AN ALGORITHM FOR COMPUTING THE JORDAN CANONICAL FORM OF A MATRIX

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ABSTRACT

Recent developments in the theory of linear dynamical systems have generated an interest in efficient ways for calculating the Jordan canonical form of a matrix. The present paper presents a computational method for finding the Jordan canonical form, based on three subprocedures, each of which performs elementary row operations. The advantage of the method is that it is simple to program and is computationally more efficient than methods based on the computation of elementary divisors.

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Introduction

Among the recently raised questions in system theory are those of controllability, observability, equivalence and the minimality of a system representation. For dynamical systems represented by a set of linear first order differential and algebraic equations, these questions are closely related to the nature of the invariant subspaces of a certain matrix entering the differential equations [4]. Since in order to construct completely controllable or completely observable or equivalent subsystems one must eventually obtain descriptions for these invariant subspaces [4], there is a great deal of interest in efficient methods for constructing the Jordan canonical form of a matrix.

This paper presents an algorithm for constructing Jordan forms which is conceptually very simple and computationally quite efficient. The programming of this algorithm is considerably expedited by the fact that it consists of only three straightforward subprocedures. The method presented is based on a derivation of the Jordan canonical form given by Godement [1], whose proofs have been modified so as to reveal the exact computations one must perform in the construction of a Jordan canonical form. Finally, it might be of interest to point out that since the method presented performs elementary row operations on matrices whose elements are numbers and not polynomials, it is simpler and faster than the ones described in [2], [3].

I. Nilpotent Transformations from $\mathcal{C}^n$ into $\mathcal{C}^n$

1 **Definition**: Let T be a linear map from $\mathcal{C}^n$ into $\mathcal{C}^n$. T is said to be nilpotent with index of nilpotency p if $T^p x = 0$ for all $x \in \mathcal{C}^n$ and there is a $x \in \mathcal{C}^n$ such that $T^{p-1} x \neq 0$.

2 **Remark**: All the eigenvalues of a nilpotent transformation must be zero, since otherwise there would be an eigenvector e with eigenvalue $\lambda \neq 0$ such that $T^k e = \lambda^k e \neq 0$ for $k = 0, 1, 2, \ldots$

3 **Lemma**: Let $T: \mathcal{C}^n \to \mathcal{C}^n$ be nilpotent with index p, and let $\eta_i = \{x: T^i x = 0\}$ be the null space of T^i with $i = 0, 1, 2, \ldots, p$. Then,

4 $$T\eta_{i+1} \subset \eta_i \quad i = 0, 1, \ldots, p-1$$

and

5 $$\{0\} = \eta_0 \subset \eta_1 \subset \eta_2 \subset \cdots \subset \eta_{p-1} \subset \eta_p = \mathcal{C}^n$$

is a strictly increasing sequence.

Proof: Let i be an integer in $\{0, 1, 2, \ldots, p-1\}$, let $x \in \eta_{i+1}$, then $T^{i+1}x = 0$, i.e., $T^i(Tx) = 0$, and therefore $Tx \in \eta_i$. Thus $T\eta_{i+1} \subset \eta_i$ for $i = 0, 1, 2, \ldots, p-1$, which proves (4).

Now if $T^i x = 0$, then $T^{i+1}x = 0$, and hence $\eta_{i+1} \supset \eta_i$ for $i = 0, 1, 2, \ldots, p-1$. Suppose therefore that for some i in $\{0, 1, 2, \ldots, p-1\}$, $\eta_{i+1} = \eta_i$. Then, for any $x \in \mathcal{C}^n$

6 $$T^p x = 0 = T^{i+1}(T^{p-i-1}x)$$

Thus for any $x \in \mathcal{C}^n$, $T^{p-i-1}x \in \eta_{i+1}$, but if $\eta_{i+1} = \eta_i$, we must have

7 $$T^i(T^{p-i-1}x) = T^{p-1}x = 0$$

for all $x \varepsilon \mathcal{C}^n$ which contradicts the assumption that p is the index of nilpotency.

8 Lemma: Let T and $\mathcal{N}_i$, i = 1, 2, ..., p-1, be defined as in lemma (3). Let $\mathcal{M}$ be a linear subspace of $\mathcal{C}^n$ such that for some $i \varepsilon \{1, 2, \ldots, p-1\}$, $\mathcal{M} \cap \mathcal{N}_i = \{0\}$. Then $(T\mathcal{M}) \cap \mathcal{N}_{i-1} = \{0\}$ and T is nonsingular on $\mathcal{M}$.

