STABILITY CRITERIA FOR NONLINEAR RLC NETWORKS

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ABSTRACT

This paper presents a stability criterion for a class of nonlinear RLC networks for which the nonlinear characteristic curves lie in a Popov sector. The nonlinear elements considered are resistors, charge-controlled capacitors and flux-controlled inductors. It is shown that if the only nonlinear elements are resistors, then for arbitrarily large sector boundaries the circuit is both asymptotically stable in the large and, for properly placed sources, bounded-input bounded-output stable. Furthermore, if the only nonlinear elements are inductors and capacitors, any set of linear inductors and capacitors which can make the circuit oscillate define the nonlinear sector boundaries. Conditions for asymptotic stability and bounded-input bounded-output stability for networks of nonlinear resistors, inductors and capacitors are developed.

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I. Introduction

The analysis of nonlinear RLC networks has been mainly concerned with the problems of placing network equations in some standard form [1, 2], demonstrating the existence and uniqueness of their solutions [2, 3], and, without actually solving the equations, determining whether they are "stable" [2, 4-6]. Chua and Rohrer [1] have demonstrated the existence of dynamic equations of the form $\dot{x} = f(x, t)$ for very general networks with "unicursal" nonlinear elements (the input and output variables are continuous functions of a parameter). Varaiya and Liu [2] have placed Kirchoff's laws into an algebraic normal form for a wide class of coupled networks and demonstrated uniqueness of solution if, in the uncoupled case, the nonlinear characteristics are monotone increasing. Desoer and Katzenelson [3] guarantee the uniqueness of the solution of network equations providing the nonmonotone nonlinear elements are properly placed in the network topology.

The methods used to determine circuit stability follow directly from stability theory developed for nonlinear systems. Brayton and Moser [4] and Stern [9] use Liapunov functions to obtain sufficient conditions in terms of the nonlinear characteristics and the network topology. Varaiya and Liu [2] show stability for their circuits if the elements are all passive, i.e., $\langle x, f(x)\rangle > 0$. Desoer and Liu [5] give an example of an unstable circuit consisting of such passive elements whose characteristics are not, however, monotone. Kuo [6] demonstrates stability for a circuit with monotone voltage-controlled capacitors, current-controlled inductors and voltage-or current-controlled resistors lying in Popov sectors and satisfying an additional constraint. Sandberg's work on nonlinear equations [7] shows stability for some circuits with nonlinear resistors.

The nonlinear elements considered in this paper are resistors, charge-controlled capacitors and flux-controlled inductors all of whose characteristics lie in a "Popov sector." Hence there is no restriction on the slopes of the nonlinear characteristics or requirement of continuity, except at the origin. It is shown that for a fairly general class of nonlinear coupled circuits the dynamic equations may be written in a canonical form, and a stability criterion similar to the Popov criterion [8] is developed which is also applicable to input-output stability. For circuits containing only nonlinear resistors or only nonlinear inductors and capacitors (all other elements being linear) the sector boundaries can be interpreted in terms of the physical circuit and actually become coincident with the boundary for the linear case.

II. Notation

The mathematical notation used here will be essentially the same as Sandberg [7] except that without further mention all functions will be assumed measurable. The set of complex N-vector-valued functions satisfying $\int_0^\infty f^*(t)f(t)dt < \infty$ is the linear space L_{2N} or, if $N = 1$, just L_2. On L_{2N} the inner product is defined by $\langle f, g\rangle_{L_2} = \int_0^\infty f^*(t)g(t)dt$, and for $f \in L_{2N}$ the norm is defined by $||f||_{L_2} = \langle f, f\rangle^{\frac{1}{2}}$. The Euclidean inner product $\langle f,g \rangle = \Sigma_{i=1}^n \overline{f}_i(t)g_i(t)$ and the Euclidean norm $||f|| = \langle f, f\rangle^{\frac{1}{2}}$ will also be used. For any $f \in L_{2N}$ the Fourier transform is defined component-wise as is the Laplace transform of a vector.

Let M be any matrix. Then M', M^*, and M^{-1} denote the transpose, complex conjugate transpose and, when it exists, the inverse of M, respectively. The smallest eigenvalue of M is denoted by $\Lambda\{M\}$. The element in the ith row and jth column of M will be written $m_{ij}(t)$. The linear space K_{pN} is defined by

$$K_{pN} = \{M| \int_0^\infty |m_{ij}(t)|^p\, dt < \infty \ \forall\, i, j\}$$

M is called positive definite (p.d.) or positive semidefinite (p.s.d.) if, for any vector x, $\langle Mx, x\rangle > 0$ or $\langle Mx, x\rangle \geq 0$, respectively.

Associated with each nonlinear element e_i will be an independent variable, $x_i(t)$, and a dependent variable denoted by $f_i(x_i(t))$. The nonlinear characteristic curves of x_i vs $f_i(x)$ will be assumed to be piece-wise continuous and to lie in a Popov sector. That is,

$$\epsilon_i \leq \frac{f_i(x_i)}{x_i} \leq k_i - \epsilon_i$$

where ϵ_i is nonnegative and k_i is positive. In particular, $f(0) = 0$. A network will be referred to as stable if when the input is zero, there exists a positive constant K_B such that $|x_i(t)| \leq K_B$ for all variables x_i associated with the network. If $x_i(t) \to 0$ as $t \to \infty$ that variable will be called asymptotically stable.

We will employ the following notation:

S_1 = maximal number of linear L's from which no cut set can be formed

S_{10} = total number of linear L's

S_2 = maximal number of linear C's from which no circuit can be formed

S_{20} = total number of linear C's

n_1 = number of nonlinear current-controlled resistors

n_2 = number of nonlinear voltage-controlled resistors

Then the following vectors will be used:

$$x = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ x_{S_2} \\ x_{S_2+1} \\ \cdot \\ \cdot \\ x_{S_1+S_2} \end{bmatrix} = \begin{bmatrix} q_1 \\ \cdot \\ \cdot \\ q_{S_2} \\ \psi_{S_2+1} \\ \cdot \\ \cdot \\ \psi_{S_1+S_2} \end{bmatrix}$$

where $q_1, \ldots, q_{S_2}$ are the tree capacitor charges and $\psi_{S_2+1}, \ldots \psi_{S_1+S_2}$ are the link inductor fluxes,

$$V_R = \begin{bmatrix} V_{R_1} \\ \cdot \\ \cdot \\ V_{R_r} \end{bmatrix}$$

where $V_{R_1}, \ldots V_{R_r}$ are the voltages across linear resistors,

V_{in} = vector of input voltages

I_{in} = vector of input currents

$u = \begin{bmatrix} V_{in} \\ I_{in} \end{bmatrix}$

I = $n_1 \times 1$ column vector of nonlinear resistor currents

V = $n_2 \times 1$ column vector of nonlinear resistor voltages

q = $n_3 \times 1$ column vector of nonlinear capacitor charges

ψ = $n_4 \times 1$ column vector of nonlinear inductor fluxes

Also, K is a diagonal matrix with k_i as the ith diagonal element and zeros elsewhere, and the matrix $\mathcal{E}$ is a diagonal matrix with ϵ_i as the ith diagonal element.

Coupling will be allowed between linear inductances and between linear capacitors. We therefore define the inductance (capacitance) matrix as follows: the i-jth element is the coupling between the ith and jth inductors (capacitors) and the diagonal elements are the self inductances (capacitances). If both these matrices are p. s. d. then the network will be said to have "realistic" coupling. It is assumed throughout that each element may be in series with a voltage source and in parallel with a current source.

III. Nonlinear Resistors

The networks to be discussed in this section consist of linear resistors (R's), inductors (L's), and capacitors (C's), with coupling between inductors and between capacitors, and nonlinear voltage and current controlled resistors. For the nonlinear resistor characteristics, k_i may be arbitrarily large, ϵ_i arbitrarily small or zero in some cases to be discussed. In this section, let $N' = (V'I')$ be an $n_1 + n_2$ column vector of voltages and currents where V is the independent voltage vector for the voltage controlled nonlinear resistors and I is the independent current vector for the current controlled nonlinear resistors. Let $R = -f(N)$ be the corresponding dependent current-voltage vector for the nonlinear resistors. We will use the term proper normal tree for any normal tree in which all current-controlled resistors are tree branches and all voltage controlled resistors are links. The following two lemmas exhibit the existence of a canonical form for the dynamic equations of the network.

Lemma 1: Let $\mathcal{N}$ be a network of nonlinear resistors in Popov sectors and linear R's, L's and C's with realistic coupling between L's and between C's. There exists an equivalent network with at least one proper normal tree.

Proof: Suppose no proper normal tree exists for the given network. Then pick any normal tree. Each I-controlled resistor which is a link is in series with a C, in a cut set of C's and I- controlled resistors, or in a circuit of I-controlled resistors. Similarly, each V-controlled resistor which is a tree branch is in parallel with an L, in a circuit of L's and V-controlled resistors, or in a cut set of V-controlled resistors. Since each of the nonlinear elements is assumed to lie in a Popov sector, the I-controlled resistors can be represented as series combinations of $\frac{\epsilon}{2}$ ohm linear resistors and new I-controlled nonlinear resistors; the V-controlled resistor can be represented as parallel combinations of $\frac{\epsilon}{2}$ mho linear resistors and new V-controlled nonlinear resistors. Figures (1a) and (1b) show this representation. Let us replace each non-linear I-controlled link resistor and V-controlled tree branch resistor by its equivalent combination of elements. The new network which is obtained has a proper normal tree. This tree consists of the original tree with each V-controlled resistor replaced by its associated $\frac{\epsilon}{2}$ ohm linear resistor and the I-controlled resistors without their associated $\frac{\epsilon}{2}$ ohm resistors. For convenience, the new nonlinear characteristics will be denoted by f.

Lemma 2: Let $\mathcal{N}$ be a network of nonlinear resistors in Popov sectors and linear R's, L's, and C's with coupling between the L's and between the C's. Then, the dynamic network equations may be placed in the form

$$\dot{x} = Ax + BR + B_1 u$$
$$N = Cx + DR + D_1 u \tag{3-1}$$

where A, B, B_1, C, D, and D_1 are appropriately defined constant matrices.

