and resistors and bounded time-variable gyrators.

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I. INTRODUCTION

A basic synthesis problem for both time-invariant and time-variable linear systems is that of constructing a system of differential equations having a prescribed input-output response. One formulation of this problem which has received a great deal of attention in recent years [1-5] can be stated as follows:

Given an $m x r$ matrix functions of two variables $H(t, \tau)$, find a set of stable equations in the form

$$\dot{x}(t) = A(t)x + B(t)\ (t) \tag{1a}$$

$$y(t) = C(t)x(t) \tag{1b}$$

where A, B, and C are, respectively, nxn, nxr and mxn matrices for some finite n, for which $H(t, \tau)$ is the impulse response matrix; that is,

$$H(t, \tau) = C(t)\,\Phi(t, \tau)\,B(\tau), \quad t \geq \tau, \tag{2}$$

where $\Phi(t, \tau)$ is the transition matrix of the homogeneous part of (1a). If (2) is satisfied, (1) is termed a realization of H.

The above problem is of importance for several reasons. Primarily, the state equations (1) yielded an immediate physical realization of H in the form of an analog computer. Furthermore, as has recently been shown for time-invariant systems [6-8] and as will be demonstrated here for time-variable systems they provide a useful starting point for network synthesis.

In the time-variable case, most previous research has centered on the more formal aspects of the realizability problem, such as questions of existence and minimality of a system of equations in the form (1) for a given matrix H, and providing a description of the class of all possible realizations [1-5]. It is clear, however, that not all realizations are equally useful if practical implementation of the system is desired. Unfortunately, the usual procedures in general yield realizations having unbounded coefficient matrices and internal stability properties which in no way reflect the external stability properties characterized by H (e.g., bounded-input bounded-output (BIBO) stability [9]). Consequently, these realizations are useless even for analog computer simulation. In the time-invariant case, the problem does not arise, since any time-invariant realization of the form (1) certainly has bounded coefficients, and if the realization is minimal (completely controllable and observable) one is assured that it is uniformly asymptotically stable [9] (exponentially stable [10]), if the impulse response matrix represents a BIBO stable system. For this reason, minimal realizations play a crucial role in the analysis and synthesis of time-invariant systems.

The difficulty in the time-variable case, is that minimality of a realization alone does not imply anything about its boundedness or stability properties. In this paper, a class of realizations which appears to be more natural for synthesis is introduced. This class (termed the class of uniform realizations), which is equivalent to the class of minimal realizations in the time-invariant and periodic cases, is examined in detail and a set of necessary and sufficient conditions for an impulse response matrix to possess a realization in the class are derived, together with a general realization procedure. A simpler alternative set of conditions and synthesis procedure are also given for the special case of periodic systems.

The problem of network synthesis is also examined. A sufficient condition for an impulse response matrix to be the "impedance" of a passive network is given and based on this result it is shown that any BIBO stable uniformly realizable system can be synthesized as the "transfer response" of a passive network composed of constant inductors and resistors and bounded time-variable gyrators.

II. PRELIMINARY DEFINITIONS

For convenience, the system representation (1) will be denoted by the triple (A, B, C). Initially, we will be concerned with properties of equivalent representations of a given realization (A, B, C). The following definition of equivalence is standard [1].

Definition 1: The representation $(\bar{A}, \bar{B}, \bar{C})$ is algebraically equivalent to (A, B, C) if there exists a nonsingular matrix T with continuous derivative $\dot{T}$ such that

$$\bar{A} = (TA + \dot{T})T^{-1}, \quad \bar{B} = TB, \quad \bar{C} = CT^{-1}. \tag{3a, b, c}$$

The above type of equivalence will be denoted symbolically as

$$(A, B, C) \xrightarrow{T} (\bar{A}, \bar{B}, \bar{C}).$$

It is easily shown that if (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ are algebraically equivalent, then they are realizations of the same impulse response matrix, and it is clear that continuity of (A, B, C) implies that of $(\bar{A}, \bar{B}, \bar{C})$, and conversely. However, this type of equivalence does not preserve internal stability (e. g. Lyapunov stability or exponential stability) or boundedness of the coefficient matrices so that the following type of equivalence [1] will be of more importance here.