Proof: Let $x \varepsilon (T\mathcal{M}) \cap \mathcal{N}_{i-1}$ be arbitrary. Then there exists a $y \varepsilon \mathcal{M}$ such that $Ty = x$ and $T^{i-1}(Ty) = 0$. Hence $y \varepsilon \mathcal{M} \cap \mathcal{N}_i$ and therefore $y = 0$, so that $x = 0$. We therefore conclude that $(T\mathcal{M}) \cap \mathcal{N}_{i-1} = \{0\}$. Now suppose there is a $y \varepsilon \mathcal{M}$, $y \neq 0$ such that $Ty = 0$. Then $T^i y = 0$ also and $y \varepsilon \mathcal{M} \cap \mathcal{N}_i$. But then $y = 0$ which contradicts our assumption that $y \neq 0$, and hence T is nonsingular on $\mathcal{M}$.

9 Lemma: Let T and $\mathcal{N}_i$, i = 1, 2, ..., p-1, be defined as in lemma (3). Then there exist subspaces $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_p$ of $\mathcal{C}^n$ such that

10 $$\mathcal{N}_i = \mathcal{N}_{i-1} \oplus \mathcal{M}_i \quad \text{for } i = 1, 2, \ldots, p^*$$

and, for i = 2, 3, 4, ..., p, T maps $\mathcal{M}_i$ into $\mathcal{M}_{i-1}$ one-to-one.

Proof: Suppose that for any $i \varepsilon \{2, 3, \ldots, p\}$, we have found a subspace $\mathcal{M}_i$ such that

12 $$\mathcal{N}_i = \mathcal{N}_{i-1} \oplus \mathcal{M}_i$$

13 Obviously $$\mathcal{N}_{i-1} \cap \mathcal{M}_i = \{0\}$$

and hence, by lemma (8),

14 $$\mathcal{N}_{i-2} \cap T\mathcal{M}_i = \{0\}$$

and T maps $\mathcal{M}_i$ onto $T\mathcal{M}_i$ one-to-one.

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* The symbol $\oplus$ denotes the direct sum operation.

Now, since $\mathcal{M}_i \subset \mathcal{N}_i$, it follows from lemma (3) that $T\,\mathcal{M}_i \subset \mathcal{N}_{i-1}$; it also follows from lemma (3) that $\mathcal{N}_{i-2} \subset \mathcal{N}_{i-1}$. Let $\mathcal{O}_{i-1}$ be the orthogonal complement of $T\,\mathcal{M}_i \oplus \mathcal{N}_{i-2}$ (which is well defined because of (14)) in $\mathcal{N}_{i-1}$, i.e.,

15 $$\mathcal{O}_{i-1} \oplus (T\,\mathcal{M}_i \oplus \mathcal{N}_{i-2}) = \mathcal{N}_{i-1}$$

Now let

16 $$\mathcal{M}_{i-1} = \mathcal{O}_{i-1} \oplus T\,\mathcal{M}_i$$

Then, T maps $\mathcal{M}_i$ into $\mathcal{M}_{i-1}$ one-to-one, and because of (15),

17 $$\mathcal{N}_{i-1} = \mathcal{N}_{i-2} \oplus \mathcal{M}_{i-1}$$

hence, $\mathcal{M}_{i-1}$ satisfies the postulates of the lemma. Now, let $\mathcal{M}_p$ be the orthogonal complement of $\mathcal{N}_{p-1}$ in $\mathcal{N}_p = \mathcal{C}^n$. Then (15) and (16) define the subspaces $\mathcal{M}_{p-1}, \mathcal{M}_{p-2}, \ldots, \mathcal{M}_1$ uniquely and they satisfy the conditions of the lemma. This completes our proof.

18 <u>Theorem</u>: If $T: \mathcal{C}^n \rightarrow \mathcal{C}^n$ is a linear, nilpotent transformation, with index of nilpotency p, then there exists a basis in $\mathcal{C}^n$ with respect to which T has a representation

19 $$\begin{bmatrix} 0 & \delta_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \delta_2 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & \delta_{n-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \delta_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where $\delta_i = 0$ or 1.