Proof: Lemma 1 proves the existence of a proper normal tree for such a network. Let $b = S_{10} + S_{20} + r + n_1 + n_2$ be the number of elements, and ν be the number of nodes. Then the fundamental loop equations are of the form (after substituting for charges on link C's and fluxes in tree L's)

$$A_1 x + B_1 \dot{x} + C_1 V_R + E_1 V = D_1 f(I) + F_1 V_{in} \tag{3-2}$$

and the node equations are of the form

$$A_2 x + B_2 \dot{x} + C_2 V_R + D_2 I = E_2 f(V) + F_2 I_{in} \tag{3-3}$$

Combining these

$$\begin{pmatrix} C_1 & 0 & E_1 \\ C_2 & D_2 & 0 \end{pmatrix} \begin{pmatrix} V_R \\ I \\ V \end{pmatrix} = \begin{pmatrix} -A_1 & -B_1 & D_1 & 0 & F_1 & 0 \\ -A_2 & -B_2 & 0 & E_2 & 0 & F_2 \end{pmatrix} \begin{bmatrix} x \\ \dot{x} \\ f(I) \\ f(V) \\ V_{in} \\ I_{in} \end{bmatrix} \tag{3-4}$$

Then by substituting into (3-4)

$$N = Cx + DR + D_1 u$$

Lemma 1 and 2 are illustrated in the following example in which we set up the dynamic equations for the network shown in Fig. 2a. Since the only normal tree consists of the capacitor and the resistor, the resistor must be replaced by its equivalent shown in Fig. 1b. The resulting network is given in Fig. 2b. The loop equations, corresponding to Eq. (3-2) are

$$\frac{1}{C_1} x_1 + \dot{x}_2 - V = V_{in}$$

$$\frac{1}{C_1} x_1 + \dot{x}_2 - V_R = -V_{in}$$

The node equations corresponding to Eq. (3-3) are

$$\frac{1}{L} x_2 - \dot{x}_1 = 0$$

$$\frac{1}{L} x_2 + \frac{\epsilon}{2} V_R = -f(V)$$

Simple algebraic manipulation put these equations in the form of (3-1) where the appropriate matrices are

$$A = \begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C_1} & -\frac{2}{\epsilon L} \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ \frac{2}{\epsilon} \end{pmatrix} \qquad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & -\frac{2}{\epsilon L} \end{pmatrix} \qquad D = \begin{pmatrix} \frac{2}{\epsilon} \end{pmatrix} \qquad D_1 = \begin{pmatrix} 0 \end{pmatrix}$$

The basis for studying stability of the circuits described by Lemma 2 will be Eq. (3-1). We will assume throughout that inductor and capacitor coupling is realistic, and that the linear subnetwork is passive. Let

$$\Phi(t) = (sI - A)^{-1}, \quad \Psi(t) = B\,\Phi(t)\,C \quad , \quad \Psi_1(t) = B_1\,\Phi(t)\,C$$

where $\Phi(t) = [\,\phi_{ij}(t)\,]$, $\Psi(t) = [\,\psi_{ij}(t)\,]$ and $\Psi_1(t) = [\,\psi_{ij}^{(1)}(t)\,]$.

First, we will consider the case where $u = 0$.

<u>Lemma 3</u>: For $T < \infty$, $N(t)$ is a bounded function on $0 < t < T$.

<u>Proof</u>: The solution to (3-1) may be written for $x(0) = x_0$ as

$$N(t) = C\,\Phi(t)\,x_0 - \int_0^t \Psi(t-\tau)\,f\left[N(\tau)\right]d\tau - Df\left[N(t)\right] \qquad (3\text{-}6)$$

and we can bound norms by

$$\|N(t) + Df\left[N(t)\right]\| \le \|C\,\Phi(t)\,x_0\| + \int_0^t \|\Psi(t-\tau)\|\;\|f\left[N(\tau)\right]\|\,d\tau$$

Since a passive linear circuit has no roots of its characteristic equation with positive real parts or multiple roots with zero real part, the $\phi_{ij}(t)$ and $\psi_{ij}(t)$ are all bounded. Therefore, if $\psi_M = \max_{i,j}\ \sup_t \psi_{ij}(t-\tau)$

$$\|N(t) + Df\left[N(t)\right]\| \le a + \int_0^t \psi_M\,K\|N(\tau)\|\,d\tau \qquad (3\text{-}7)$$

where a is a constant. By definition,

$$||N(t) + Df[N(t)]||^2 = [N + Df(N)]'[N + Df(N)]$$
$$= ||N(t)||^2 + 2N'(t)\,Df[N(t)] + ||Df[N(t)]||^2 \tag{3-8}$$

At any instant of time t_0, $f[N] = \Delta N$ where

$$\Delta = \begin{bmatrix} \delta_1(t_0) & & 0 \\ & \ddots & \\ 0 & & \delta_n(t_0) \end{bmatrix} ; \; k_i > \delta_i(t_0) > \epsilon_i$$

so that $N'(t)\,Df[N(t)] = N'(t)(D\Delta)N(t)$. However, D is a driving point matrix for the circuit obtained by short circuiting all C's and open circuiting all L's. Hence D is positive semidefinite. It follows that $D\Delta$ is p.s.d. so that $N'(t_0)(D\Delta)N(t_0) \geq 0$. Then in (3-8), all terms are nonnegative. Therefore,

$$||N(t) + Df[N(t)]|| \geq ||N(t)||$$

Substituting into (3-7) for b an appropriately defined constant

$$||N(t) + Df[N(t)]|| \leq a + \int_0^t b\,||N(t) + Df[N(t)]||\,dt$$

Hence, by the Bellman-Gronwall Lemma [10]

$$||N(t)|| \leq ||N(t) + Df[N(t)]|| \leq ae^{bt}$$

Thus no component of N(t) grows faster than an exponential, and the Lemma is proved.

Before stating the main stability result of this section, it is interesting to note that this result may be proven by a direct application of Sandberg's results [7]. The proof given here, however, will be applicable to the nonlinear RLC case as well as leading to input-output stability results. The following result is needed in the proof:

Lemma 4: Let f_1, f_2, f_3 be real N-vector valued functions in L_{2N} and h be a real $N \times N$ K_{2N} matrix. Let F_1, F_2, F_3, and H be their Fourier transforms. If $F_1 = -HF_3 + F_2$ and if $\Lambda\{H + H^*\} \geq \delta > 0$, then $\int_0^\infty f_1{}' f_3 \, dt \leq \frac{1}{2\delta} \int_0^\infty f_2{}' f_2 \, dt$.

The proof is a simple exercise in completing the square.

Theorem 1: Let $\mathcal{N}$ be a network of linear, positive R's, L's, and C's with realistic coupling, and nonlinear R's in Popov sectors. Then for $u = 0$, N(t) and f [N(t)] are bounded L_{2N} functions which approach zero as $t \to \infty$. Moreover, x(t) is bounded, and if all eigenvalues of A have negative real parts, $x(t) \to 0$ as $t \to \infty$.

Proof: After using the representations described in Lemma 1, open circuit all V-controlled resistors. Then, successively short circuit all I-controlled resistors except one, and compute the input admittance at the terminals of the remaining I-controlled resistor. If this admittance has a pole on the jw-axis decompose the I-controlled resistor as shown in Fig. 1a. Now, reverse the process by short circuiting all I-controlled resistors and successively compute the open circuit input impedance at the terminals of each V-controlled resistor. If any input impedance has a pole on the jw-axis, decompose the corresponding V-controlled resistor as shown in Fig. 1b. The network obtained after this procedure is completed still has dynamic equations of the form (3-1). For the general solution (3-6), both $\Phi(t)$ and $\Psi(t) \in K_{2N}$.

At $t = T$ assume all I-controlled resistors are short circuited and all V-controlled resistors are open circuited (which is equivalent to setting $f[N(t)] = 0$ for $t > T$). Let $N_T(t)$ denote the new output variable,

$$N_T(t) = \begin{cases} N(t) & \text{for } t \leq T \\ C\,\Phi(t)\,x_0 & \text{for } t > T \end{cases}$$

After adding $-K^{-1} f[N_T(t)]$ to both sides of (3-6), the equation becomes

$$N_T(t) - K^{-1} f[N_T(t)] = C\,\Phi(t)\,x_0 - \int_0^t \Psi(t-\tau)\, f[N_T(\tau)]d\tau$$

$$- Df[N_T(t)] - K^{-1} f[N_T(t)] \qquad (3\text{-}10)$$

By Lemma 3, $N_T(t) \in L_{2N}$ so all terms in (3-10) are Fourier transformable. Let us take Fourier transforms

$$F_1(jw) = \mathcal{F}\{N_T(t) - K^{-1} f[N_T(t)]\}, \qquad H(jw) = \mathcal{F}\{\Psi(t) + D\}$$

$$F_2(jw) = \mathcal{F}\{C\,\Phi(t)\,x_0\} \qquad F_3(jw) = \mathcal{F}\{f[N_T(t)]\}$$

Then, if we take the Fourier transform of (3-10), we obtain

$$F_1(jw) = -H(jw)\, F_3(jw) + F_2(jw)$$

By Lemma 4, if $\Lambda\{H(jw) + H^*(jw)\} \geq \delta > 0$

$$\int_0^\infty \{N_T(t) - K^{-1} f[N_T(t)]\}' f[N_T(t)]\, dt \leq ||C\, \Phi(t)\, x_0||_{L_2}$$

or

$$\int_0^\infty \{N(t) - K^{-1} f[N(t)]\}' f[N(t)] dt \leq \gamma$$

where γ is independent of T.

