STUDIES IN PURSUIT-EVASION DIFFERENTIAL GAMES

by

Noboru Sugino

Memorandum No. ERL-M 230

27 November 1967

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

The research reported herein was supported by the U. S. Army Research Office--Durham under Grant DA-ARO-D-31-124-G776 and Contract DAHCO4-67-C-0046.

TABLE OF CONTENTS

I. PRELIMINARIES 1

A. Introduction 1

B. Characterization of Dynamics 8

C. Attainability Sets and Escapability Sets 12

D. Rules of the Game 16

II. TIME-DISCRETE APPROXIMATIONS AND CONVERGENCE PROBLEMS 20

A. Time-discrete Approximations 21

B. Convergence Problems 26

C. Minimax Theorem 47

D. Time-continuous Game 63

III. CAPTURE CONDITIONS 71

A. General Capture Conditions 73

B. Sufficient Capture Conditions 83

IV. SUFFICIENT STRATEGIES 94

A. Motivations and Definition of Sufficient Strategies 95

B. Constructive Algorithms for Sufficient Pursuit Strategies 98

C. Existence of Sufficient Pursuit Strategies for Linear Systems 108

D. Some Examples 114

V. CONCLUSIONS 126

REFERENCES 128

I. PRELIMINARIES

A. INTRODUCTION

There are optimization problems in the control of dynamical systems for which the game-theoretic concept of an adversary is appropriate. A typical example of this is the pursuit-evasion problem. For these problems the classical approach to the theory of games due to von Neumann is inadequate because for dynamical systems a continuum of moves is possible. A reformulation of this problem will be done with great care in a later section. For now, the situation that we wish to consider can be roughly described as follows. Let t denote time and $z(t) \in E^s$ denote the state of the system at time t. There are two persons, called Player I and Player II, who can exert controls on the state of the system through the differential equation

$$\frac{dz(t)}{dt} = f(z(t),u(t),v(t),t)$$

where $u(t) \in E^m$ and $v(t) \in E^m$ characterize the controls of Player I and Player II, at time t, respectively. Let there be given an initial time t_o, an initial state z_o, and a subset T of $E^s \times [t_o,\infty)$, called the target set or the end zone. Let there also be given a payoff or a criterion function $J(u(\cdot),v(\cdot))$. At each time t, both players can completely observe the state z(t). The purpose of Player I (Player II) is to decide the value of his control, at each time t, based upon observation of the state z(t), such that the state of the system transfers from z_o to the target set T with the minimum (maximum) payoff. Since the choice of the value of controls is made based upon the value of the state z(t) observed at each instant of the game, these games are sometimes called the "closed-loop" (or feed-back) games. Thus, the game mentioned above is a class of two-person, zero-sum, infinite multimove games with perfect information. This will be called the general differential game.

If there is only one person (Player I or II) concerned with the system, the above problem can be reduced to the classical optimization problem. On the other hand, if observation of the state z(t) is made at discrete instants, the above problem can be treated by classical game theory, at least in theory.

Thus, differential game problems differ from classical optimization problems in that the latter contains only one control, whereas, the former contains two different sorts of controls under conflict situations. On the other hand, differential game problems also differ from classical game problems in that the latter is mainly concerned with discrete processes, whereas, the former is concerned with continuous processes. Thus, the first difficulty in differential games is the precise formulation of games with a continuum of moves.

A possible approach is to approximate the time-continuous game by a sequence of time-discrete games with observation time intervals $h>0$, and to consider the limit as h goes to zero.

W. H. Fleming [F2], [F3] successfully applied these techniques to a special class of general differential games, viz., differential games with integral payoffs and a fixed duration. In games of this type, the payoff to Player I is given by

$$J(u(\cdot),v(\cdot)) = g_o(z(t_1)) + \int_{t_o}^{t_1} g(z(t),u(t),v(t),t)dt$$

and the target set is

$$T_1 = \left\{ (z(t_1),t_1) \in E^{s+1} : z(t_1) \in E^s,\ t_1 \text{ is fixed} \right\}$$

Player I's (Player II's) objective is to decide the value of his control at each instant of the game, based upon observation of $z(t)$, so as to minimize (maximize) the payoff J. Games of this type will be called fixed duration games. He showed that J_h converges to a certain limit J_o and J_o coincides with the appropriately defined "value" of the time-continuous game. Moreover, the "minimax" theorem holds for these games [F3], [F4].

Another important class of differential games arises from the study of pursuit-evasion problems. In this case, Player I (Pursuer) pursues Player II (Evader) who is moving away from Pursuer. The game is considered terminated as soon as the Euclidean distance between the state x(t) of Pursuer and the state y(t) of Evader becomes less than or equal to some prescribed non-negative number ϵ. In this case, the target set T_2 is given by

$$T_2 = \left\{(z(t),t) \in E^{s+1} : z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \ \|x(t) - y(t)\| \leq \epsilon \atop t_o \leq t \right\}$$

and the payoff is the time up to the termination of the game. Pursuer (Evader) tries to choose the value of his control at each instant of the game, based upon observation of x(t) and y(t), so as to terminate the game in the shortest (longest) period of time. Games of this type will be called pursuit-evasion games.