Proof: With $\mathfrak{N}_i$, $\mathfrak{M}_i$ defined as in lemmas (3), (8) and (9), we find

$$\mathfrak{N}_0 = \{0\}$$

$$\mathfrak{N}_1 = \mathfrak{M}_1 \oplus \mathfrak{N}_0 = \mathfrak{M}_1$$

$$\mathfrak{N}_2 = \mathfrak{M}_2 \oplus \mathfrak{N}_1 = \mathfrak{M}_2 \oplus \mathfrak{M}_1$$

....

$$\mathfrak{N}_p = \mathfrak{M}_p \oplus \mathfrak{N}_{p-1} = \mathfrak{M}_p \oplus \mathfrak{M}_{p-1} \oplus \mathfrak{M}_{p-2} \oplus \cdots \oplus \mathfrak{M}_1$$

and $\mathfrak{N}_p = \mathcal{C}^n$.

Now, by the proof of lemma (9), we may take $\mathfrak{M}_p$ to be $\mathcal{O}_p$ the orthogonal complement of $\mathfrak{N}_{p-1}$ in $\mathfrak{N}_p$.

Let $\xi_{p,1}, \xi_{p,2}, \ldots, \xi_{p,k_p}$ be a basis for $\mathfrak{M}_p$ and let, $\xi_{p-1,1}$, $\xi_{p-1,2}, \ldots, \xi_{p-1,k_{p-1}}$ be a basis for $\mathcal{O}_{p-1}$, the orthogonal complement of $T\mathfrak{M}_p \oplus \mathfrak{N}_{p-2}$ in $\mathfrak{N}_{p-1}$. Then, by lemma (9),

21 $$\mathfrak{M}_{p-1} = T\mathfrak{M}_p \oplus \mathcal{O}_{p-1}$$

and $T\xi_{p,1}, T\xi_{p,2}, \ldots, T\xi_{p,k_p}, \xi_{p-1,1}, \xi_{p-1,2}, \ldots, \xi_{p-1,k_{p-1}}$ is a basis for $\mathfrak{M}_{p-1}$. Continuing this construction, we obtain the following result. For $i = 1, 2, \ldots, p-1$, let $\xi_{p-i,1}, \xi_{p-i,2}, \ldots, \xi_{p-i,k_{p-i}}$ be a basis for $\mathcal{O}_{p-i}$, the orthogonal complement of $T\mathfrak{M}_{p-i+1} \oplus \mathfrak{N}_{p-i-1}$ in $\mathfrak{N}_{p-i}$ then the resultant bases for $\mathfrak{M}_p, \mathfrak{M}_{p-1}, \ldots, \mathfrak{M}_1$ are

$$
(22)\quad \begin{cases}
\mathfrak{M}_p: & \xi_{p,1}, \xi_{p,2}, \ldots, \xi_{p,k_p} \\
\mathfrak{M}_{p-1}: & T\xi_{p,1}, \ldots, T\xi_{p,k_p}, \xi_{p-1,1}, \ldots, \xi_{p-1,k_{p-1}} \\
\mathfrak{M}_{p-2}: & T^2\xi_{p,1}, \ldots, T^2\xi_{p,k_p}, T\xi_{p-1,1}, \ldots, T\xi_{p-1,k_{p-1}}, \xi_{p-2,1}, \ldots, \xi_{p-2,k_{p-2}} \\
 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathfrak{M}_1 = \mathfrak{N}_1: & T^{p-1}\xi_{p,1} \ldots T^{p-1}\xi_{p,k_p}, T^{p-2}\xi_{p-1,1}, \ldots, T^{p-2}\xi_{p-1,k_{p-1}},
\end{cases}
$$

$$
T^{p-3}\xi_{p-2,1}, \ldots, T^{p-3}\xi_{p-2,k_{p-2}}, \ldots, \xi_{1,1}, \xi_{1,2}, \ldots, \xi_{1,k_1}
$$