Definition 2: The representation $(\bar{A}, \bar{B}, \bar{C})$ is topologically equivalent to (A, B, C) if $(A, B, C) \xrightarrow{T} (\bar{A}, \bar{B}, \bar{C})$ and T is a Lyapunov transformation [9] (i.e., T, T^{-1} and $\dot{T}$ are continuous and bounded on $(-\infty, \infty)$).

Definition 3: The system representation (A, B, C) is said to be bounded if there exists a constant K such that

$$||A(t)|| \leq K, \quad ||B(t)|| \leq K, \quad ||C(T)|| \leq K \tag{4a, b, c}$$

where $||\cdot||$ denotes the Euclidian norm.

It is obvious from (3), that if T is a Lyapunov transformation, then boundedness of (A, B, C) implies that of $(\bar{A}, \bar{B}, \bar{C})$, and conversely, and as is well known [9], internal stability is invariant under such a transformation.

The fundamental constraints we will impose on system realizations, in addition to boundedness, are uniform complete controllability and observability, concepts introduced by Kalman [11]. For bounded realizations they may be defined as follows [11, 12]:

Definition 4: A bounded system realization (A, B, C) is said to be uniformly completely controllable if there exists $\delta > 0$ such that for all t^1,

$$M(t-\delta, t) \geq \alpha_1(\delta)\, I > 0, \tag{5}$$

where,

$$M(t-\delta, t) = \int_{t-\delta}^{t} \Phi(t, \tau) B(\tau) B'(\tau) \Phi'(t, \tau)\, d\tau. \tag{6}$$

Definition 5: A bounded system realization (A, B, C) is said to be uniformly completely observable if there exists $\delta > 0$ such that for all t,

$$N(t, t+\delta) \geq \alpha_2(\delta)\, I > 0 \tag{7}$$

where,

$$N(t, t+\delta) = \int_{t}^{t+\delta} \Phi'(\tau, t) C'(\tau) C(\tau) \Phi(\tau, t)\, d\tau \tag{8}$$

The above definitions are equivalent (for bounded realizations) to Kalman's original definitions [11], since if (4a) is satisfied, then [11]

$$||\phi(t, \tau)|| \leq \alpha_3(|t-\tau|) \tag{9}$$

and (4a)-(4c) imply [12]

$$M(t-\delta, t) \leq \alpha_4(\delta) I \tag{10}$$

and

$$N(t, t+\delta) \leq \alpha_5(\delta) I \tag{11}$$

The matrices $M(t-\delta, t)$ and $N(t, t+\delta)$ will be referred to as the controllability and observability matrices of (A, B, C) and, when the context is clear, their arguments will be suppressed.

III. UNIFORM REALIZATIONS

The class of realizations proposed for synthesis is delineated by

Definition 6: (a) A system representation (A, B, C) is said to be uniform if it is continuous[2], bounded, and uniformly completely controllable and observable. (b) If an impulse response matrix H can be realized by a uniform system representation, it is said to be uniformly realizable.

The class of uniform realizations of an impulse response matrix (when such exist) appears to be the most general for which time-variable synthesis can be put on a systematic basis. In the time-invariant and periodic cases it is equivalent to the class of minimal realizations [12] and as indicated by the following theorem, proven in [12], members of this class are exponentially stable if the prescribed impulse response matrix represents a BIBO stable system.

Theorem 1: If (A, B, C) is a uniform realization of an impulse response matrix satisfying the BIBO stability constraint

$$\int_{-\infty}^{t} ||H(t,\tau)||\,d\tau \leq K_1 < \infty \qquad \text{for all } t ,$$

then there exist positive constants K_2 and K_3 such that

$$||\Phi(t,\tau)|| \leq K_2 e^{-K_3(t-\tau)} \qquad \text{for } t \geq \tau \quad .$$

Boundedness and stability may not be the only properties required of a system one is trying to synthesize. It is quite possible that other constraints, such as passivity, may be imposed. In the time-invariant case, one procedure for handling additional constraints [6-8] is to start with an arbitrary minimal realization and then generate (when possible) equivalent realizations which possess the desired properties via constant coordinate transformations. One is assured that all minimal realizations are considered by this procedure since the class of minimal time-invariant realizations of an impulse response matrix is closed under constant transformations. [1, 3]

A similar closure property is derived below for the class of uniform realizations.