Here, we encounter analogous difficulties in the precise formulation of these games with the additional difficulties that the duration of the pursuit-evasion game is not prescribed a priori and that the target set T_2 may not be attained. At present, there is no satisfactory general mathematical formulation of these games. Only some preliminary results have been obtained [H3], [K8], [P2]. It is these games that are studied extensively in this thesis.

Now, a general theory of differential games should answer the following questions:

(a) Do there exist admissible "strategies" for Player I which transfer the initial state z_o to the target set T against any admissible strategy of Player II?

(b) If such strategies for Player I exist, does there exist an "optimal" strategy?

(c) How can we construct the optimal strategy?

In a fixed duration game, the target set T_1 is always attained. However, in a pursuit-evasion game, the target set T_2 may or may not be attained. The conditions that T_2 is attained are called capture conditions and studied in Chapter III of this thesis.

Now, let us assume that the target set T can be attained. In this case, our next job is to select some "optimal" strategy pairs from those which are attainable to T. The only compromizing definition of the optimality which is satisfactory for both players under conflict situations is the von Neumann's saddle-point optimality. But, it is known that the saddle-point definition is valid if and only if the "minimax" theorem holds. The "minimax" theorem for the pursuit-evasion game, which is,in general, still an open question at present, is studied in Chapter II-C under some restrictive conditions.

Now, let us further assume that the target set T is attainable and that the "minimax" theorem holds. In this case, it has been conjectured that the saddle-point "optimal" strategies can be obtained by writing down formally a "modified" Hamilton-Jacobi-Bellman partial differential equation (this equation was called the "Main equation I" by R. Isaacs [I1]). However, this approach is restricted by many technical difficulties, above all by the fact that the domains of regularity (in which the partial derivative of the payoff is continuous) are, in general, extremely difficult to obtain. Moreover, at present, mathematical validity of this approach is not well established except for special cases [F5].

Another difficulty with the saddle-point "optimal" strategy is that it requires continuous observation of the states of both players. This is undesirable from the practical point of view (see Chapter IV).

In this thesis, instead of trying to find "optimal" strategies, we introduce the concept of "sufficient" strategies. A sufficient pursuit strategy as introduced in Chapter IV guarantees capture within some finite, but possibly not the shortest, period of time. This strategy requires neither the verification of the "minimax" theorem, nor continuous observation by Pursuer of the states. Constructive algorithms given in this thesis are geometric in nature and are straightforward. It will be seen that the existence of such strategies is closely related to the capture condition. By applying this method, we have obtained some results which subsume those obtained by L. S. Pontryagin [P2] and Y. C. Ho et al. [H3].

The organization of this thesis is as follows: In Chapter I, we introduce some notation and definitions which will be used in the thesis. The rules of the game are explained. In Chapter II, we formulate pursuit-evasion, time-continuous games by means of time-discrete approximations

(Section A). Time-discrete approximations used here are different from those used by W. F. Fleming [F2], [F3] and make our discussions simpler. In Section B, we prove that optimal capture times for approximating discrete games converge to a limit as observation intervals h go to zero. The discussions make essential use of geometrical attainability sets and iterative relations by dynamic programming. In Section C, we prove the minimax theorem under certain restrictive assumptions by applying attainability sets, dynamic programming, and the Kakutani's fixed point theorem. In Section D, the relation between the limit of discrete games and the continuous game is discussed. In Chapter III, capture and escape conditions for discrete games and continuous games are derived. These are closely related to algorithms for constructing sufficient strategies given in Chapter IV. In Chapter IV, a new concept of sufficient strategies is introduced. Algorithms for constructing these strategies are introduced. Existence theorems for sufficient strategies are given and some examples for L_p controls are shown. Lower dimensional projections of capture are discussed.

B. CHARACTERIZATION OF DYNAMICS

1. Dynamics of Pursuer and Evader

We shall consider a system (P) whose state at time t is described by a vector x(t) in an Euclidean space E^n, n=1,2,..., and whose control at time t is characterized by a vector u(t) in E^m, m=1,2,.... . We shall assume that the dynamics of this system (P) is described by a differential equation

$$\frac{dx(t)}{dt} = f(x(t),u(t),t) \tag{1:1}$$

where $f(\cdot,\cdot,\cdot)$ is a function from $E^n \times E^m \times E$ into E^n. This system (P) will be called Pursuer.