Now, proceeding in the array (22) from bottom to top and from left to right, we make the following substitutions:

$$
(23)\quad \begin{cases}
\zeta_1 = T^{p-1}\xi_{p,1}; \quad \zeta_2 = T^{p-2}\xi_{p,1}; \ldots, \zeta_p = \xi_{p,1} \\
\vdots \\
\zeta_{(k_p-1)p+1} = T^{p-1}\xi_{p,k_p}; \zeta_{(k_p-1)p+2} = T^{p-2}\xi_{p,k_p}; \ldots; \zeta_{k_p p} = \xi_{p,k_p}
\end{cases}
$$

$$
\begin{cases}
\zeta_{k_p p+1} = T^{p-2}\ {}_{p-1,1}; \ldots; \zeta_{k_p(p+1)-1} = \xi_{p-1,1} \\
\vdots
\end{cases}
$$

$$
\vdots
$$

$$
\left\{ \zeta_{n-k_1} = \xi_{1,1}; \zeta_{n-k_1+1} = \xi_{1,2}; \ldots; \zeta_n = \xi_{1,k_1} \right. .
$$

Note that each vector $\zeta_{p,i}$ $(i = 1, 2, \ldots, k_p)$ is in $\mathfrak{N}_p$ but not in $\mathfrak{N}_{p-1}$; for this reason it is called a generalized eigenvector of order p. To each such generalized eigenvector corresponds a chain of p basis vectors; to each such chain corresponds (in the Jordan form) a Jordan block of order p. Similarly, for a generalized eigenvector of order k, say $\xi_{k,1}$, corresponds a chain of k basis vectors and a Jordan block of order k (k = 2, 3, ..., p).

Then, by inspection, $\zeta_1, \zeta_2, \ldots, \zeta_n$ is the desired basis. This completes our proof.

II. Arbitrary Transformations from $\mathcal{C}^n$ into $\mathcal{C}^n$

We shall now give without proof the remaining theorems which are necessary to establish the existence of the Jordan canonical form for a matrix.

24 Lemma: Let T: $\mathcal{C}^n \to \mathcal{C}^n$ be a linear transformation and let $\eta_i = \{x | T^i x = 0\}$ for i = 0, 1, 2, ... Then there exists a positive integer $p \leq n$ such that

25 $$\{0\} = \eta_0 \subset \eta_1 \subset \eta_2 \subset \cdots \subset \eta_p$$

is a strictly monotonic sequence and

26 $$\eta_p = \eta_i \quad \text{for all } i \geq p ,$$

i. e., η_p is invariant under T.

Furthermore, the dimension of η_p is equal to the multiplicity of zero as a root of the characteristic polynomial of T.

27 Definition: Let T: $\mathcal{C}^n \to \mathcal{C}^n$ be a linear transformation and let p satisfy the conditions of lemma (24). We shall call the subspace η_p, the generalized null space of T.

28 Lemma: Let T: $\mathcal{C}^n \to \mathcal{C}^n$ be a linear transformation; let its distinct eigenvalues be $\lambda_1, \lambda_2, \ldots, \lambda_s$, $(s \leq n)$. For i = 1, 2, ..., s, let $\eta^i_{p_i}$ be the generalized null space of $(T - \lambda_i I)$, where I is the identity operator, then,

29 $$\mathcal{C}^n = \eta^1_{p_1} \oplus \eta^2_{p_2} \oplus \cdots \oplus \eta^s_{p_s}$$

30 Theorem: Let $T: \mathcal{C}^n \to \mathcal{C}^n$ be a linear transformation, let its distinct eigenvalues be $\lambda_1, \lambda_2, \ldots, \lambda_s$ ($s \leq n$), and their respective multiplicity as roots of the characteristic equation is m_i, $i = 1, 2, \ldots, s$. Then there exists a basis in $\mathcal{C}^n$ with respect to which T has a representation of the form

31 $$\begin{bmatrix} J_1 & 0 & \cdots\cdots & 0 \\ 0 & J_2 & \cdots\cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots\cdots & J_s \end{bmatrix}$$

where, for $i = 1, 2, \ldots, s$, J_i is a $m_i \times m_i$ matrix of the form

32 $$\begin{bmatrix} \lambda_i & \delta_1 & 0 & \cdots & 0 \\ 0 & \lambda_i & \delta_2 & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & \delta_{m_i-1} \\ 0 & \cdots\cdots & & & \lambda_i \end{bmatrix}$$

with $\delta_j = 0$ or 1 for $j = 1, 2, \ldots, m_i-1$.