Then, since $N(t) - K^{-1} f[N(t)] \geq [N(t) - K^{-1}(K - \mathcal{E})N(t)] = K^{-1} \mathcal{E} N(t)$

and $f[N(t)] \geq \mathcal{E} N(t)$,

$$\int_0^\infty [N(t)]' N(t)\, dt \leq k_{max} \gamma / \epsilon^2_{min} \quad (\text{where } \epsilon_{min} = \min_i \epsilon_i \text{ and } k_{max} = \max_i k_i)$$

That is, $N(t) \epsilon L_{2N}$.

Now consider the condition $\Lambda\{H(jw) + H^*(jw)\} \geq \delta > 0$. The matrix $H(j\omega)$ is a driving point matrix and hence is positive real. Thus, for any $k_i > 0$, the condition is fulfilled, so N(t) is a bounded L_{2N} function as is f[N(t)]. Returning to (3-9), the Riemann-Lebesgue Lemma[16] implies that $|| N(t) + Df[N(t)]|| \to 0$ as $t \to \infty$ and since $||N(t)|| \leq ||N(t) + Df[N(t)]||$, $||N(t)|| \to 0$ as $t \to \infty$, and hence $N(t) \to 0$ as $t \to \infty$. From (3-1)

$$x(t) = \Phi(t)\, x_0 - \int_0^t \Psi(t-\tau)\, Bf[N(\tau)]\, d\tau$$

Although $f[N(t)] \in L_{2N}$, there may be constants or sinusoidal terms in $\Phi(t)$ which did not appear in $C\,\Phi(t)$. In any case $f[N(t)] \in L_{2N}$ and $\Phi(t)$ bounded insures that $x(t)$ is bounded. Furthermore, if all eigenvalues of A have negative real parts, $\phi_{ij}(t)$ will be linear combinations of exponentials with negative real parts so again by the Riemann-Legesgue Lemma, $x(t) \to 0$ as $t \to \infty$. This completes the proof of the theorem.

Theorem 2: Let $\mathcal{N}$ be a network of linear, positive R's, L's, and C's with realistic coupling and nonlinear resistors in Popov sectors. Let $u(t)$ be bounded. Suppose the I-controlled resistors are shorted and V-controlled resistors are opened. Then, if the driving point matrix with respect to the input ports has no poles on the jw-axis, $x(t)$, $N(t)$ and $f[N(t)]$ are bounded.

Proof: The solution to (3-1) is now

$$N(t) = C\,\Phi(t)\, x_0 + \int_0^t \Psi_1(t-\tau)\, u(\tau)\, d\tau + D_1\, u(t) - \int_0^t \Psi(t-\tau)\, f[N(\tau)]\, d\tau - Df[N(t)] \qquad (3\text{-}11)$$

First, let $Z(t) = C\,\Phi(t)\, x_0$, $r(t) = \int_0^t \Psi_1(t-\tau)\, u(\tau)\, d\tau + D_1\, u(t)$ and note that $Z(t)$ and $r(t)$ are bounded since $\mathcal{L}\,[\Psi(t)]$ has no poles on the jw-axis. Then, we can rewrite (3-11) as

$$N(t) = Z(t) + r(t) - \int_0^t \Psi(t - \tau) f[N(\tau)] \, d\tau - Df[N(\tau)] \tag{3-12}$$

The remainder of the proof is a modification of Bergen, et. al. [11]. Set $u = 0$ for $t > T$. Then, if we multiply (3-12) by $e^{\alpha t}$, we obtain

$$\{N_T(t) - K^{-1} f[N_T(t)]\} \, e^{\alpha t} = [Z(t) + r_T(t)] \, e^{\alpha t} - Df[N_T(t)] \, e^{\alpha t}$$

$$- \int_0^t e^{(t-\tau)} \Psi(t - \tau) \, e^{\alpha \tau} f[N_T(\tau)] \, dt - K^{-1} f[N_T(t)] \, e^{\alpha t}$$

where $\alpha > 0$ but sufficiently small that $Z(t) \, e^{\alpha t} \epsilon \, L_{2N}$ and $e^{\alpha t} \, \Psi(t) \, \epsilon \, L_{2N}$. Then taking Fourier transforms,

$$F_1(jw) = F_2(jw) - [H(jw - \alpha) + D + K^{-1}] \, F_3(jw - \alpha)$$

where

$$F_1(jw) = \mathcal{F}\{N_T(t) - K^{-1} f[N_T(t)]\} \, e^{\alpha t}\},$$

$$F_2(jw) = \mathcal{F}\{[Z(t) + r_T(t)] \, e^{\alpha t}\}$$

$$F_3(jw) = \mathcal{F} \, \{f[N_T(t)]\}$$

$$H(jw) = \mathcal{F}\{\Psi(t)\}$$

Bergen et. al. have shown that since $H(jw) + D + K^{-1}$ satisfies Lemma 4 with parameter $\delta > 0$, $H(jw - \alpha) + D + K^{-1}$ satisfies it with parameter $\delta^*, \delta > \delta^* > 0$. Hence

$$\int_0^T \{N(t) - K^{-1} f[N(t)]\}' f[N(t)] \, e^{2\alpha t} dt \leq \frac{1}{4\delta^*} \int_0^T e^{2\alpha t} \, |Z(t) + r(t)|^2 \, dt$$

Let $k_{max} = \max_i k_i$ and $\epsilon_{min} = \min_i \epsilon_i$. Since $\{N(t) - K^{-1}f[N(t)]\}'f[N(t)] \geq \frac{1}{k_{max}} \epsilon_{min}^2 N'(t)N(t)$

$$\int_0^T N'(t)\,N(t)\,e^{2\alpha t}\,dt \leq \frac{k_{max}}{4\epsilon_{min}^2\,\delta^*} \int_0^T e^{2\alpha t}|Z(t) + r(t)|^2\,dt \qquad (3\text{-}13)$$

We will use the subscript (i) to denote the $\underline{i}$th row of a matrix. From (3-12)

$$|N(t) + Df[N(t)]|_{(i)} \leq |Z_{(i)}(t) + r_{(i)}(t)| + \left|\int_0^t e^{\alpha(t-\tau)}\,\Psi_{(i)}(t-\tau)e^{-\alpha(t-\tau)}f[N(\tau)]\,d\tau\right|$$

Let $|N(t) + Df[N(t)]|_{(i)} = |\sigma_i(t)|$. By the Schwartz inequality,

$$|\sigma_i(t)| \leq |Z_{(i)}(t) + r_{(i)}(t)|$$
$$+ \left[\int_0^\infty e^{2\alpha x}\,|\Psi_{(i)}(x)|^2\,dx\right]^{\frac{1}{2}} \epsilon^{-\alpha t}\left[k_{max}\int_0^t e^{2\alpha t}\,N'(\tau)\,N(\tau)\,d\tau\right]^{\frac{1}{2}}$$

Then, using (3-13)

$$|\sigma_i(t)| \leq |Z_{(i)}(t) + r_{(i)}(t)|$$
$$+ \left[\int_0^\infty e^{2\alpha x}\,|\Psi_{(i)}|^2\,dx\right]^{\frac{1}{2}} M \left[\int_0^t e^{-2\alpha(t-\tau)}\,|Z(t) + r(t)|^2\,dt\right]^{\frac{1}{2}}$$

The first integrand is an L_{2N} function, and the second integral is a convolution of a strictly stable function and a bounded one. Thus each term is bounded and so is $|N(t) + Df[N(t)]|$. But $N'(t)N(t) \leq [N(t) + Df[N(t)]]' \cdot [N(t) + Df[N(t)]]$ which is bounded implies $N(t)$ bounded. From (3-1), $\dot{x} = Ax - Bf[N(t)] + B_1 u(t)$. Thus $x(t)$ is bounded since this is a strictly stable linear differential equation with a bounded forcing term. Therefore, the theorem is proved.

Based on the proof of Theorem 1, it is now possible to interpret the meaning of the sector boundary ϵ for the nonlinear resistors. In all cases with $\epsilon > 0$, ϵ may be arbitrarily small. Suppose we follow the procedure of Theorem 1 in order to find a proper normal tree. It may be that there are nonlinear resistors which need not be represented as the series on parallel combination of an $\frac{\epsilon}{2}$ ohm or mho resistor as shown in Fig. 1. Instead, we can represent these elements as the series or parallel combination of $R = -\epsilon/2$ ohm or $G = -\frac{\epsilon}{2}$ mho linear resistors and nonlinear resistors with characteristics $-f(I) - \frac{\epsilon}{2} I$ or $-f(V) - \frac{\epsilon}{2} V$, respectively. For ϵ small enough all statements made in the proof still hold and for these newly modified resistors, the Popov sector need be only $[0, \infty)$. That is, any nonlinear characteristic lying in the first and third quadrants and passing through the origin of the I-V plane will not alter the conclusions of the above theorems. The only resistors for which it is necessary to require $\epsilon > 0$ are those "facing" resonant circuits, V-controlled resistors in V-controlled resistor cut sets or in parallel with an inductor, and I-controlled resistors in I-controlled resistor circuits or in series with a capacitor. If $\epsilon = 0$ for a

V-controlled resistor in parallel with a capacitor or an I-controlled resistor in series with an inductor, it is impossible to write the dynamic equations in canonical form. This is illustrated by the network shown in Fig. 3 taken from Stern [9]. Also, for a non-monotonic non-linear curve there is not a unique solution to the network equations. Suppose a V-controlled resistor is in a V-controlled resistor cut set or an I-controlled resistor is in an I-controlled resistor circuit. If $\epsilon = 0$, each such V-controlled resistor may be replaced by an open circuit or each such I-controlled resistor may be replaced by a short circuit. The network response may then sustain oscillations or constant terms and no guarantee about stability can be given. Such a situation is illustrated in Fig. 3b.

Another interesting case arises when the nonlinear characteristics are bounded. As long as such elements are not situated so as to require the representation described in Theorem 1, and they also satisfy $\langle x, f(x) \rangle > 0$, then the conclusions of Theorems 1 and 2 can be shown to hold.