Remark 1: In this thesis, the following definitions of a mapping or a function are used [B4]. Let X and Y be two sets. Corresponding to each element x of X, if we associate a subset F(x) of Y, the correspondence x to F(x) will be called a mapping from X into Y. If the mapping $F(\cdot)$ from X into Y is such that the set F(x) always consists of a single element, $F(\cdot)$ will be called a single-valued function (or a single-valued mapping) from X into Y. Where no confusion is possible, single-valued functions from X into Y will be denoted by small Latin letters and called simply functions from X into Y. General or multi-valued mappings will be denoted by capital Latin letters.

Remark 2: Let $F(\cdot)$ be a mapping from X into Y. By $F(\cdot)$, we represent a mapping and by $F(x)$, $x \in X$, we represent a subset of Y. Similarly, let $f(\cdot)$ be a single-valued function from X into Y. By $f(\cdot)$, we represent a function and by $f(x)$, $x \in X$, we represent a point of Y.

Similarly, we shall consider a system (E) whose dynamics is described by a differential equation

$$\frac{dy(t)}{dt} = g(y(t),v(t),t) \tag{1.2}$$

where $y(t) \in E^n$ is a state, $v(t) \in E^m$ is a control, and $g(\cdot,\cdot,\cdot)$ is a function from $E^n \times E^m \times E$ into E^n. This system (E) will be called Evader.

2. Admissible controls

Let t_o be a real number called the initial time. Let U and V be non-empty, compact subsets of E^m. A measurable function $u(\cdot)$ from $[t_o,\infty)$ into U will be called a Pursuer's admissible control. The set of all Pursuer's admissible controls will be denoted by $\mathcal{U}$, i.e.,

$\mathcal{U} = \{u(\cdot) : u(\cdot)$ is a measurable function from $[t_o,\infty)$ into $U\}$.

A measurable function $v(\cdot)$ from $[t_o,\infty)$ into V will be called an Evader's admissible control. The set of all Evader's admissible controls will be denoted by $\mathcal{V}$. From this definition, we see that if $u_1(\cdot)$ is admissible, then for any vector $u \in U$ and any t_1, $t_o < t_1 < \infty$, the function $u_2(\cdot)$ defined by

$$u_2(t) = \begin{cases} u_1(t) & t_o \leq t < t_1 \\ u & t_1 \leq t < \infty \end{cases} \tag{1.3}$$

belongs to $\mathcal{U}$.

Similar fact holds for $\mathcal{V}$.

In what follows, we shall assume that the sets U and V are fixed.

3. Trajectories and graphs

Let $I = [t_1, t_2]$ be an arbitrary finite time interval such that $t_o \leq t_1 < t_2 < \infty$. Let $u(\cdot)$ be a Pursuer's admissible control. A function $x(\cdot)$ from I into E^n will be called a Pursuer's trajectory on I corresponding to a control $u(\cdot)$ and an initial condition

$$x(t_1) = x_1 \in E^n$$

if

(a) $x(\cdot)$ is absolutely continuous on I

(b) $\frac{dx(t)}{dt} = f(x(t),u(t),t)$ a.e. in I

(c) $x(t_1) = x_1$

An Evader's trajectory is similarly defined.

A Pursuer's graph on I, denoted by $f^u_{x_1}$ corresponding to an admissible control $u(\cdot)$ and an initial condition $x(t_1) = x_1$ is defined by

$$f^{u}_{x_1} = \left\{ (x(t),t) \in E^{n+1} : t \in I \quad \begin{array}{l} x(\cdot) \text{ is a Pursuer's trajectory on } I \\ \text{corresponding to } u(\cdot) \text{ and } x(t_1) = x_1 \end{array} \right\} \tag{1.4}$$

When there is no possibility of misunderstanding, the superscript u will be dropped and the domain of definition I will not be specified. An Evader's graph is similarly defined.

In what follows, we assume that the following conditions are satisfied for any finite time interval I.

<u>A1-1</u> $f(\cdot,\cdot,\cdot)$ is continuous in $(x(t),u(t))$ on $E^n \times U$ for each $t \in I$.

<u>A1-2</u> $f(\cdot,\cdot,\cdot)$ is integrable with respect to t on I for each $(x(t),u(t)) \in E^n \times U$.

<u>A1-3</u> There exists a Lipschitz constant $K < \infty$ such that

$$\begin{aligned} &\| f(x(t),u(t),t) - f(x'(t),u(t),t) \| \\ &\qquad \leq K \| x(t) - x'(t) \| \end{aligned} \tag{1.5}$$

for any $x(t),\ x'(t) \in E^n,\ u(t) \in U$, and $t \in I$

Here, $\| \cdot \|$ implies the Euclidean norm.

