# STABILITY OF MULTIPLE-LOOP FEEDBACK LINEAR TIME-INVARIANT SYSTEMS

by

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## I. INTRODUCTION

In this paper we give sufficient conditions for the stability of multiple-input multiple-output linear time-invariant feedback systems. In this sense it is generalization of the results which started with Nyquist [1, 3, 7]. The class of open loop systems considered is broader than those studied heretofore. The open-loop impulse response may contain an infinite sequence of impulses subject to the requirement that the open-loop transfer function be stable in the sense of Zadeh-Desoer ([2], p. 413). In contrast to previous work on multiple feedback systems, the gain matrix K is not assumed to be diagonal. It is not necessarily symmetric either. Furthermore the specialization of the results of this paper to the case of single-input single-output systems are more powerful than those we obtained recently [7] because of some technical improvements in the method of proof.

Since the dynamical systems under consideration are described in terms of a convolution operator, the stability results are expressed in terms of input-output properties. This is particularly important nowadays when most of the results concerning nonlinear systems are expressed in this form [8, 9, 10]. Means for applying the results of this paper to the study of stability of nonlinear time-varying systems are indicated in the conclusions.

## II. DESCRIPTION OF THE SYSTEM

Consider the linear, time-invariant, multiple-input, multiple-output system shown in Fig. 1. The vectors $u, e, \xi, \eta, z$ and $y$ have $n$ components. The symbol $\underset{\sim}{K}$ denotes a linear time-invariant gain block: its input-output relation is described by the equation

$$e(t) = K\,\xi(t)$$

where $K$ is an $n \times n$ constant real matrix. $\underset{\sim}{G}$ is a linear, time-invariant, nonanticipative subsystem: its input-output relation is described in terms of its impulse response matrix $G$ by

$$\eta(t) = \begin{cases} (G * \xi)(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{1}$$

We assume throughout that the input $u(t) = 0$ for $t < 0$ and that $z(t)$ (which represents either the zero-input response or some outside disturbance) is also zero for $t < 0$. We think of $e$ as the "error" and $y$ as the output.

The equations of the system are

$$e = u - y \tag{2}$$

$$y = \eta + z \tag{3}$$

and, since $u$ and $z$ are identically zero for $t < 0$,

$$e(t) = \begin{cases} u(t) - z(t) - (G * Ke)(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{4a}$$

Using (2) to eliminate e in (4a), we obtain the relation between the input u and the output y of the closed loop system

$$y(t) = \begin{cases} z(t) + [G * K(u-y)](t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{4b}$$

## III. ASSUMPTIONS

The $n \times n$ identity matrix is denoted by I. The symbol $|\cdot|$ applied to a vector denotes a norm in $R^n$ and, applied to a matrix, it denotes the induced matrix norm. $\|\cdot\|$ applied to any vector denotes $\sup_{t\geq 0} |\cdot|$.

Let g be a distribution whose support is in $[0, \infty)$. We say that g is an element of $\mathfrak{A}$ if

$$g(t) = g_a(t) + \sum_{i=0}^{\infty} g_i \delta(t-t_i) \tag{5}$$

where $g_a: [0, \infty) \rightarrow R^n$ is in $L^1(0, \infty)$; the sequence $\{t_i\}_0^\infty$ is in $[0, \infty)$ with $0 = t_0 < t_1 < t_2 \ldots$ ; the sequence of constant vectors in $R^n$ $\{g_i\}$ is subject to $\sum_{i=0}^{\infty} |g_i| < \infty$. The set of all elements in $\mathfrak{A}$ constitutes a commutative Banach algebra with the usual definition for addition, the product defined by convolution, and the norm defined by ([4], Chapt. VI, sec. 5;[5], sec. 6.2;[6], sec. IV.4)

$$\|g\| = \int_0^\infty |g_a(t)|dt + \sum_{i=0}^{\infty} |g_i| . \tag{6}$$

Similarly we shall say that the $n \times n$ matrix $G$ is in $\mathcal{A}$ whenever each of its column vectors is in $\mathcal{A}$. With these notations in mind we formulate the following assumption:

(G). The open loop impulse response matrix $G$ is of the form

$$G(t) = \begin{cases} R + G_\ell(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{7}$$

where $R$ is an $n \times n$ constant real matrix and $G_\ell \in \mathcal{A}$, i.e.

$$G_\ell(t) = G_a(t) + \sum_{\nu=0}^{\infty} G_\nu \delta(t-t_\nu) \quad \text{for } t \geq 0 \tag{8}$$

with $G_a \in L^1$ and the constant matrices $G_\nu$ satisfy $\sum_{\nu=0}^{\infty} |G_\nu| < \infty$.