<u>Corollary 1:</u> Let $\mathcal{N}$ be a network satisfying the hypotheses of Theorems 1 and 2. Suppose there exists a positive number f_{max} such that for all nonlinear resistor characteristics, $|f(x)| \leq f_{max}$ and $\langle x, f(x) \rangle > 0$. If no nonlinear resistor is situated so as to require the representation described in Lemma 1, then the conclusions of Theorems 1 and 2 hold.

Proof: From (3-1)

$$||N(T)|| \leq ||C\,\Phi(t)x_0|| + || \int_0^t \Psi(t-\tau)f[N(\tau)]\,d\tau|| + ||Df[N(t)]\,||$$

and since each term on the right-hand side is bounded, $||N(T)|| \leq N_{max}$ where N_{max} is a constant which depends only on the initial conditions. Hence, given the initial conditions, N(t) will be the same as it would be if the nonlinear characteristics were given by

$$\tilde{f}_i(x_i) = \begin{cases} f_i(x_i) & x_i < N_{max} \\ \dfrac{f_i(N_{max})}{N_{max}}\,x_i & x_i > N_{max} \end{cases}$$

and $\tilde{f}(x)$ lies in a Popov sector with lower bound

$$\epsilon = \frac{\min\limits_{x<N_{max}} f(x)}{N_{max}}$$

Since the stability behavior is the same whether the nonlinear characteristics are f(x) or $\tilde{f}(x)$, the analyses of Theorems 1 and 2 are now valid.

IV. Nonlinear Inductors and Capacitors

For the networks to be considered next, all resistors are linear, capacitors are linear or charge controlled, and inductors are linear or flux controlled. Coupling will be allowed between linear inductors or between linear capacitors only. For all nonlinear characteristices $\epsilon_i > 0$ and k_i will be a finite (possibly arbitrarily large) positive number. As in Sec. III, the dynamic equations are first put into a canonical form. In this section, $N \triangleq \begin{pmatrix} q \\ \psi \end{pmatrix}$, where q is the vector of charges on the nonlinear capacitors and ψ is the vector of fluxes in the nonlinear inductors. As before, $R \triangleq -f(N)$. Now, the components of R represent voltages on charge controlled capacitors or currents in flux controlled inductors.

Lemma 5: Let $\mathcal{N}$ be a network of linear R's, L's and C's with coupling between L's and between C's, and nonlinear flux controlled inductors and charge controlled capacitors in Popov sectors. Then there exists an equivalent network for which the dynamic equations may be written as

$$\begin{aligned} \dot{x} &= Ax + BR + B_1 u \\ \dot{N} &= Cx + DR + D_1 u \end{aligned} \tag{4-1}$$

where A, B, B_1, C, D, D_1 are appropriately defined matrices.

__Proof__: Pick any normal tree. Represent each nonlinear capacitor which is a __link__, by the series combination of an $\frac{\epsilon}{2}$ farad linear capacitor and a nonlinear charge controlled capacitor with characteristic $V = -f(q) - \frac{\epsilon}{2} q$ (see Fig. 4a). Represent each __branch__ nonlinear inductor by the parallel combination of a $\frac{2}{\epsilon}$ henry linear inductor and a flux controlled nonlinear inductor with characteristic $I = -f(\psi) - \frac{\epsilon}{2} \psi$ (see Fig. 4b). Then the original normal tree with the nonlinear inductors replaced by the $\frac{2}{\epsilon}$ linear inductors and with the nonlinear capacitors added is a normal tree of the network. For this network the fundamental loop and cut set equations are of the form

$$A_1 x + B_1 \dot{x} + C_1 V_R + D_1 \dot{\psi} = E_1 f(q) + F_1 V_{in} \tag{4-2}$$

$$A_2 x + B_2 \dot{x} + C_2 V_R + D_2 \dot{q} = E_2 f(\psi) + F_2 I_{in} \tag{4-3}$$

where the vector x contains the charges and fluxes of all (linear) C's and L's respectively. It may be assumed that variables corresponding to charges on link capacitors and fluxes in branch inductances, being linear combinations of other variables, have already been eliminated. Then using the same arguments as in Lemma 1, it is possible to solve uniquely for V_R, $\dot{\psi}$ and $\dot{q}$ in terms of the other variables. Making this substitution in (4-2) and (4-3), the equations take the general form

$$A_3 x + B_2 \dot{x} = E_3 f(N) + F_3 u$$

which upon algebraic manipulation can be reduced to the form

$$\dot{x} = Ax + BR + B_1 u$$

Substituting this into the expressions for $\dot{\psi}$ and $\dot{q}$ gives

$$\dot{N} = Cx + DR + D_1 u$$

Equation (4-1) may be interpreted from a systems approach with the aid of Fig. 5a. The block labeled N. L. receives the vector N(t) as an input, produces the vector f[N(t)] as an output, and feeds this back to the input of the linear plant as shown. For this system we define the following functions G(s) and $G_u(s)$, which may be calculated by means of (4-1), by

$$\dot{N}(s) = G(s)\, f[N(s)] \Big|_{u=0} \qquad \dot{N}(s) = G_u(s) u(s) \Big|_{f[N] = 0} \tag{4-4}$$

Refer now to Fig. 5b where each component of N(t) has been fed back around the linear plant and fed forward around the nonlinearity. Clearly this produces no change in N(t), so in Fig. 5c a new linear plant (the old one with feedback) and new nonlinearities has been drawn.

The new nonlinear characteristics are given by $f[N(t)] - \frac{\epsilon}{4} N(t)$ and will be labeled $g[N(t)]$. In the frequency domain, the system equations become

$$N(s) = \frac{G(s)}{s} [g[N(s)] - \epsilon N(s)] + \frac{G_u(s)}{s} u(s)$$

Then, if $G(s) = [g_{ij}(s)]$, $H(s) = [h_{ij}(s)] \triangleq [sI + \epsilon G(s)]^{-1} G(s)$, and

$H_u(s) \triangleq [I + \epsilon \frac{G(s)}{s}]^{-1} \frac{G_u(s)}{s}$,

$$N(s) = [I + \epsilon \frac{G(s)}{s}]^{-1} \frac{G(s)}{s} g[N](s) + [I + \epsilon \frac{G(s)}{s}]^{-1} \frac{G_u(s)}{s} u(s)$$

$$= [sI + \epsilon G(s)]^{-1} G(s) g[N](s) + [sI + \epsilon G(s)]^{-1} G_u(s) U(s)$$

$$N(s) = H(s) g[N](s) + H_u(s) u(s) \tag{4-5}$$

The following properties of $H(s)$ are of interest:

(1) Since $g_{ij}(s)$ is a proper rational fraction, $h_{ij}(s)$ is strictly proper.

(2) $H(0) = (\epsilon I)^{-1}$ so there will be no pole of $H(s)$ at $s = 0$.

(3) At a pole of $g_{ij}(s)$,

$$H(s) = [sI + \epsilon G(s)]^{-1} G(s) = [sG^{-1}(s) + \epsilon I]^{-1}$$

Hence the poles of $H(s)$ are poles of $[sI + \epsilon G(s)]^{-1}$ which are zeros of $\det[sI + \epsilon G(s)]$.

(4) $[(\sigma + jw) I + \epsilon G(s)]^{-1} = \{[\sigma I + \epsilon \operatorname{Re} G(s)] + j[wI + \epsilon \operatorname{Im} G(s)]\}^{-1}$

Since $\operatorname{Re} G(s)$ is p.s.d., for $\sigma > 0$, $[\sigma I + \epsilon \operatorname{Re} G(s)]$ is p.d.

It then follows [12] that

$$|\det[(\sigma + jw) I + \epsilon G(s)]| \geq |\det[\sigma I + \epsilon \operatorname{Re} G(s)]| > 0$$

Thus $H(s)$ has no poles for $\sigma > 0$

(5) $H(jw) = \{\epsilon \operatorname{Re} G(jw) + j[wI + \epsilon \operatorname{Im} G(jw)]\}^{-1}$. Then for all w for which $\operatorname{Re} G(jw)$ is p.d., there can be no pole of $G(jw)$ as in remark 4. If $\det[\operatorname{Re} G(jw_0)] = 0$, there exists a linear dependence among the rows of $\operatorname{Re} G(jw_0)$. Hence, $\det[H(jw_0)] = 0$ only if $\det[w_0 I + \epsilon \operatorname{Im} G(jw_0)] = 0$. But for any w_0 except arbitrarily near a pole of $G(jw)$, $\operatorname{Im} G(jw)$ is bounded. Let this arbitrary distance be α and define $S = \{w: |w - w_0| < \alpha\}$. Then ϵ can be chosen small enough to insure $\det[w_0 I + \epsilon G(jw_0)] > 0$, and this distance, α, can be chosen small enough to insure that $\operatorname{Re} G(jw_0)$ is p.d. for w within S. (The exception to this statement is if $\operatorname{Re} G(jw) \equiv 0$, i.e., all poles and zeros of $G(s)$ on the jw-axis, or if $G(jw) \equiv 0$). Thus all entries of $H(s)$ are proper rational fractions with poles in the open left half plane. The general solution to Eq. (4-1) can then be written as

$$N(t) = N_0(t) - \int_0^t h(t - \tau)\, g[N(\tau)]\, d\tau + \int_0^t h_u(t - \tau)\, u(\tau)\, d\tau \qquad (4\text{-}6)$$

where $N_0(t) \in L_{2N}$ since all observable modes are strictly stable.

$$h(t) = \mathcal{L}^{-1} H(s) \in K_{2N}$$

and

$$h_u(t) = \mathcal{L}^{-1} H_u(s)$$

The zero input case is considered first.

Theorem 3: Let $\mathcal{N}$ be a network of linear, positive R's, L's, and C's, with realistic coupling, and nonlinear charge controlled capacitors and flux controlled inductors in Popov sectors. Suppose the nonlinear capacitors and inductors are replaced with linear capacitors and inductors, respectively. Let $\{C_1, C_2, \ldots, C_{n_3}, L_1, L_2, \ldots, L_{n_4}\}$ be this set of linear elements, and suppose that with these terminations the network oscillates. Then, if the upper bounds on the Popov sectors for the corresponding nonlinear capacitors are $k_i = C_i^{-1} (i = 1, \ldots, n_3)$ and for the nonlinear inductors are $k_j = L_j^{-1} (j = 1, 2, \ldots, n_4)$, $N(t)$ and $f[N(t)]$ are bounded, L_{2N} functions which go to zero as $t \to \infty$. Furthermore, $x(t)$ and $\dot{N}(t)$ are bounded and if all eigenvalues of A have negative real parts, both go to zero as $t \to \infty$.