Theorem 2: (i) If (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ are uniform realizations of the same impulse response matrix, then they are topologically equivalent

(ii) If (A, B, C) is a uniform realization of an impulse response matrix H and $(\bar{A}, \bar{B}, \bar{C})$ is topologically equivalent to (A, B, C) then $(\bar{A}, \bar{B}, \bar{C})$ is a uniform realization of H.

Proof: (i) It is clear that a uniform realization of an impulse response matrix is also globally reduced in the sense of Youla [3], so that any two such realizations are algebraically equivalent. Let T represent the transformation between the two realizations, let P be any matrix such that $M = PP'$, and let $\bar{P} = TP$. If $\bar{M}$ is the controllability matrix of $(\bar{A}, \bar{B}, \bar{C})$ it is straightforward to show that,

$$\bar{M} = TMT' \tag{12}$$

Consequently, $\bar{M} = \bar{P}\bar{P}'$. By our assumption of uniformity, the matrices M, $\bar{M}$, M^{-1} and $\bar{M}^{-1}$ are bounded which in turn implies $P, \bar{P}, P^{-1}$ and $\bar{P}^{-1}$ are bounded. But since $\bar{P} = TP$, it follows that T and T^{-1} are bounded. Continuity and boundedness of $\dot{T}$ follows from the relationship

$$\dot{T} = -TA + \bar{A}T ,$$

and the continuity and boundedness of A and $\bar{A}$. Thus, T is a Lyapunov transformation.

(ii) The second part of the theorem follows directly the relationships (3) and (12), and the dual of (12)

$$\bar{N} = (T^{-1})' N T^{-1} . \tag{13}$$

The following lemma and its corollary established in the appendix prove to be basic in constructing uniform and passive realizations of an impulse response matrix.

Lemma 1: If a symmetric matrix V is continuously differentiable and positive definite on $(-\infty,\infty)$, and if the maximum eigenvalue of $V^{-1}\dot{V}$ is bounded on $(-\infty,\infty)$ then the matrix $U^{-1}\dot{U}$ is continuous and bounded on $(-\infty,\infty)$, where $U = V^{1/2}$, the unique positive definite square root of V.

Corollary 1: If V is a symmetric positive definite Lyapunov transformation then $U = V^{1/2}$ is also a Lyapunov transformation.

IV. CONDITIONS FOR UNIFORM REALIZABILITY

It is well known [1] that a matrix $H(t,\tau)$ is realizable as a system of the form (1) if and only if it is separable in the form

$$H(t,\tau) = \psi(t)\,\Theta(\tau) \quad , \qquad t \geq \tau \quad . \tag{14}$$

Corresponding to any such separation, is the realization $(0,\Theta,\psi)$ for which we can define the controllability and observability matrices

$$M(t-\delta,t) = \int_{t-\delta}^{t} \Theta(\tau)\,\Theta'(\tau)\,d\tau \tag{15}$$

$$N(t,t+\delta) = \int_{t}^{t+\delta} \psi'(\tau)\psi(\tau)\,d\tau \quad . \tag{16}$$

It is also useful to define the product of these two matrices

$$W(t,\delta) = N(t,t+\delta)\,M(t-\delta,t) \quad . \tag{18}$$

It is apparent that an elementary realization of the form $(0,\Theta,\psi)$ will rarely be uniform even when H is time-invariant. Of course, this does not preclude the existence of algebraically equivalent realizations which are uniform. It can also be shown [13] that not all realizable impulse response matrices possess uniform realizations. Hence, it is of interest to determine precise conditions under which an impulse response matrix is uniformly realizable. In [13] a sufficient condition for uniform realizability of single-input single output systems was obtained. A complete solution to the realizability problem is given by