## IV. MAIN RESULT

<u>Theorem 1.</u> Let the system S satisfy the assumption (G). Let $\hat{G}(s)$ denote the Laplace transform of G. Under these conditions, if

$$\inf_{\mathrm{Re}\, s \geq 0} \left| \det(I + \hat{G}(s)K) \right| > 0 \tag{9}$$

and if either $R = 0$ or all the eigenvalues of $RK$ are in the open right half plane, then the impulse response matrix $H$ of the closed loop system $S$ is also in $\mathcal{Q}$, i.e., $H(t) = H_a(t) + \sum_{\nu=0}^{\infty} H_\nu \delta(t-\tau_\nu)$ with $H_a \epsilon L^1$, the constant matrices $H_\nu$ satisfies $\sum_{\nu=0}^{\infty} |H_\nu| < \infty$ and $0 = \tau_0 < \tau_1 < \tau_2 < \ldots$.

Proof. Note that in view of (G), the elements of $\hat{G}(s)$ are analytic functions of $s$ in $\mathrm{Re}\, s > 0$ which are bounded as $|s| \to \infty$ with $|\measuredangle s| \leq \frac{\pi}{2}$. Now by definition of the impulse response of S, if we set $z = 0$ and $u(t) = d_j(t)$,[1] then the corresponding output is the $j^{\underline{th}}$ column of H, $(j = 1, 2, \ldots, n)$. Therefore, from (4b), the equation for H reads

$$H + G * KH = GK \,. \tag{10}$$

We think of this equation as a convolution equation in $\mathcal{D}'_+$ which has to be solved for H. Since in $\mathcal{D}'_+$, the convolution product has no divisors of zero ([4], p. 173; [5], p. 150) the solution is unique. Suppose now that H is Laplace transformable, then (10) gives

$$\hat{H}(s) = (I + \hat{G}(s)K)^{-1}\hat{G}(s)\, K \tag{11}$$

where $\hat{H}(s)$ denotes the Laplace transform of H. In view of the properties of $\hat{G}(s)$ and (9), $\hat{H}(s)$ is an analytic function of $s$ in $\mathrm{Re}\, s > 0$

which is bounded as $|s| \to \infty$ with $|\measuredangle s| \le \frac{\pi}{2}$. Therefore $\hat{H}(s)$ as given by (11), satisfies the necessary and sufficient condition for being the Laplace transform of a distribution in $\mathcal{D}'_+$ ([4], p. 306; [5], p. 237). Therefore H, the solution of (10), is uniquely defined by

$$H = \mathcal{L}^{-1}\left\{(I + \hat{G}(s)K)^{-1}\right\} * \mathcal{L}^{-1}\left\{\hat{G}(s)K\right\} \tag{12}$$

Case I. $R = 0$. Then G reduces to $G_\ell$ (see(7)), and $GK \in \mathcal{Q}$. We assert that the 1st factor in (12) is also in $\mathcal{Q}$. Calculate $(I + G_\ell(s)K)^{-1}$ by Cramer's rule:

$$(I + \hat{G}_\ell(s)K)^{-1} = \hat{A}(s)\,[\det(I + \hat{G}_\ell(s)K)]^{-1} \tag{13}$$

where $\hat{A}(s)$ is the $n \times n$ matrix whose i-j element is the cofactor of the j-i element of $I + \hat{G}_\ell(s)K$. Since all elements of $I + \hat{G}_\ell(s)K$ are transforms of elements in $\mathcal{Q}$, it follows, from the properties of the algebra $\mathcal{Q}$, that all elements of $\hat{A}(s)$ are transforms of elements in $\mathcal{Q}$. Since $R = 0$, assumption (9) becomes

$$\inf_{\mathrm{Re}\, s \ge 0} |\det(I + \hat{G}_\ell(s)K)| > 0 \,. \tag{14}$$

Hence by a result of Hille and Phillips ([6], p. 150)[2], the second factor in (13), is the Laplace transform of an element in $\mathcal{Q}$. Therefore by the closure property of $\mathcal{Q}$, $(I + \hat{G}_\ell(s)K)^{-1}$ is the transform of an element in $\mathcal{Q}$. And by (12), so does H.