Remark: If no set of linear C's and L's can make the circuit oscillate, all k_i can become arbitrarily large. An alternative condition for this is that whenever $\operatorname{Re} G(jw)$ is only p.s.d., $\operatorname{Im} G(jw)$ is at least p.s.d. The proof of Theorem 3 will follow from the next lemma.

Lemma 5: For the network described in the hypothesis of Theorem 3, if there exist $\delta, \mu > 0$ and a p.s.d. K^{-1} such that $\Lambda\{P + P^*\} \geq \delta > 0$ where $P(jw) = [I + jw\mu]\, H(jw) + K^{-1}$, then the conclusions of Theorem 3 hold.

Proof: For $u \equiv 0$, (4-6) becomes

$$N(t) = N_0(t) - \int_0^t h(t - \tau)\, g[N(\tau)]\, d\tau \qquad (4\text{-}7)$$

This equation is the standard vector form to which the Popov criterion can be applied, and meets the usual restrictions on $N_0(t)$, $h(t)$ and the nonlinearities [13]. Thus, defining $P(jw) = [I + jw\mu] H(jw) + K^{-1}$, if $\Lambda\{P + P^*\} \geq \delta > 0$ then $N(t)$, and hence $f[N(t)]$ are bounded, L_{2N} functions which go to zero as $t \to \infty$. Now, let us return to the original circuit (as shown in Fig. 4a). From Eq. (4-1), since $x(t)$ is the solution of a stable differential equation with an L_{2N} driving term, $x(t)$ is stable and if the roots of its characteristic equation all have negative real parts, $x(t) \to 0$ as $t \to \infty$. Also, (4-6) shows that $N(t)$ is continuous since $g[N(t)]$ is bounded. This implies that $N(t)$ is also bounded. Then from Eq. (4-1) it is clear that $N(t)$ goes to zero as $t \to \infty$ if $x(t)$ does and this will be true if all eigenvalues of A have negative real parts.

<u>Proof of Theorem 3:</u> Based on Lemma 5, it is only necessary to check the condition $\Lambda\{P + P^*\} \geq \delta > 0$. By definition,

$$(P + P^*) = [\operatorname{Re}(H + H') - w\mu \operatorname{Im}(H + H') + 2K^{-1}]$$

$$+ j\,[w\mu \operatorname{Re}(H - H') + \operatorname{Im}(H - H')]$$

where $$H(jw) = [jwI + \epsilon\, G(jw)]^{-1} G(jw)$$

Let $M = w\mu \operatorname{Re}(H - H') + \operatorname{Im}(H - H')$. Then $M = -M'$, and $\langle Mx, x\rangle \equiv 0$.

Thus, $\Lambda\{P + P^*\} \geq \delta > 0$ if

$$\frac{1}{2} \operatorname{Re}(H + H') - \frac{1}{2} w\mu \operatorname{Im}(H + H') + K^{-1}$$

is positive definite. Since

$$H(jw) = \left\{\epsilon \operatorname{Re} G(jw) + j[wI + \epsilon \operatorname{Im} G(jw)]\right\}^{-1} [\operatorname{Re} G(jw) + j \operatorname{Im} G(jw)]$$

$$= \{(\epsilon \operatorname{Re} G)(\epsilon \operatorname{Re} G)' + [wI + \epsilon \operatorname{Im} G][wI + \epsilon \operatorname{Im} G]'\}^{-1}$$

$$\cdot \{\epsilon[(\operatorname{Re} G)(\operatorname{Re} G)' + (\operatorname{Im} G)(\operatorname{Im} G)'] + w \operatorname{Im} G$$

$$+ j[\epsilon[(\operatorname{Re} G)' \operatorname{Im} G + (\operatorname{Re} G' \operatorname{Im} G)'] - w \operatorname{Re} G\}$$

Then

$$\frac{1}{2}[H(jw) + H'(jw)] = \{(\epsilon \operatorname{Re} G)(\epsilon \operatorname{Re} G)' + [wI + \epsilon \operatorname{Im} G][wI + \epsilon \operatorname{Im} G]'\}^{-1}$$

$$\cdot \{\epsilon [\operatorname{Re} G \operatorname{Re} G' + \operatorname{Im} G \operatorname{Im} G'] + \frac{1}{2} w[\operatorname{Im}(G + G')]$$

$$- jw \frac{1}{2} \operatorname{Re}(G + G')\}$$

$$= \{I + \frac{\epsilon}{w^2} [w \operatorname{Im}(G + G') + \epsilon(\operatorname{Im} G \operatorname{Im} G' + \operatorname{Re} G \operatorname{Re} G')]\}^{-1}$$

$$\cdot \{\frac{\epsilon}{w^2} [\operatorname{Re} G \operatorname{Re} G' + \operatorname{Im} G \operatorname{Im} G'] + \frac{1}{2w}[\operatorname{Im}(G + G')]$$

$$- j \frac{1}{2w} \operatorname{Re}(G + G')\}$$

Finally

$$\frac{1}{2}\,\mathrm{Re}(H + H') - \frac{1}{2} w\mu\, \mathrm{Im}(H + H') + K^{-1}$$

$$= (I - \epsilon M_1)^{-1} \{(I + \epsilon K^{-1}) \frac{\epsilon}{w^2}[\mathrm{Re}\,G\,\mathrm{Re}\,G' + \mathrm{Im}\,G\,\mathrm{Im}\,G'] + \frac{1}{2}\mu\,\mathrm{Re}(G + G')$$

$$+ \frac{1}{2w}[\mathrm{Im}[(G + K^{-1}) + (G + K^{-1})']]\}$$

$$= (I - \epsilon M_1)^{-1} M_2$$

Suppose $\mathrm{Re}(G + G')$ is p.d. Then for w anywhere except arbitrarily close to a pole, $G(jw)$ is bounded. Therefore, for μ large enough M_2 is p.d. and M_1 is bounded. Then choosing ϵ small enough so that the magnitude of all the eigenvalues of ϵM_1 (denoted by $\epsilon\lambda_i$) are less than 1,

$$(I - \epsilon M_1)^{-1} M_2 = (I + (\epsilon M_1) + (\epsilon M_2)^2 + \ldots,) M_2$$

$$= M_2 + \epsilon M_1 M_2 + (\epsilon M_1)^2 M_2 + \ldots$$

Since M_2 is p.d., there exists a nonsingular transformation V such that if $X = VY$ then

$$X'(M_2 + \epsilon M_1 M_2 + \ldots) X = \sum_{i=1}^{n} (1 + \epsilon\lambda_i + \ldots) y_i^2 = \sum_{i=1}^{n} \frac{1}{1 - \epsilon\lambda_i} y_i^2 > 0$$

and $(I - \epsilon M_1)^{-1} M_2$ is p.d. At a pole of $G(jw)$, $\frac{1}{2}(H + H')$ is positive definite since H has no poles on the jw-axis. Hence, arbitrarily close to a pole of $G(jw)$ the continuity of $H(jw)$ implies $\Lambda\{P + P^*\} \geq \delta > 0$. For $Re(G + G')$ only p.s.d., the first two terms of M_2 are p.s.d. Then for any K^{-1} such that $Im[(G + K^{-1})']$ is p.d., M_2 is positive definite, and by the same argument as above so is $(I - \epsilon M_1)^{-1} M_2$. Thus, if $Im(G + G')$ is p.s.d. whenever $Re[G(jw) + G'(jw)]$ is only p.s.d., $\Lambda\{P + P^*\} \geq \delta > 0$ for arbitrarily large k_i. For $Im(G + G')$ not p.s.d. but $Im(G + G' + 2K^{-1})$ p.s.d., replace the ith nonlinear capacitor by a linear capacitor of capacitor $1/k_i$ and replace the jth nonlinear inductor by a linear inductor of inductance $1/k_j$ (for all i, j). For this circuit $\tilde{G}(jw) = Re\,G(jw) + j\,Im[G(jw) + K^{-1}]$ and $\tilde{G}(jw_0)$ is p.s.d. Therefore, $\tilde{G}^{-1}(jw)$ has a pole at $w = w_0$. Consequently, the characteristic equation for the linear circuit has a root at $s = \pm jw_0$ so there exist initial conditions for which oscillations can occur.

To illustrate the meaning of Theorem 3, the circuit shown in Fig. 6a will be analyzed to determine the largest value of k for which V(t) can be guaranteed to be an L_2 function. For this circuit

$$G(s) = y(s) = \frac{2s(s^2 + s + 1)}{2s^4 + 4s^3 + 3s^2 + 3s + 1}$$

is the input admittance at the capacitor terminals. Since there is only one nonlinear element, $\Lambda\{P + P^*\} \geq \delta > 0$ reduces to

$$Re\{[I + jw\mu]\ H(jw) + \frac{1}{k}\} \geq \delta > 0$$

or

$$\operatorname{Re} H(jw) - \mu \, w \operatorname{Im} H(jw) + \frac{1}{k} \geq \delta > 0 \tag{4-8}$$

From the plot of $\operatorname{Re} H(jw)$ vs $w \operatorname{Im} H(jw)$ shown in Fig. 6b, it is clear that (4-8) can be satisfied for $\frac{1}{k} > 2$.

Now replace the nonlinear capacitor with a linear 2 farad capacitor. Writing the state equations in the usual manner for this network [14], it is found that the characteristic polynomial of the "A" matrix is $s(s^2 + 1)(s + 1)^2$. Thus there are initial conditions which cause oscillations of frequency j1. In fact if a charge q_0 is placed on the linear capacitor which replaced the nonlinear one, the voltage across its terminals contains the term $q_0 \sin(t)$. Clearly it would be futile to hope for asymptotic stability for a nonlinear sector $[\epsilon, k - \epsilon]$ for any $k > 2$.