<u>Theorem 3</u>: $H(t, \tau)$ is a uniformly realizable impulse response matrix if and only if it is separable in the form (14), where ψ and Θ are continuous matrices of finite order, and there exists $\delta > 0$ such that for all t [3]

(i) $\sigma_m \{W(t, \delta)\} \geq \beta_1(\delta) > 0$

(ii) $\sigma_M \{\Theta'(t) M^{-1}(t-\delta, t) \Theta(t)\} \leq \beta_2(\delta) < \infty$

(iii) $\sigma_M \{\psi(t) M(t-\delta, t)\psi'(t)\} \leq \beta_3(\delta) < \infty$

(iv) $\sigma_M \{M^{-1}(t-\delta, t) \partial / \partial t \, M(t-\delta, t)\} \leq \beta_4(\delta) < \infty$

<u>Proof (Sufficiency)</u>: Since ψ is continuous, N is finite on $(-\infty, \infty)$ so that (i) implies $M > 0$ on $(-\infty, \infty)$ (but not necessarily <u>uniformly</u> positive definite). Hence, we may define the unique positive definite square root of M, $P = M^{1/2}$. Let

$$\overline{A} = -P^{-1}\dot{P}, \quad \overline{B} = P^{-1}\Theta, \quad \overline{C} = \psi P.$$

It is now claimed that $(\overline{A}, \overline{B}, \overline{C})$ is a uniform realization of H.

To prove this assertion, it is first noted that since M has a continuous derivative (this is clear from (15)) then P must also have a continuous derivative so that $(\overline{A}, \overline{B}, \overline{C})$ is a continuous realization of H. Since $\overline{B}\,\overline{B}'$ is symmetric, (ii) implies $\overline{B}\,\overline{B}'$ and, therefore, $\overline{B}$ is bounded. Similarly, (iii) implies $\overline{C}$ is bounded. Furthermore, by Lemma 1, (iv) implies A is bounded.

It remains to show that $(\overline{A}, \overline{B}, \overline{C})$ is uniformly completely controllable and observable. But, if $\overline{M}$ is the controllability matrix of $(\overline{A}, \overline{B}, \overline{C})$ it follows from (12) that $\bar{M} = I$ so that the realization is certainly uniformly completely controllable. Also, note that from (13) and (17), the observability matrix $\bar{N}$ of $(\bar{A}, \bar{B}, \bar{C})$ is given by $\bar{N} = PWP^{-1}$. Since $\bar{N}$ is symmetric and has the same eigenvalues as W, (i) implies the realization is uniformly completely observable. This completes the proof of sufficiency.

(Necessity): Let (A, B, C) be a uniform realization of H, and let T be a fundamental matrix for $\dot{x} = Ax$, then $(0, \bar{\Theta}, \bar{\psi})$ is an algebraically equivalent representation of (A, B, C), where $\bar{\Theta} = T^{-1}B$ and $\bar{\psi} = CT$. Since (A, B, C) is uniform, there must exist a $\delta > 0$ such that the inequalities (4), (5), (7), (9), (11) and (12) are satisfied. It will be shown below that this implies conditions (i)-(iv) are satisfied for the seperation $H(t,\tau) = \bar{\psi}(t)\bar{\Theta}(\tau)$.

(i) If $\bar{M}$ and $\bar{N}$ are the controllability and observability matrices of $(0, \bar{\Theta}, \bar{\psi})$, then it follows by (12) and (13) that $\bar{W} = \bar{N}\bar{M} = T'NM(T')^{-1}$. Let $P = M^{1/2}$ and note that the eigenvalues of $(NP^{1/2})P^{1/2}$ are equal to those of $P^{1/2}NP^{1/2}$. It is clear by (5) and (7), however, that the eignvalues of the latter matrix, and consequently those of $\bar{W}$, are uniformly bounded away from zero.

(ii), (iii): To establish (ii) and (iii), it suffices to observe that $\bar{\Theta}'\bar{M}^{-1}\bar{\Theta} = B'M^{-1}B$ and $\bar{\psi}\bar{M}\bar{\psi}' = CMC'$.