Proof: For $i = 1, 2, \ldots, s$, let $\eta^i_{p_i}$ be the generalized null space of $(T - \lambda_i I)$. Then, by lemma (24) the m_i dimensional subspace $\eta^i_{p_i}$ is invariant under $(T - \lambda_i I)$ and hence also under T, therefore the restrictions of these operators to $\eta^i_{p_i}$ are well defined. Now, let

$(T - \lambda_i I)_i$ be the restriction of $(T - \lambda_i I)$ to $\eta^{i}_{p_i}$. Then $(T - \lambda_i I)_i$ is nilpotent with index p_i and according to theorem (18) there exists a basis in $\eta^{i}_{p_i}$ with respect to which $(T - \lambda_i I)_i$ has a representation

33
$$N_i = \begin{pmatrix} 0 & \delta_1 & 0 & \cdots & 0 \\ 0 & 0 & \delta_2 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & \delta_{m_{i-1}} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where N_i is a $m_i \times m_i$ matrix and $\delta_j = 0$ or 1 for $j = 1, 2, \ldots, m_{i-1}$. But then, with respect to the same basis, T_i, the restriction of T to $\eta^{i}_{p_i}$ has a representation

34
$$J_i = \begin{pmatrix} \lambda_i & \delta_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & \delta_2 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & & & 0 & \lambda_i & \delta_{m_{i-1}} \\ 0 & & & 0 & 0 & \lambda_i \end{pmatrix}$$

The existence of the representation (31) now follows from (29) and (34). This completes the proof.

We shall now show how the above indicated calculations can be mechanized.

III. Three Basic Procedures

We begin by describing three elementary procedures from which we shall build up the algorithm for constructing Jordan canonical forms.

(P1) <u>Procedure for Computing a Basis for the Subspace $\{x \mid Ax = 0\}$ of $\mathcal{C}^n$</u>

Let A be a n × n matrix of rank m with real or complex components (it may have any number of zero rows). Consider the subspace

35 $$L = \{x \mid Ax = 0\}$$

and let D by any n × n nonsingular matrix. Then $x \varepsilon L$ if and only if $DAx = 0$. We make use of this fact in the construction of a basis for L. For $i, j \varepsilon \{1, 2, \ldots, n\}$. Let U_{ij} be a n × n matrix which is obtained from the n × n identity matrix by interchanging the $i^{\underline{th}}$ and $j^{\underline{th}}$ rows. For $i, j \varepsilon \{1, 2, \ldots, n\}$ let $V_{ij}(\alpha)$ be a n × n matrix which is obtained from the n × n identity matrix by adding α times the $i^{\underline{th}}$ row to the $j^{\underline{th}}$ row, and for $i \varepsilon \{1, 2, \ldots, n\}$ let $W_i(\beta)$ be a n × n matrix obtained from the n × n identity matrix by multiplying the $i^{\underline{th}}$ row by β, with $\beta \neq 0$. Thus,

36 $$U_{ij} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \begin{matrix} \\ i \\ \\ \\ j \\ \\ \end{matrix}$$

37 $$V_{ij}(\alpha) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \alpha & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ i \\ \\ j \\ \\ \\ \end{matrix}$$

$$W_i(\beta) = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \beta & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ i \\ \\ \\ \\ \end{matrix}$$

The matrices U_{ij}, $V_{ij}(\alpha)$, $W_i(\beta)$ are nonsingular. By inspection, the premultiplication of A with any one of them will perform the corresponding elementary row operation (i.e., the one which was performed on the identity matrix). Hence, if $\tilde{A}$ is a $n \times n$ matrix obtained from A by means of elementary row operations, i.e., by successive left multiplication by matrices U_{ij}, $V_{ij}(\alpha)$, $W_i(\beta)$, then x satisfies $Ax = 0$ if and only if $\tilde{A}x = 0$.

To obtain a basis for L proceed as follows.

Step 1: Use elementary row operations to obtain from A an upper triangular matrix $\tilde{A}$

38
$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ 0 & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \tilde{a}_{nn} \end{bmatrix}$$

Step 2: Let $i \in \{1, 2, \ldots, n\}$ be the smallest integer such that $\tilde{a}_{ii} = 0$. If $\tilde{a}_{i+1,i+1} = 0$, interchange the i^{th} and $(i+1)^{th}$ rows to make the element $\tilde{a}_{i,i+1} = 0$. If $\tilde{a}_{i+1,i+1} \neq 0$, subtract a multiple of the $(i+1)^{th}$ row from the i^{th} row to make $\tilde{a}_{i,i+1} = 0$. Proceed in a similar fashion to make the rest of the i^{th} row zero, with $\tilde{A}$ remaining upper triangular.