It is useful to know that for the circuits described in Theorem 3 that if a bounded input is applied, all variables will remain bounded as long as the impedance at the terminals of each current source and the admittance at the terminals of each voltage source contains no poles on the jw-axis.

Theorem 4: Let $\mathcal{N}$ be a network of linear, positive R's, L's, and C's with realistic coupling, nonlinear charge controlled capacitors and flux controlled inductors in Popov sectors and voltage and current sources. Then, if the driving point matrix with respect to the input ports, with all nonlinear inductors removed and nonlinear capacitors shorted, has no poles on the jw-axis, all variables are bounded.

Proof: By the discussion before Theorem 3 and the statement of Theorem 4, $H_u(s)$ has all roots in the open left half plane. Then

$$N(t) = N_0(t) - \int_0^t h(t-\tau)\, g[N(\tau)]\, d\tau + \int_0^t h_u(t-\tau)\, u(\tau)\, d\tau .$$

where $N_0(t) \in L_{2N}$

$h(t) = \mathcal{L}^{-1}[H(s)] \in K_{2N}$

$h_u(t) = \mathcal{L}^{-1}[H_u(s)] \in K_{2N}$

The rest of the proof is the same as that of Theorem 2 except that here it is directly shown that $N(t)$ is bounded.

V. Generalization

When we combine the results of Secs. III and IV to determine the stability of circuits containing nonlinear R's, L's and C's, we find that while the mathematical results are still valid, it is no longer simple to physically interpret these results. To simplify notation, in this section we make the following definitions:

$$N_1 = \begin{pmatrix} q \\ \psi \end{pmatrix} \quad N_2 = \begin{pmatrix} V \\ I \end{pmatrix} \qquad R_1 = -f(N_1) \quad R_2 = -f(N_2)$$

Lemma 6: Let $\mathcal{N}$ be a network of linear R's, L's and C's with realistic coupling, and nonlinear R's, charge controlled C's and flux controlled L's in Popov sectors. Then, the dynamic equations may be written as

$$\begin{aligned} \dot{x} &= Ax + B_1 R_1 + B_2 R_2 + B_3 u \\ \dot{N}_1 &= C_1 x + D_{11} R_1 + D_{12} R_2 + D_{13} u \\ N_2 &= C_2 x + D_{21} R_1 + D_{22} R_2 + D_{23} u \end{aligned} \tag{5-1}$$

Proof: As in Lemmas 1 and 5 pick a normal tree and use the representations shown in Figs. 1 and 4 to I-controlled resistor links, V-controlled resistor branches, nonlinear capacitor links and nonlinear inductor branches. The loop and cut set equations are then of the form

$$A_1 x + B_1 \dot{x} + C_1 V_R + D_1 \dot{\psi} + E_1 V = G_1 f(I) + H_1 f(q) + F_1 V_{in}$$

$$A_2 x + B_2 \dot{x} + C_2 V_R + D_2 \dot{q} + E_2 I = G_2 f(V) + H_2 f(\psi) + F_2 I_{in}$$

The rest of the proof is the same as that of Lemmas 1 and 5.

To apply the Popov criterion to this network we make a transformation similar to the one discussed in Sec. IV and shown in Fig. 5. The original network is now represented in Fig. 7a and the transformed one in Figs. 7b and 7c. From Fig. 7a, a driving point matrix is defined by

$$\begin{pmatrix} \dot{N}_1(s) \\ N_2(s) \end{pmatrix} = \begin{pmatrix} G(s) & H_1(s) \\ H_2(s) & G_R(s) \end{pmatrix} \begin{pmatrix} R_1(s) \\ R_2(s) \end{pmatrix} \tag{5-2}$$

Then for Fig. 6b we find that

$$\begin{pmatrix} N_1(s) \\ N_2(s) \end{pmatrix} = \begin{pmatrix} T(s)\,G(s) & T(s)\,H_1(s) \\ H_2(s)[I - \epsilon\, T(s)\,G(s)] & -H_2(s)\,\epsilon\, T(s)\,H_1(s) + G_R(s) \end{pmatrix} \begin{pmatrix} R_1(s) \\ R_2(s) \end{pmatrix} \tag{5-3}$$

where $T(s) = [sI + \epsilon G(s)]^{-1}$.

Note that the matrix $G_R(s)$ is equal to the matrix $H(s) + D$ obtained for the nonlinear resistor case if all nonlinear inductors and capacitors are removed, and $G(s)$ is the same as for the nonlinear LC case if all nonlinear resistors are removed. Thus by the same reasoning as in Secs. III and IV, if we represent Eq. (5-3) as

$$N(s) = M(s)\, R(s) \tag{5-4}$$

$N(s)$ has no poles on the jw-axis or in the right half plane. The main result of this section gives a sufficient condition for L_2 stability and appears to be the most general result that can be obtained directly by the Popov approach for the nonlinear RLC case.

Theorem 5: Let $\mathcal{N}$ be a network containing linear R's, L's and C's with realistic coupling, and nonlinear R's, charge controlled C's and flux controlled L's in Popov sectors. If there exists $\delta, \mu > 0$ such that $\Lambda\{(P + P^*)\} \geq \delta > 0$, where $P(jw) = (I + jw\mu)\, M(jw) + K^{-1}$, then nonlinear resistor voltages and currents, nonlinear capacitor charges and inductor fluxes are bounded, L_{2N} functions which go to zero as

$t \to \infty$; all other network variables are bounded and, if the eigenvalues of A have negative real parts, go to zero as $t \to \infty$.

Proof: The general solution of Eq. (5-4) is

$$N(t) = N_0(t) - \int_0^t m(t-\tau)\, g[N(\tau)]\, d\tau$$

where $\quad m(t) = \mathcal{L}^{-1}[M(s)] \in K_{2N}$

$$N_0(t) \in L_{2N}$$

The rest of the proof is the same as that of Theorem 3.

Unfortunately, no general conclusions about the network defined by Theorem 5 can be drawn from the theorem not only because of the complicated form of $M(jw)$, but also because all that is known about its submatrices is that $G(jw)$ and $G_R(jw)$ are p.s.d. We conclude with an extension to input-output stability.

Theorem 6: Let $\mathcal{N}$ be a network containing linear R's, L's and C's with realistic coupling, and nonlinear R's, charge controlled C's and flux controlled L's. Suppose the driving point matrix $M_u(s)$ at the input ports with all nonlinear inductors and V-controlled resistors removed and nonlinear capacitors and I-controlled resistors shorted has no poles on the jw-axis. If there exists $\delta, \mu > 0$ such that $\Lambda\{(P+P^*)\} \geq \delta > 0$ where $P(jw) = [(I + jw\mu)\, M(jw) + K^{-1}]$, then $N(t)$, $f[N(t)]$ and $x(t)$ are bounded.

Proof: The general solution of (5-4) is now

$$N(t) = N_0(t) - \int_0^t m(t - \tau)\, g[N(\tau)]\, d\tau + \int_0^t m_u(t - \tau)\, u(\tau)\, d\tau$$

where
$$m(t) = \mathcal{L}^{-1}[M(s)] \in K_{2N}$$
$$m_u(t) = \mathcal{L}^{-1}[M_u(s)] \in K_{2N}$$
$$N_0(t) = \in L_{2N}$$

The rest of the proof is the same as that of Theorem 4.

VI. Conclusions

The main results of this paper have been to demonstrate the existence of canonical forms for the dynamic equations of nonlinear RLC circuits and from these equations derive some stability criteria for nonlinearities which are neither necessarily monotone nor necessarily continuous. The former category includes devices like tunnel diodes while the latter covers relays with deadzone. It should be noted that all results hold if the nonlinear elements have time-varying characteristics as long as these remain within the Popov sector for all time. It can also be shown that in Theorems 3, 4, 5 and 6, ϵ_i may be zero for certain elements using the same type of arguments as following Theorem 2. The results of Secs. IV and V are valid for any system containing nonlinear elements with memory as long as their dynamic equations can be put in the general form of Eqs. (4-1) or (5-1), or more generally if their outputs can be expressed as a convolution of a stable function with their inputs.

The results presented here should be useful to the problem of designing circuits containing nonlinear elements. Because of the generality of Theorems 1-4, the designer, by simply noting the sector boundary, can take advantage of the properties of nonlinear devices without having to worry about producing unwanted oscillations.

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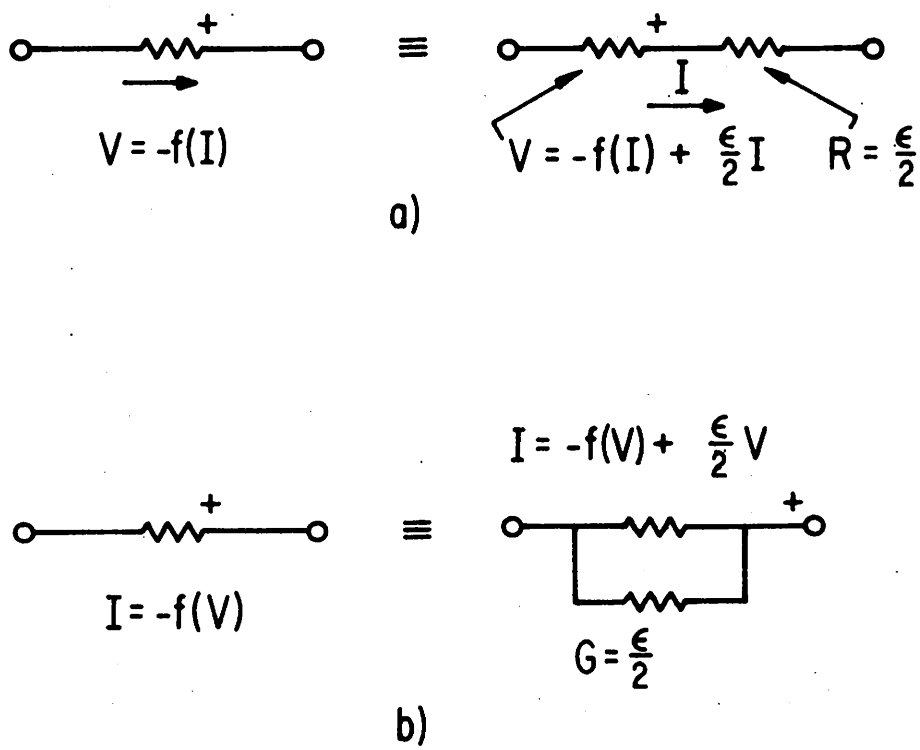


Fig. 1 Equivalent representation for nonlinear resistors.