(iv) Consider first the relationship

$$\partial/\partial t \; M(t-\delta, t) = B(t)B'(t) - \Phi(t, t-\delta)B(t-\delta)B'(t-\delta)\Phi'(t, t-\delta) + A(t)M(t-\delta, t) + M(t-\delta, t)A'(t). \tag{19}$$

It follows the above and equations (4a, b), (9) and (10) that if (A, B, C) is uniform then $\partial/\partial t \; M(t-\delta, t) = \dot{M}$ is bounded. Using (12), it can also be shown that $\dot{\bar{M}} = T^{-1}\left[-AM-MA'+\dot{M}\right](T')^{-1}$ so that

$$\bar{M}^{-1}\dot{\bar{M}} = T'\left[-M^{-1}AM-A'+M^{-1}\dot{M}\right](T')^{-1}$$

From (4a), (5), (10) and (19), however, the bracketed quantity in the above equation is bounded. This in turn implies that the eigenvalues of $\bar{M}^{-1}\bar{M}$ are bounded, which completes the proof.

The general synthesis procedure for obtaining uniform realizations is apparent from the sufficiency proof of the above theorem, but several clarifying comments can be made.

It should first be noted that it is not necessary to consider all possible separations of the form (14). In fact, it suffices to consider any globally reduced decomposition [3] ; that is, one in which the rows of Θ and the columns of ψ are linearly independent on the real line (such a decomposition can be obtained in a straightforward manner from an arbitrary one [3]). The reason for this is that if one globally reduced decomposition satisfies the criteria (i)-(iv) in Theorem 3, then all such decompositions must satisfy the criteria. This follows from the fact [3] that if

$$H(t,\tau) = \psi(t)\Theta(\tau) = \bar{\psi}(t)\bar{\Theta}(\tau)$$

and both decompositions are globally reduced, then there exists a constant nonsingular matrix K such that $\bar{\psi}(t) = \psi(t)K$ and $\Theta(t) = K^{-1}\Theta(t)$.

One drawback of the above synthesis procedure should also be pointed out; when the impulse response matrix possess periodic or time-invariant realizations, the realizations obtained in the in the prrof of Theorem 3 will not in general be periodic or time-invariant for the corresponding case. However, an alternate procedure is available in these cases. This procedure, which was first outlined in [14] , is given in the following section.

V. PERIODIC AND TIME-INVARIANT SYSTEMS

It was shown in [12], that minimal periodic realizations (A, B, C periodic with the same period) are necessarily uniform, so that the synthesis problem in this case reduces to finding criteria under which an impulse response matrix has a periodic realization. As would be expected, these conditions are considerably simpler than those of Theorem 3.

From the known form of the transition matrix of a period system, and equation (2), it is clear that a necessary condition for H to possess a periodic realization is that a constant s exist such that for all $t \geq \tau$

$$H(t+s, \tau+s) = H(t, \tau) \tag{19}$$

It will be established below, by construction of an explicit periodic realization, that this condition together with (14) is also sufficient.[5]

<u>Theorem 4</u>: $H(t, \tau)$ is realizable by a continuous, periodic system of the form (1) If and only if it is separable in the form (14), where ψ and Θ are continuous matrices of finite order, and there exists a constant s such that (19) is satisfied for all $t \geq \tau$.

<u>Proof</u>: (Sufficiency) Without loss of generality it can be assumed that (14) is a global decomposition, so that a finite interval Δ exists such that

$$M_1 = \int_\Delta \Theta(\tau)\Theta'(\tau)\,d\tau \quad \text{and} \quad N_1 = \int_\Delta \psi'(\tau)\psi(\tau)\,d\tau$$

are nonsingular matrices [3]. Let

$$M_2 = \int_\Delta \Theta(\tau-s)\Theta'(\tau)\,d\tau \quad \text{and} \quad N_2 = \int_\Delta \psi'(\tau)\psi(\tau+s)\,d\tau$$