Case II. R is not the zero matrix but RK has all its eigenvalues in the open right half plane. For this case,

$$\hat{G}(s) = s^{-1}R + \hat{G}_\ell(s) \qquad \text{for } \mathrm{Re}\, s > 0 \tag{15}$$

and (11) becomes successively

$$\begin{aligned}
\hat{H}(s) &= [I + s^{-1}RK + \hat{G}_\ell(s)K]^{-1} \; [s^{-1}RK + \hat{G}_\ell(s)K] \\
&= \left\{(sI + RK)\;[I + (sI + RK)^{-1}s\hat{G}_\ell(s)K]\right\}^{-1} s[s^{-1}RK + \hat{G}_\ell(s)K] \\
&= [I + (sI + RK)^{-1}s\,\hat{G}_\ell(s)K]^{-1} \; [s(sI + RK)^{-1}(s^{-1}RK + \hat{G}_\ell(s)K)]
\end{aligned} \tag{16}$$

Call the first bracket $\hat{M}_1(s)$ and the second $\hat{M}_2(s)$, then

$$\hat{H}(s) = \hat{M}_1(s)^{-1}\hat{M}_2(s) \tag{17}$$

Now by assumption RK has all its eigenvalues in the open right half plane; the elements of $(sI + RK)^{-1}$ are rational functions of s with poles in the <u>open left half plane</u> and these rational functions $\to 0$ as $|s| \to \infty$. Consequently the elements of $\mathcal{L}^{-1}[s(sI + RK)^{-1}]$ are in $\mathcal{Q}$: the factor s, which indicates differentiation, may at most create impulses at $t = 0$. Going back to (16), we see that $\hat{M}_2(s)$ is of the form

$$(sI + RK)^{-1}\, RK + [s(sI + RK)^{-1}]\; [\hat{G}_\ell(s)K] \tag{19}$$

The inverse transform of the first term is in $\mathcal{Q}$; that of the second term is also in $\mathcal{Q}$ because it is the convolution of two elements in $\mathcal{Q}$. It remains to show that $\mathcal{L}^{-1}\{[\hat{M}_1(s)]^{-1}\} \in \mathcal{Q}$, for then by (17) it follows that $H \in \mathcal{Q}$. We calculate as before the inverse of $\hat{M}_1(s)$ by Cramer's rule.

In view of the quoted result of Hille and Phillips, the claim will be established if $\det[\hat{M}_1(s)]$ is bounded away from zero in the closed right half plane. To prove this consider

$$\det[\hat{M}_1(s)] = \det[(sI + RK)^{-1}(sI + s\hat{G}(s)K)]$$

$$= \det[sI + s\hat{G}(s)K]\,[\det(sI + RK)]^{-1} \tag{20}$$

The second factor is bounded away from zero in $\operatorname{Re} s \geq 0$. By assumption (9), the same is true for the first factor except possibly, in the neighborhood of the origin. Now at $s = 0$ the matrix in the first factor reduces to $RK$. But $\det(RK) > 0$ since $RK$ is a real matrix with eigenvalues in the open right half plane. Consequently the first factor in (20) is bounded away from zero for all $s$ in $\operatorname{Re} s \geq 0$. Therefore $\inf_{\operatorname{Re} s \geq 0} |\det(M_1(s)| > 0$, hence $\mathcal{L}^{-1}\{[M_1(s)]^{-1}\} \in \mathcal{Q}$ and so does $H$. This completes our proof.

## V. INPUT OUTPUT PROPERTIES

We now use Theorem 1 to describe the input-output properties of the closed loop system. Since in Eq. (4a), the input $u$ and the disturbance $z$ play symmetric roles, we shall exclusively consider the case where $z \equiv 0$. Note also that our results are stated in terms of the input u and the output y; the concurrent properties of the error e are readily obtained from Eq. 2.

Theorem 2. Let the system S satisfy assumption (G). Let $z \equiv 0$. Let inequality (9) hold and let either R = 0 or RK have all its eigenvalues in the open right half plane. Under these conditions,

(a) $u \in \mathcal{A} \Rightarrow y \in \mathcal{A}$;

(b) for $1 \leq p \leq \infty$, $u \in L^p \Rightarrow y \in L^p$;

(c) provided $RK \neq 0$, for any constant vector $a \in R^n$, if $u(t) = a1(t)$, then $y(t) \rightarrow a$ as $t \rightarrow \infty$;

(d) $u(0) = 0$ with u continuous on $[0, \infty)$ implies that y is continuous;

(e) $u \in L^\infty$ and $u(t) \rightarrow 0$ as $t \rightarrow \infty$ implies that $y \in L^\infty$ and $y(t) \rightarrow 0$ as $t \rightarrow \infty$.

Comments. (I) conclusions (c), (d) and (e) imply that, provided $RK \neq 0$, the feedback system is a position servo with zero steady state error: let $u(0) = 0$ and u be any continuous function bounded on $[0, \infty)$ with $u(t) \rightarrow u_\infty$ (a constant) as $t \rightarrow \infty$, then, by superposition, the output y is also continuous, bounded and $y(t) \rightarrow u_\infty$ as $t \rightarrow \infty$.