Step 3. Let j be the first integer greater than i such that $\tilde{a}_{jj} = 0$ after step 2 has been completed. Proceed as in step 2 to make the j^{th} row zero.

Step 4: Continue the procedure implied by steps 2 and 3 to obtain a triangular matrix $\tilde{A}$ with the maximum number of $(n - m)$ zero rows. Note that the m nonzero rows are now linearly independent.

Step 5. Contract the matrix $\tilde{A}$ by deleting the zero rows to produce an $m \times n$ rectangular matrix $\tilde{A}_c$. Let $K \subset \{1, 2, \ldots, n\}$ be the index set characterized by $i \in K$ if $\tilde{a}_{ii} \neq 0$ in $\tilde{A}$ after step 4 is completed, and let $\tilde{c}_i$ be the i^{th} column of $\tilde{A}_c$. Then, by inspection, the m columns $\tilde{c}_i$ $i \in K$ are linearly independent and form a $m \times m$ matrix. Now, $x \in L$ if and only if

39
$$\tilde{A}_c x = 0$$

Rearrange the components of x and the columns of $\tilde{A}_c$ in such a way that (39) becomes

40 $$\tilde{A}'_c x_1 + \tilde{A}''_c x_2 = 0$$

where $\tilde{A}'_c$ is a m × m square matrix whose columns are $\tilde{c}_i$, $i \in K$, while $\tilde{A}''_c$ is a m × (n-m) rectangular matrix whose columns are $\{\tilde{a}_{ci}\}$ $i \in \bar{K}$, the complement of K in {1, 2, ..., n}. Then, from (40),

41 $$x_1 = -\left(\tilde{A}'_c\right)^{-1} \tilde{A}''_c x_2$$

which is best computed by back substitution in (40).

Now, for i = 1, 2, ..., n-m, let x_{2i} = (0, 0, ..., 0, 1, 0, ...0) i.e., the $i^{\underline{th}}$ unit vector in $\mathcal{C}^{n-m}$ and let x_{1i} be the corresponding solution of (41). Then the vectors (x_{1i}, x_{2i}), i = 1, 2, ..., n-m, are a basis for L. (The components must be rearranged again, of course.)

(P2) <u>Procedure for Computing a Basis for the Orthogonal Complement of the Subspace $\{x|Ax = 0\}$ in the Subspace $\{x|Bx = 0\}$</u>

Let A be a n × n matrix of rank m and let B be a n × n matrix of rank ℓ, with the components of A, B real or complex. Suppose that the subspace

42 $$\mathcal{N}_A = \{x|Ax = 0\}$$

is contained in the subspace

43 $$\mathcal{N}_B = \{x|Bx = 0\}$$

Then $\ell \leq m$. Note that there is no restriction on the number of zero rows in A or B and hence B may be the zero matrix, i.e., $\mathcal{N}_B = \mathcal{C}^n$.

To obtain the orthogonal complement of L_1 in L_2 proceed as follows.

Step 1. Use the procedure (P1) to compute the $m \times n$ matrix $\tilde{A}_c$. Then, the columns of $(\tilde{A}_c)^*$ span the orthogonal complement of L_1 in $\mathcal{C}^n$.†

Step 2. Let $C = B(\tilde{A}_c)^*$, i.e., C is a $n \times m$ matrix of rank $\ell \leq m$. Let M be the $m - \ell$ dimensional subspace of $\mathcal{C}^m$ defined by

44 $$M = \{y \in \mathcal{C}^m | C\, y = 0$$

Then the orthogonal complement, $\mathcal{O}$, of L_1 in L_2 is obviously given by

45 $$\mathcal{O} = \{x \in \mathcal{C}^n | \; x = (\tilde{A}_c)^* \, y, \; y \in M\}$$

Use procedure (P1) (modified trivially to account for the fact that C is not square) to construct a basis for the $m - \ell$ dimensional subspace M, say $y_1, y_2, \ldots, y_{m-\ell}$. Now compute a basis for $\mathcal{O}$ $x_1, x_2, \ldots, x_{m-\ell}$, according to the formula $x_i = (\tilde{A}_c)^* \, y_i$, for $i = 1, 2, \ldots, m-\ell$.