and observe that (14) and (19) together imply

$$\psi(t+s)\,\Theta(\tau) = \psi(t)\,\Theta(\tau-s) \quad . \tag{20}$$

Then, if (20) is post-multiplied by $\Theta'(\tau)$ and integrated with respect to τ over the interval Δ the relation

$$\psi(t+s) = \psi(t)\, T_1 \tag{21}$$

is obtained, where $T_1 = M_2 M_1^{-1}$ is a unique real constant matrix. Similarly,

$$\Theta(t-s) = T_2 \Theta(t), \tag{22}$$

where $T_2 = N_1^{-1} N_2$. Substituting (21) and (22) in (20), it is seen that $\psi(t)\, T_1\, \Theta(\tau) = \psi(t)\, T_2\, \Theta(\tau)$ which in turn implies that $N_1 T_1 M_1 = N_1 T_2 M_1$ or, since N_1 and M_1 are nonsingular, $T_1 = T_2 = T$. Since the rows of $\psi(t)$ are linearly independent on $(-\infty, \infty)$ the same must be true of $\psi(t+s)$ so that (21) implies T is nonsingular. Consequently, T^2 is positive definite so that there exists a unique, real, constant matrix A such that $T^2 = e^{2sA}$. Let $C(t) = \psi(t)e^{-At}$ and $B(t) = e^{At}\Theta(t)$, then it follows easily from (21) and (22) that $C(t) = C(t+2s)$ and $B(t) = B(t+2s)$. Hence, $(A, B(t), C(t))$ is a continuous periodic (with period 2s) realization of H. This completes the proof of the theorem.

The realization obtained in the above proof, in addition to being uniform, is also canonical in the sense that it has a constant A-matrix. As shown in [14] , this implies that the system can be synthesized directly with passive RLC elements and periodic controlled sources. A further advantage of the realization procedure is that when $H(t,\tau)$ represents a time-invariant system ($H(t+s, \tau+s) = H(t,\tau)$ for all s and for all $t \geq \tau$ [1]) the realization obtained will be time-invariant (any choice of s leads to the same realization). Hence, we have an alternate method to that of Youla [3] for realizing time-invariant impulse response matrices. The present method has the computational advantage of not requiring differentiation of ψ and Θ.

VI. PASSIVE REALIZATIONS

If $u = i$, an N-vector is the current entering an N-port network and $y = v$ is the corresponding voltage vector then $H(t, \tau)$ will be termed an impedance matrix.

Definition 7: An N-port network is said to be passive if for zero initial conditions at time t_0

$$\int_{t_0}^{t} v'(\tau)i(\tau)d\tau \geq 0 \ , \ \forall t_0, \forall t \geq t_0 \ . \tag{23}$$

In the time-invariant case, passivity is equivalent to positive reality of the laplace transform of H. Recently [6, 15], algebraic conditions for positive reality (passivity) have been given in terms of the matrices A, B, and C (and the direct transmission matrix in the more general case). It will now be shown that these conditions extend quite simply to the time-variable case.[4]

Lemma 2: H is the impedance matrix of a passive N-port if for any realization (A, B, C) of H there exists a symmetric matrix $K > 0$ and a matrix L such that

$$-\dot{K} + AK + KA' = -LL' \tag{24}$$

and

$$CK = B' \tag{25}$$

Proof: Let $J = K^{1/2}$ and consider the realization defined by $(A, B, C) \xrightarrow{J^{-1}} (\bar{A}, \bar{B}, \bar{C})$. It is straightforward to show that

$$\bar{A} + \bar{A} = -\bar{L}\bar{L}' \tag{26}$$

where $\bar{L} = J^{-1}L$, and

$$\bar{C} = \bar{B}' \tag{27}$$

Let x denote the state of $(\bar{A}, \bar{B}, \bar{C})$, then it may be shown with the aid of (26) and (27) that

$$\frac{d}{dt}(x'x) = -x'(\bar{L}\bar{L}')x + 2v'i$$

so that if $x(t_0) = 0$,

$$2\int_{t_0}^{t} v'(\tau)i(\tau)d\tau = x'(t)x(t) + \int_{t_0}^{t} x'(\tau)\bar{L}(\tau)\bar{L}'(\tau)x(\tau)d\tau$$

This relationship clearly implies (23), which completes the proof.