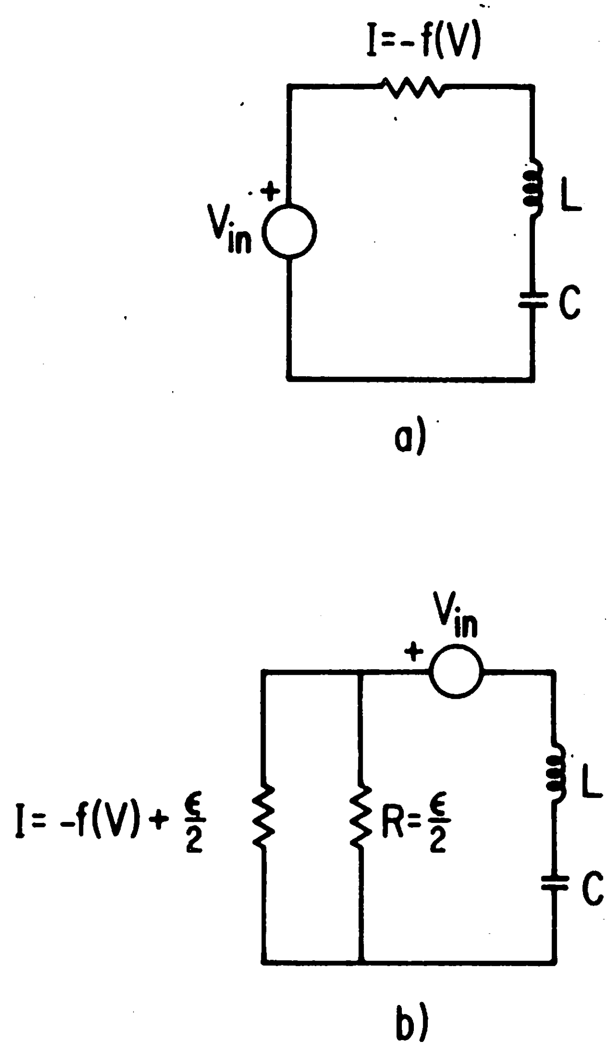


Fig. 2 Circuit analyzed in example of application of Lemmas 1 and 2.

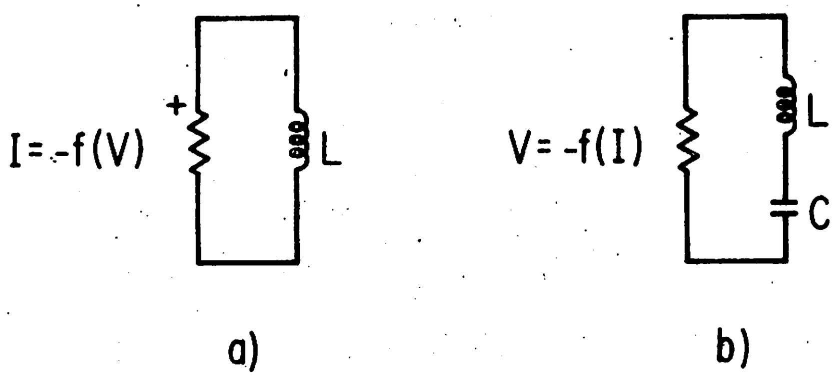


Fig. 3 Circuits for which no proper normal tree exists.

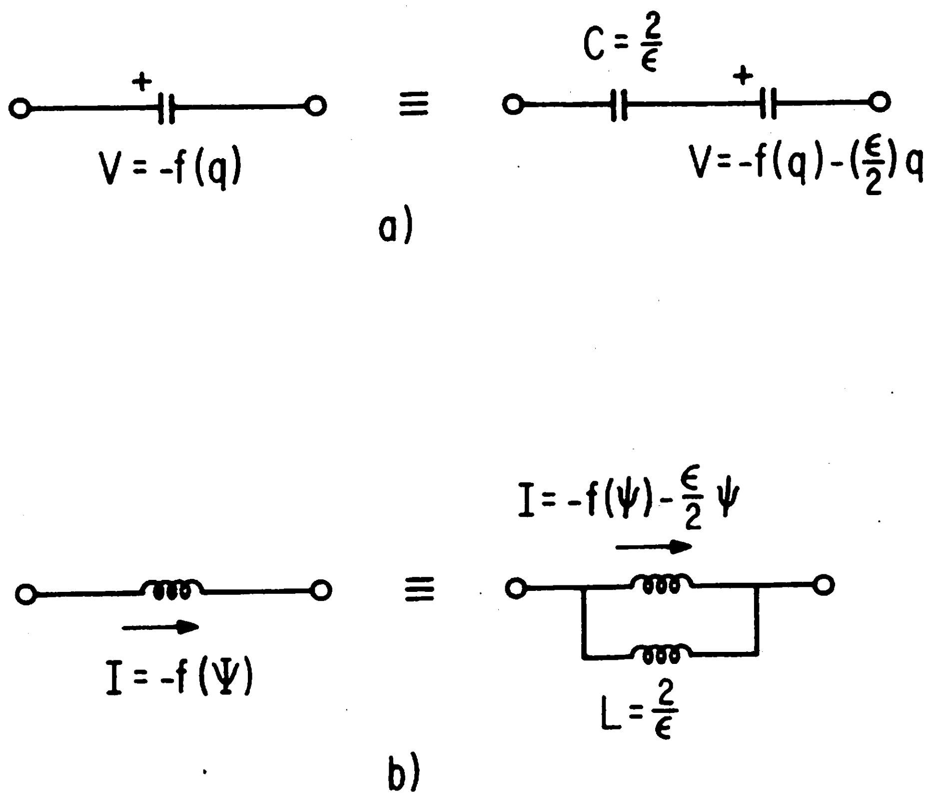
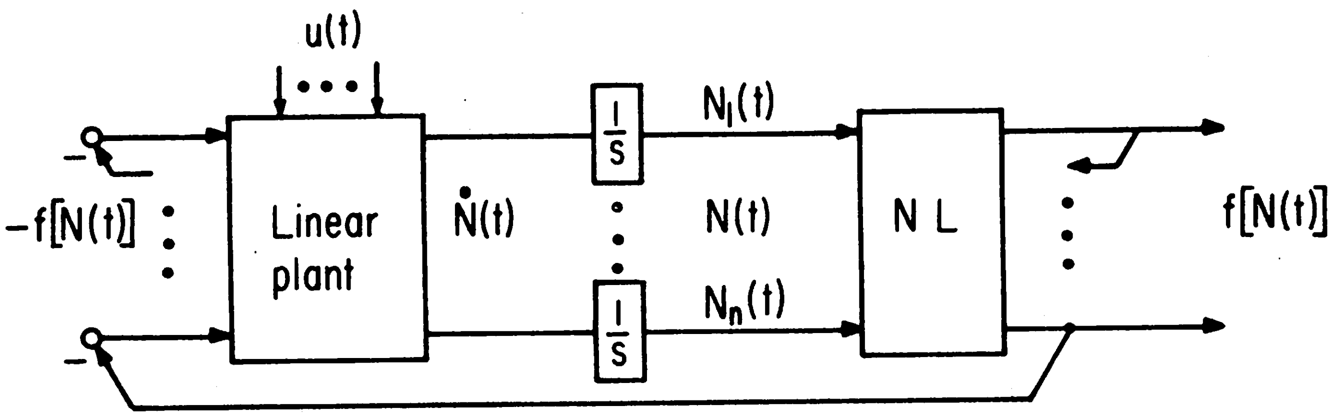
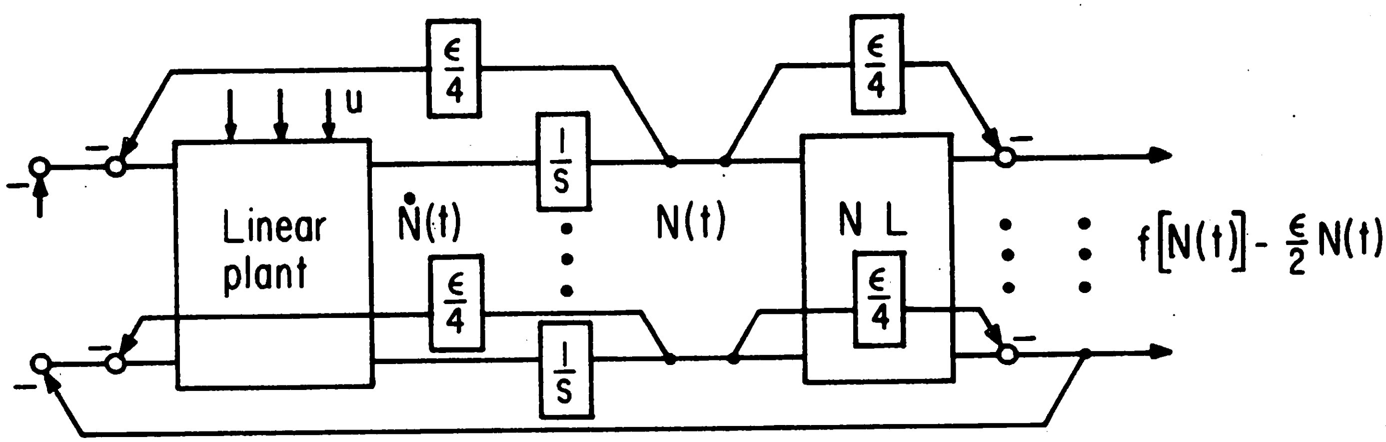