(P3) Procedure for Computing a Basis for the Orthogonal Complement of the Subspace $\{x | x = \Sigma\, \alpha^i \, v_i\}$ in the Subspace $\{x | Bx = 0\}$

Let B be a $n \times n$ matrix of rank ℓ and let $v_1, v_2, \ldots, v_m$ be a set of linearly independent vectors in the subspace

46 $$L_2 = \{x | Bx = 0\}$$

Let L_1 be the subspace defined by

47 $$L_1 = \left\{ x | x = \sum_{i=1}^{m} \alpha^i \, v_i, \; \alpha^i \in \mathcal{C}^1 \right\}$$

To obtain the orthogonal complement of L_1 in L_2 proceed as follows.

Step 1. Form a $m \times n$ matrix V whose i^{th} row is v_i. Use procedure (P1) to compute a basis for the complex conjugate of the orthogonal complement of L_1, i.e., for the subspace

† The symbol * denotes the complex conjugate transpose.

48 $$\{x|Vx = 0\}$$

Call this basis $w_1, w_2, \ldots, w_{n-m}$. Let W be a (n-m) × n matrix whose i^{th} row is w_i, then an alternate description for L_1 is

49 $$L_1 = \{x|Wx = 0\}$$

Step 2. Use procedure (P2) to find a basis for the orthogonal complement of L_1 (49) in L_2.

IV. Algorithm: Jordan Canonical form for Nilpotent Operators from $\mathcal{C}^n$ into $\mathcal{C}^n$

Let T: $\mathcal{C}^n \to \mathcal{C}^n$ be a nilpotent linear operator for which we have a representation with respect to some basis in the form of a matrix, say A. To compute the basis established in theorem (18) (see (22)) proceed as follows.

Step 1. Compute $A^2, A^3, \ldots, A^{p-1}$ (where $A^p = 0$).

Step 2. Use procedure (P1) to find bases for the null spaces $\mathcal{N}_i = \{x|A^i x = 0\}$ i = 1, 2, ..., p-1. Call these vectors $\mathfrak{n}_{i1}$, $\mathfrak{n}_{i2}, \ldots, \mathfrak{n}_{i\ell_i}$, respectively.

Step 3. Use procedure (P2) to find a basis for $\mathcal{M}_p$, the orthogonal complement of $\mathcal{N}_{p-1}$ in $\mathcal{C}^n$. Then $\mathcal{M}_p = \mathcal{O}_p$. Call the basis constructed $\xi_{p,1}, \xi_{p,2}, \ldots, \xi_{p,k_p}$.

Step 4. (i) Compute the vectors $A\xi_{p,1}, \ldots, A\xi_{p,k_p}$. Then $A\xi_{p,1}, \ldots, A\xi_{p,k_p}, \mathfrak{n}_{p-2,1}, \mathfrak{n}_{p-2,2}, \ldots, \mathfrak{n}_{p-2,k_{p-2}}$ are a basis for $T\mathcal{M}_p \oplus \mathcal{N}_{p-2}$.

(ii) Use procedure (P3) to compute a basis for $\mathcal{O}_{p-1}$, the orthogonal complement of $T\mathcal{M}_p \oplus \mathcal{N}_{p-2}$ in $\mathcal{N}_{p-1}$. Call this basis $\xi_{p-1,1}$, $\xi_{p-1,2}, \ldots, \xi_{p-1,k_{p-1}}$. Then

$$A\xi_{p,1},\ A\xi_{p,2},\ \ldots,\ A\xi_{p,k_p},\ \xi_{p-1,1},\ \xi_{p-1,2},\ \ldots,\ \xi_{p-1,k_{p-1}}$$

is the required basis for $\mathfrak{M}_{p-1}$.

<u>Step 5</u>. Continue the construction of the vectors in (22) using the procedure (P3) in the manner indicated above until the entire basis is obtained.

<u>Example</u>: Consider the nilpotent matrix A given below.

50 $$A = \begin{pmatrix} 0 & -1 & -1 & 1 & 0 \\ -1 & 1 & 2 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \end{pmatrix}; \quad A^2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \ A^3 = \underline{0}$$

The index of nilpotency p = 3.

a) <u>To find $\mathfrak{N}_1$</u>

(i) interchange first and last rows of A, then add the first row to the second row, then add the third row to the fourth row and subtract the third row from the fifth row. We get in succession -

51 $$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We therefore get

$$\eta_1 = \left\{x: \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix} = 0 \right\}$$

Thus η_1 is two dimensional.