A procedure for synthesizing a passive network follows directly from the representation $(\bar{A}, \bar{B}, \bar{C})$ obtained in the above proof precisely as in the time-invariant case. For example, we may use the technique of Anderson and Brockett [18] by observing that an alternate representation for $(\bar{A}, \bar{B}, \bar{C})$ is

$$\dot{x} = 1/2(\bar{A}-\bar{A}')x + \bar{B}i - 1/\sqrt{2}\,\bar{L}i^*$$

$$v = \bar{B}'x$$

$$v^* = -1/\sqrt{2}\,\bar{L}'x$$

together with the constraint $v^* = -i^*$. A network realization is then obtained by terminating the time-variable $(N + n + k)$-port gyrator (k = number of columns of L) defined by

$$\begin{array}{cc} \begin{array}{ccc} (N) & (k) & (n) \end{array} & \\ \begin{bmatrix} O & O & -\bar{B}' \\ O & O & 1/\sqrt{2}\,\bar{L}' \\ \bar{B} & -1/\sqrt{2}\,\bar{L} & 1/2(\bar{A}-\bar{A})' \end{bmatrix} & \begin{array}{c} (N) \\ (k) \\ (n) \end{array} \end{array}$$

as follows: the last n ports are terminated in unit inductors, the preceding k ports are terminated in unit resistors.

<u>Corollary 2</u>: If (A, B, C) is a bounded realization of an impulse response

matrix H and there exists a symmetric matrix K with $\alpha_1 I \leq K(t) \leq \alpha_2 I$ for all t and a bounded matrix L such that

$$-\dot{K} + AK + KA' = -LL'$$

and

$$CK = B'$$

then H can be realized as passive network composed of constant inductors and resistors and <u>bounded</u> time-variable gyrators.

The proof of this corollary follows directly from Lemma 2 and the realization given above together with the fact that $J = K^{1/2}$ is a Lyapunov transformation by Corollary 1.

It should be noted that time-variable gyrators are not merely theoretical circuits; practical methods for constructing them have been given (see, for example, [18]). Previous synthesis methods employing time-variable gyrators have not, however, guaranteed circuits with bounded elements and internal stability (see, for example [17] and [19]).

In the following lemma, an extension of passive network synthesis to include transfer responses is given. The technique employed is believed to be novel even in the time-invariant case.

<u>Lemma 3</u>: If H has a bounded exponentially stable realization, then it can be synthesized as the transfer response of a passive network composed of constant inductors and capacitors and bounded time-variable gyrators.

<u>Proof</u>: Let (A, B, C) be a bounded exponentially stable realization of H with transition matrix Φ, then it is easily shown that

$$V(t) = \int_{-\infty}^{t} \Phi(t,\tau)\, \phi'(t,\tau) d\tau$$

is a positive definite Lyapunov transformation. Hence, by Corollary 1, $U = V^{1/2}$ is also a Lyapunov transformation. Consequently, if

$(A, B, C) \xrightarrow{U^{-1}} (\bar{A}, \bar{B}, \bar{C})$, then $(\bar{A}, \bar{B}, \bar{C})$ is bounded and exponentially stable. Furthermore, $\bar{A} + \bar{A}' = -V^{-1}$

Consider now the augmented system

$$\dot{x} = \bar{A}x + [\,\bar{B} \quad \bar{C}'\,] \begin{bmatrix} u \\ u^* \end{bmatrix}$$

$$\begin{vmatrix} y^* \\ y \end{vmatrix} = \begin{vmatrix} \bar{B}' \\ \bar{C} \end{vmatrix} x$$

It is clear that this system satisfies the hypothesis of Corollary 2 so that it defines an (r+m)-port passive network composed of unit inductors and resistors and a bounded time-variable gyrator, where u represents the current input to the first r ports; u^* represents the current input to the last in ports; y^* represents the voltage output of the first r ports; y represents the voltage output of the last m ports. Hence, if the last m ports are open circuited the network realizes H as the current - voltage transfer response from the first r ports to the last m ports.