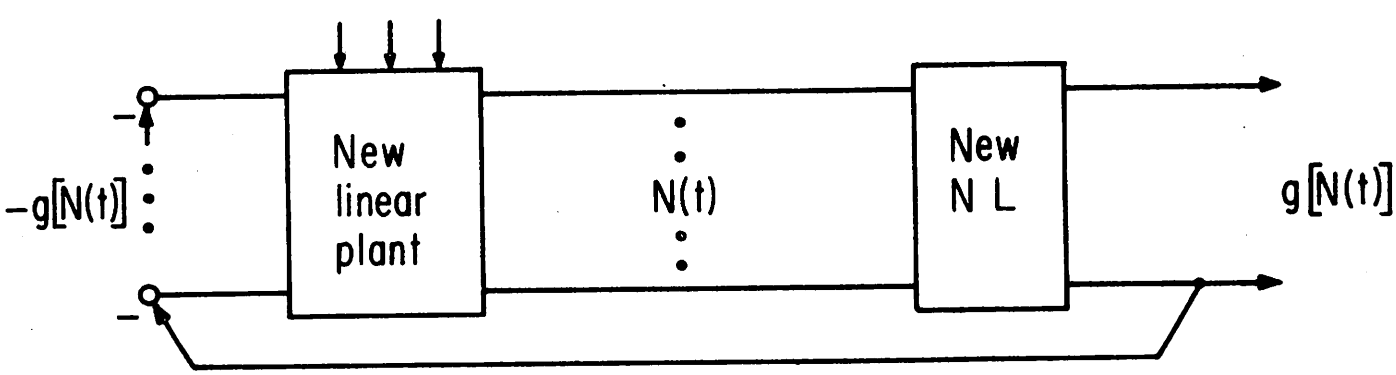
Fig. 4. Equivalent representation for nonlinear capacitors and inductors.



a)



b)



c)

Fig. 5 Representation of network containing nonlinear capacitors and inductors.

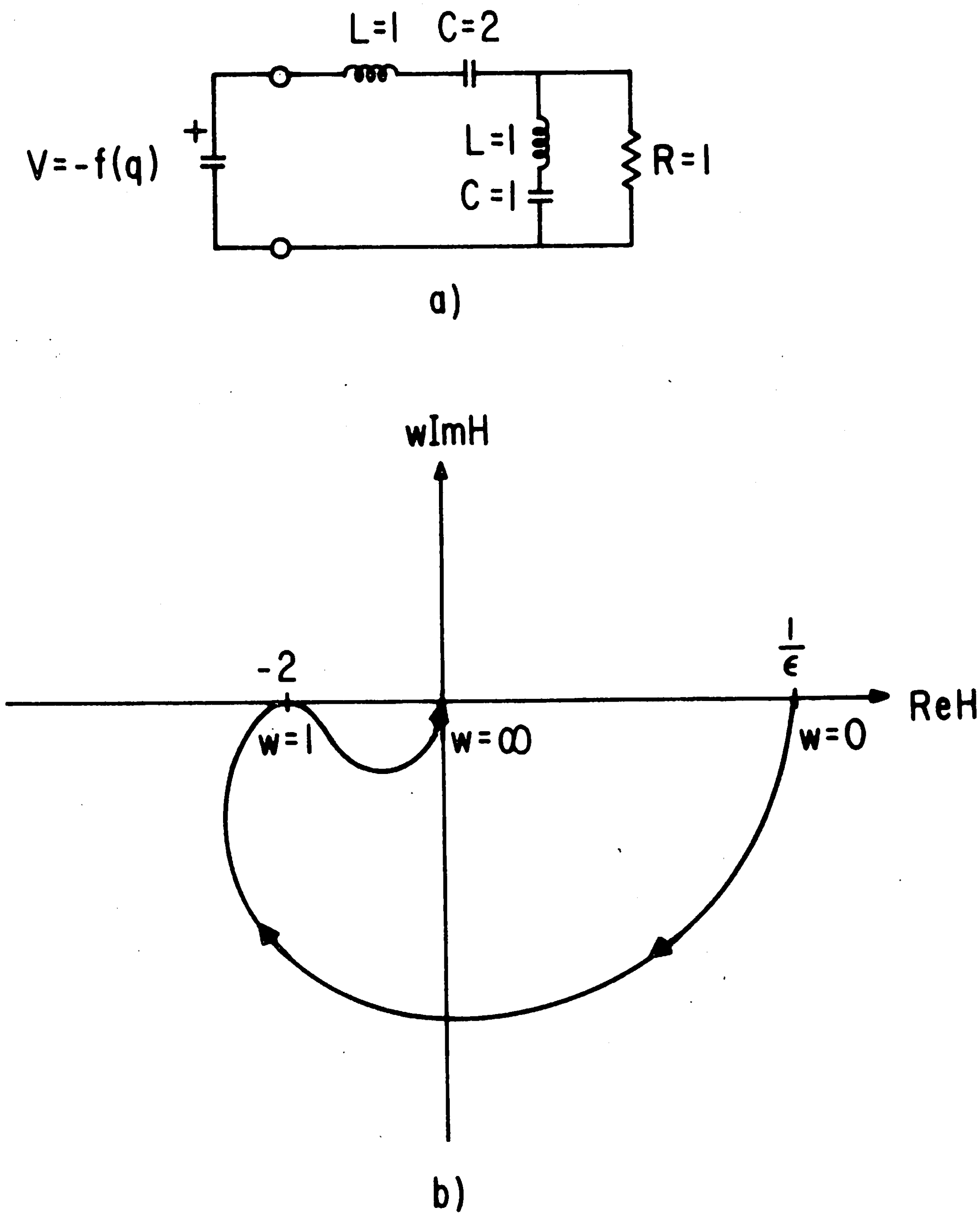


Fig. 6 Circuit with finite Popov sector boundary and its modified frequency response plot.

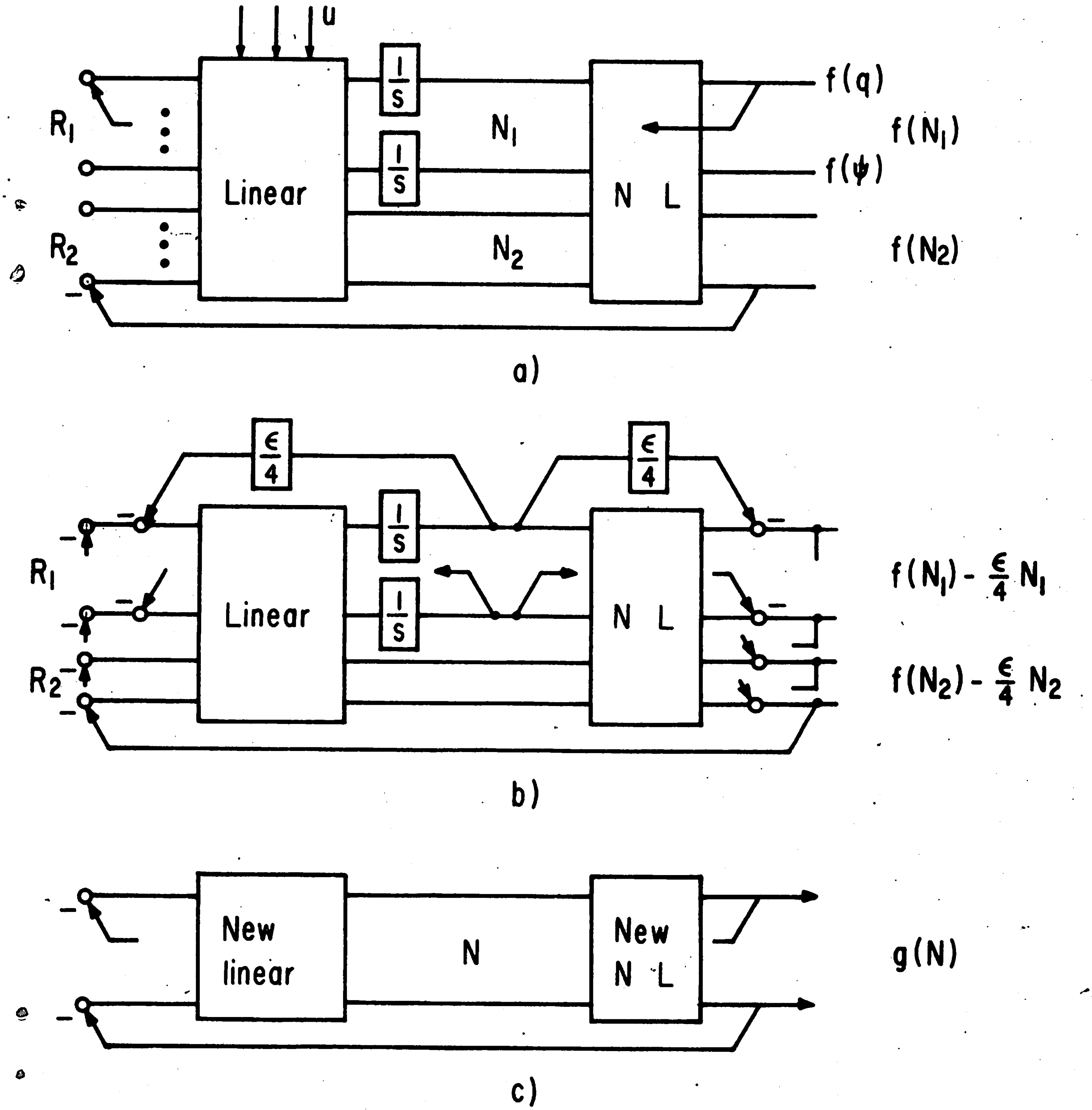


Fig. 7 Representation of a nonlinear RLC network.