(ii) Letting $x^1 = 1$, $x^2 = 0$, and $x^1 = 0$, $x^2 = 1$, we find that a basis for η_1 is (1, 0, 1, 1, -1) and (0, 1, 0, 1, -1).

b) <u>To find η_2</u>

(i) add to the first row of A^2 to the second and subtract the first row from the third to get

53
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence

54
$$\eta_2 = \{x: (1 \quad 1 \quad 0 \quad 0 \quad 1) \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix} = 0\}$$

This is obviously a four dimensional space.

(ii) We do not need a basis for η_2.

c) From (54) a <u>basis for $\mathcal{M}_3$</u> is the vector (1, 1, 0, 0, 1) = $\xi_{3,1}$

d) <u>To find $\mathcal{M}_2$:</u>

55 (i) A basis for A $\mathcal{M}_3$ is $A\xi_{3,1}$ = (-1, 0, -1, 2, 1)

(ii) Combining the basis for η_1 with $A\xi_{3,1}$, we get

56 $$(\eta_1 \oplus A\,\mathcal{M}_3)^{\perp} = \{x: \begin{pmatrix} -1 & 0 & -1 & 2 & 1 \\ 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix} = 0\}$$

This is a two dimensional space, for which we compute a basis by putting $x_3 = 1$, $x_5 = 0$ and $x_3 = 0$, $x_5 = 1$ to get

57 $$(1, 1, 0, 0, 1), \quad (-1, 0, 1, 0, 0)$$

Thus, the orthogonal complement of $\eta_1 \oplus A\,\mathcal{M}_3$ relative to η_2 is the set

$$\{x: (1\ 1\ 0\ 0\ 1) \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix} = 0 \text{ and } x = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix}\}$$

Our only choice for μ^1, μ^2 is $\mu^1 = 1$, $\mu^2 = 3$ (within a scalar multiple) and hence a basis for $(\eta_1 \oplus A\,\mathcal{M}_3)^{\perp} \cap \eta_2$ is the vector

58 $$\xi_{2,1} = (-2, 1, 3, 0, 1)$$

Hence a basis for $\mathcal{M}_2$ is

59 $$A\xi_{3,1} = (-1, 0, -1, 2, 1), \ \xi_{2,1} = (-2, 1, 3, 0, 1)$$

e) <u>To find A basis for $\mathcal{M}_1$</u>

Since we have already found three vectors for our bases: ξ_{31}, $A\xi_{31}$, and ξ_{21}, and since $A^2\xi_{3,1}$, $A\xi_{2,1}$ must be part of the basis for $\mathcal{M}_1$, we find that we already have five basis vectors for our five dimensional space, and hence $\mathcal{M}_1$ must be spanned by

60 $$A^2\xi_{3,1} = (3, -3, 3, 0, 0), \ \ A\xi_{2,1} = (-4, 9, -4, 5, -5)$$

V. <u>Algorithm: Jordan Canonical Form for Linear Operators for $\mathbb{C}^n$ into $\mathbb{C}^n$.</u>

Let T: $\mathbb{C}^n \to \mathbb{C}^n$ be a linear operator for which the n n matrix A is a representation with respect to a given basis. To compute the basis with respect to which T will have a Jordan canonical form representation proceed as follows to implement the proof of theorem (30)

<u>Step 1</u>. Compute the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$ of A.

<u>Step 2</u>. Compute $(A - \lambda_1 I) = \tilde{D}$. Use procedure (P1) up to (39) to compute $\tilde{D}_c$. Compute D^2 and $(\widetilde{D^2})_c$ as before. If $(\widetilde{D^2})_c$ has the same number of rows as $\tilde{D}_c$, stop. If $(\widetilde{D^2})_c$ has fewer rows than $\tilde{D}_c$, compute D^3 and $(\widetilde{D^3})_c$. Continue this until the first index p_1 such that $(\widetilde{D^{p_1}})_c$ and $(\widetilde{D^{p_1+1}})_c$ have the same number of rows. Then p_1 is the index of nilpotency of $(T - \lambda_1 I)_1$.

<u>Step 3.</u> Carry out the steps 2, 3, 4, and 5 of algorithm (IV) to obtain the desired basis for $\eta^1_{p_1}$.

<u>Step 4</u>. Compute $(A - \lambda_2 I)$ and repeat the above four steps. Continue until the entire required basis is constructed.

This concludes our presentation of the algorithm for computing the Jordan canonical form of a matrix.

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