(II) All conclusions of Theorem 2 also apply to the error e, as can readily be seen by Eq. (2).

**Proof.** Given the assumption,

$$y = H * u \tag{21}$$

and $H \in \mathcal{Q}$.

(a) follows from Theorem 1 and that $\mathcal{Q}$ is closed under convolution ;

(b) case I: $p = \infty$, i.e. $u \in L^{\infty}$; thus $||u|| < \infty$.

We obtain successively,

$$||y|| = ||H*u|| \leq \sup_{t \geq 0} \left| \int_0^{\infty} H_a(t-\tau)u(\tau)d\tau \right| + \sum_{\nu=0}^{\infty} |H_{\nu}| \; ||u||$$

$$\leq ||u|| \left[ \int_0^{\infty} |H_a(t)| dt + \sum_{\nu=0}^{\infty} |H_{\nu}| \right]$$

Finally by (6) we obtain

$$||y|| \leq ||u|| \quad ||H|| < \infty.$$

Case II: $1 \leq p < \infty$. Let $L^p(y)$ denote the $L^p$ norm of $y$.

$$L^p(y) = L^p(H*u) = L^p \left[ H_a * u + \sum_{\nu} H_{\nu} u(t-\tau_{\nu}) \right]$$

$$\leq L^p(|H_a * u|) + \sum_{\nu} |H_{\nu}| \; L^p(|u|)$$

The first term can be bounded above ([11], p. 99 Thm. 53), and

$$L^p(y) \le L^1(|H_a|)\, L^p(|u|) + \sum_{\nu}^{\infty} |H_\nu|\, L^p(|u|)$$

$$\le \|H\|\, L^p(u)\ . \tag{23}$$

(c) For this case

$$y(t) = \int_0^t H(\tau)\, a\, d\tau \tag{24}$$

Since $H \in \mathcal{A}$, for any $\epsilon$ there is a $T$ such that $t > T$ implies

$$\int_t^{\infty} |H_a(\tau)|\, d\tau + \sum_{\substack{\nu \\ t_\nu \in (t,\infty)}} |H_\nu| < \epsilon \tag{25}$$

Therefore, from (24), we conclude that $\lim_{t\to\infty} y(t)$ exists. To calculate this limit we note that the integral of every element of $H$ is a locally integrable <u>function</u> hence we may use the final value theorem of the Laplace transform ([5], p. 250; [2], p. 542)

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \int_0^t H(\tau)\, a\, d\tau = \lim_{s\to 0}\ s\,\frac{1}{s}\,\hat{H}(s)\, a \tag{26}$$

Referring to the second line of (16), as $s \to 0$, $\hat{H}(s) = I$. Hence $y(t) \to a$ as $t \to \infty$.

(d) $$y(t') = \int_0^{t'} H_a(t'-\tau)u(\tau)d\tau + \sum_{\tau_\nu \in [0,t)} H_\nu u(t'-\tau_\nu) \tag{27}$$

For any $t' \geq 0$, the first term is continuous and equal to 0 at $t = 0$. If, for the $t'$ under consideration, the summation indicated in (27) has a finite number of terms then the second term of (27) is also continuous in $t'$. Suppose now that the summation is over an infinite number of terms; since for any finite $t'$, $\sup_{t \leq t'} |u(t)|$ is finite, by $\sum_{\nu=0}^{\infty} |H_\nu| < \infty$ the sum is uniformly convergent on $[0, t']$ hence it is continuous. Therefore y is continuous on $[0, \infty)$.

(e) Since $u(t) \to 0$ as $t \to \infty$ by assumption, for any $\epsilon > 0$ there is a $T_u(\epsilon)$ such that $t > T_u(\epsilon)$ implies

$$|u(t)| < \epsilon \quad . \tag{28}$$

Now since $H \in \mathcal{Q}$, for any $\epsilon > 0$ there is a $T_H(\epsilon)$ such that for $t > T_H(\epsilon)$

$$\int_t^\infty |H(t')| dt' < \epsilon \tag{29}$$

Note that in (29) we use an informal notation but the meaning is quite clear. For any $\epsilon > 0$, let $t > T_H(\epsilon) + T_u(\epsilon)$, then still using the informal notation,

$$|y(t)| \leq \int_0^{t-T_H} |H(t-\tau)| \; |u(\tau)| d\tau + \int_{t-t_H}^{t} |H(t-\tau)| \; |u(\tau)| d\tau$$

In the first integral, the argument of H varies from $T_H$ to $t > T_H + T_u$, hence, by (29), the integral is smaller than $\epsilon \; ||u||$. In the second integral, the argument of u is larger than $t-T_H > T_u$ hence, by (28), the integral is smaller than $\epsilon \; ||H||$. Therefore $t > T_u + T_H$ implies that $|y(t)| < \epsilon \; (||u|| + ||H||)$. In other words $y(t) \to 0$ as to $\to \infty$.