Finally, combining Theorems 1 and 3 and Lemma 3 we have

Theorem 5: If H is uniformly realizable and there exists a constant K such that for all τ

$$\int_{-\infty}^{t} ||H(t, \tau)|| d\tau \leq K$$

then it can be synthesized as the transfer response of a passive network composed of constant inductors and resistors and bounded time-variable gyrators.

If the hypothesis of Theorem 5 is satisfied, we can also give a closed form expression for the network realization. By defining

$$V(t) = \lim_{s \to -\infty} M(s,t) = \int_{-\infty}^{t} \Theta(\tau)\Theta'(\tau)\, d\tau$$

and $U = V^{1/2}$ it may be verified that $(\bar{A}, \bar{B}, \bar{C})$, where

$$\bar{A} = -\dot{U}^{-1}U, \quad \bar{B} = U^{-1}\Theta, \quad \bar{C} = \psi U \quad ,$$

is a uniform realization of H with the property that $\bar{A} + \bar{A}' = -\bar{B}\bar{B}'$ so that a passive network realization obtains directly as in the proof of Lemma 3.

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APPENDIX

Proof of Lemma 1: Since V is positive definite, there must exist an orthogonal matrix T such that $V = T \Lambda T$ where Λ is a diagonal matrix with diagonal elements $\lambda_i(t) > 0$ on $(-\infty, \infty)$. Therefore, $U = V^{1/2} = T \Lambda^{1/2} T'$. Continuity of $\dot{U}$ follows from that of $\dot{V}$ since $U = \exp(1/2 \, \ell n V)$ is an analytic function of V on $(-\infty, \infty)$.

To show that $U^{-1}\dot{U}$ is bounded requires a somewhat more complex argument. Let $U^{-1}\dot{U} = D$, and observe that

$$V^{-1}\dot{V} = U^{-2}(U\dot{U} + \dot{U}U)$$
$$= D + U^{-1}DU$$

If the above equation is multiplied on the left by $T'U$ and on the right by $U^{-1}T$, it is seen that

$$T'U^{-1}\dot{V}U^{-1}T = T'DT + \Lambda^{1/2}(T'DT)\Lambda^{-1/2} \tag{28}$$

Let $F = T'DT$ and $G = T U^{-1}\dot{V}U^{-1}T$, and let f_{ig} and g_{ij} denote the ij^{th} elements of F and G, respectively. Then, it is clear from (28) that

$$g_{ij} = f_{ij}\left[1 + (\lambda_i/\lambda_j)^{1/2}\right]. \tag{29}$$

Since $\lambda_i(t) > 0$ for all $t \in (-\infty, \infty)$ and all i, (29) implies $f_{ij}(t) \leq g_{ij}(t)$ for all t. Hence, if G is bounded then F is bounded. But G is symmetric and has eigenvalues equal to those of $V^{-1}\dot{V}$ so that G is bounded. Since T is orthoganal, boundedness of D follows from that of F.

Proof of Corollary 1: If V is Lyapunov then $V^{-1}\dot{V}$, and consequently the maximum eigenvalue of $V^{-1}\dot{V}$ is bounded. Hence, by Lemma 1 $U^{-1}\dot{U}$ is bounded. Since U and U^{-1} are bounded it follows that $\dot{U}$ is bounded so that U is a Lyapunov transformation.

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Footnotes

1. If A and B are symmetric, $A > B$ $(A \geq B)$ means $A-B$ is positive (nonnegative) definite, $\alpha_1(\delta)$ is a positive constant determined solely by δ, I denotes the identity matrix of appropriate order, and $'$ denotes matrix transposition.

2. Continuity is included in the definition for simplicity rather than necessity, all of the results in the sequal hold, with slight modification, for piecewise continuous systems.

3. $\sigma_m\{A\}$ and $\sigma_M\{A\}$ denote the minimum and maximum eigenvalues, respectively, of a matrix A.

4. Conditions for passivity of time-variable systems have also been given recently by Rohrer [16], who considered the case when a nonsingular direct transmission matrix was present. Anderson [17] has also considered this case.