## VI. CONCLUSIONS

The application of the theorems above require the testing of inequality (9). This can be done by graphical methods $\bar{a}$ la Nyquist: indeed the principle of the argument ([12], p. 252) applies to $\det(I+\hat{G}(s)K)$ since it is an analytic function in the closed right half plane.

The theorems of this paper give simple means for checking the sufficient conditions for stability of broad classes of nonlinear time-varying systems by the use of, for example, Sandberg's general theory ([8], sec. 5, theorem 8 in particular).

## LIST OF FOOTNOTES

1. $d_j(t)$ denotes the n-vector all of whose components are identically zero except for the $j^{th}$ which is $\delta(t)$.

2. In the notation of Hille and Phillips, our algebra $\mathcal{A}$ is denoted by $L(1(\cdot)) + A(1(\cdot))$.

## REFERENCES

[1] Desoer, C. A., "A General Formulation of the Nyquist Criterion," IEEE Trans. on Circuit Theory, Vol. CT-12, No. 2, pp. 230-234, June, 1965.

[2] Zadeh, L. A. and Desoer, C. A., Linear System Theory, McGraw-Hill Book Company, New York, 1963.

[3] Nyquist, H., "Regeneration Theory," Bell System Tech. J., Vol. 2, pp. 126-147, January, 1932.

[4] Schwartz, L., Théorie des Distributions (Revised Edition), Hermann Company, Paris, 1966.

[5] Zemanian, A. H., Distribution Theory and Transform Analysis, McGraw-Hill Book Company, New York, 1965.

[6] Hille, E. and Phillips, R. S., Functional Analysis and Semi-Groups, (Revised Edition) AMS, Providence, R.I., 1957.

[7] Desoer, C. A. and Wu, M. Y., "Stability of Linear Time-Invariant Systems," (To appear in Trans. IEEE Circuit Theory).

[8] Sandberg, I. W., "Some Results on the Theory of Physical Systems Governed by Nonlinear Functional Equations," Bell Syst. Tech. Jour., Vol. 44, pp. 871-898, May-June, 1965.

[9] Zames, G., "On the Input-Output Stability of Nonlinear Time-Varying Feedback Systems, Part I and II," Trans. IEEE, AC-11, No. 2, pp. 228-238, April, 1966, and No. 3, pp. 465-477, July, 1966.

[10] Lee, C. T. and Desoer, C. A., "Stability of Single-Loop Nonlinear Feedback Systems," Proc. III Annual Allerton Conference, Oct. 20-22, 1965, pp. 259-269; also in complete form, ERL - Report 66-13, Electronics Research Laboratory, University of California, Berkeley.

[11] Bochner, S. and Chandrasekharan, K., Fourier Transform, Princeton Study, No. 19, Princeton University Press, Princeton, 1949.

[12] Hille, E., Analytic Function Theory, Vol I, Ginn and Company, New York, 1959.

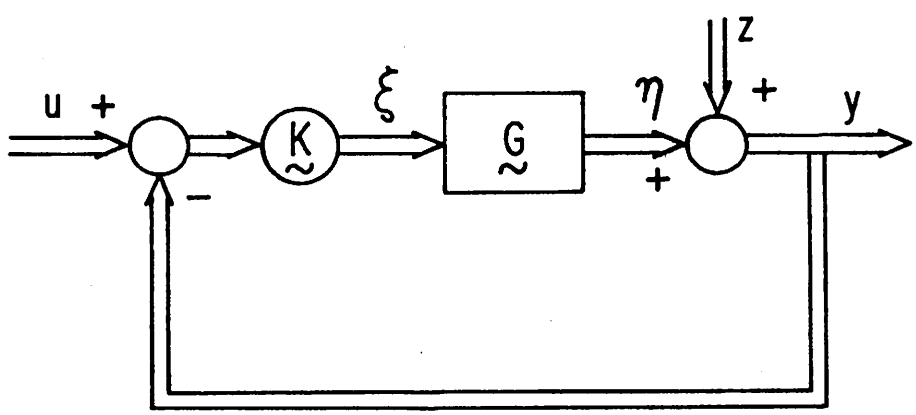


Fig. 1 System S