<u>A1-4</u> There exists a constant $M < \infty$ such that

$$\| f(x(t),u(t),t) \| \leq M(\| x(t) \| + 1) \tag{1.6}$$

It is known (see [C1], [H1], and [S1]) that Assumptions A1-1 to 4 guarantee the existence and uniqueness of a global trajectory $x(\cdot)$ on I for any admissible control $u(\cdot)$ and initial condition $x(t_1) = x_1 \in E^n$. A finite escape time is ruled out by A1-4.

Similar assumptions are made on $g(\cdot,\cdot,\cdot)$. For simplicity, Assumptions A1-1 to 4 on $f(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot,\cdot)$ will be called Assumption A1.

C. ATTAINABILITY SETS AND ESCAPABILITY SETS

1. Attainability sets

A point $(x(t_2),t_2) \in E^{n+1}$ is called attainable from $(x(t_1),t_1) \in E^{n+1}$, $t_o \leq t_1 < t_2 < \infty$, if there exists a Pursuer's admissible control $u(\cdot)$ such that

$$(x(t_2),t_2) \in f^u_{x(t_1)}$$

In other word, $(x(t_2),t_2)$ is attainable from $(x(t_1),t_1)$ if there exists an admissible control $u(\cdot)$ such that Eq. (1.1) with the initial condition $x(t_1)$ at time t_1 has the solution $x(t_2)$ at time t_2.

The attainability set for Pursuer from $(x(t_1),t_1) \in E^{n+1}$, $t_o \leq t_1 < \infty$ is defined by

$$A_{x(t_1)} = \{(x(t),t) \in E^{n+1} : t \in [t_1,\infty), (x(t),t) \text{ is attainable from } (x(t_1),t_1)\} \quad (1.7)$$

The fixed-time cross section of $A_{x(t_1)}$ at time t, $t_o \leq t_1 \leq t < \infty$, is defined by

$$A_{x(t_1)}(t) = \{x(t) \in E^n : (x(t),t) \in A_{x(t_1)}\} \quad (1.8)$$

The attainability set $A_{y(t_1)}$ for Evader from $(y(t_1),t_1) \in E^{n+1}$,

$t_o \le t_1 < \infty$, and the fixed-time cross section of $A_{y(t_1)}$ at time t, $t_o \le t_1 \le t < \infty$, are similarly defined.

Now, by Assumption A1, it is seen that $A_{x(t_1)}(t)$, $t_o \le t_1 \le t < \infty$, satisfies the following properties.

<u>At-1</u> (Boundedness) For each $x(t_1) \in E^n$, t_1, and t, $t_o \le t_1 \le t < \infty$, $A_{x(t_1)}(t)$ is a bounded subset of E^n.

<u>At-2</u> (Continuity) For each $x(t_1) \in E^n$, $A_{x(t_1)}(t)$ is continuous in t, $t_o \le t_1 \le t < \infty$, namely, for each $x(t_1) \in E^n$, t_1, t, and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left.\begin{array}{l} |t - t'| \le \delta \\ t_1 \le t' \end{array}\right\} \text{ imply } \left\{\begin{array}{l} A_{x(t_1)}(t) \subset A_{x(t_1)}(t') + B_\epsilon \\ A_{x(t_1)}(t') \subset A_{x(t_1)}(t) + B_\epsilon \end{array}\right.$$

where $B_\epsilon \in E^n$ represents the ball of center $0 \in E^n$ and radius ϵ and

$$A_{x(t_1)}(t) + B_\epsilon = \left\{ a+b : a \in A_{x(t_1)}(t),\ b \in B_\epsilon \right\}$$

<u>At-3</u> (Semi-group property) For each $x(t_1) \in E^n$, t_1, t_2, and t_3, $t_o \le t_1 \le t_2 \le t_3 < \infty$,

$$A_{x(t_1)}(t_3) = \bigcup_{x(t_2) \in A_{x(t_1)}(t_2)} A_{x(t_2)}(t_3) \qquad (1.9)$$

Similar properties hold for $A_{y(t_1)}(t)$.

Now, any attainability set generated by a differential equation of the form (1.1) has properties At-1, 2, and 3 under Assumption A1. But we sometimes require further restrictive properties for attainability sets, especially in

Chapter II-C and Chapter IV. They are

<u>At-4</u> (Compactness) For each $x(t_1) \in E^n$, t_1 and t, $t_o \leq t_1 \leq t < \infty$, $A_{x(t_1)}(t)$ is a compact subset of E^n.

<u>At-5</u> (Convexity) For each $x(t_1) \in E^n$, t_1, and t, $t_o \leq t_1 \leq t < \infty$, $A_{x(t_1)}(t)$ is a convex subset of E^n.

Sufficient conditions for At-4 and 5 have been extensively examined in connection with the existence of optimal controls.

<u>Remark</u>: Standard results which guarantee properties At-4 and 5 above are as follows:

(a) Nonlinear dynamics

Let $f(\cdot,\cdot,\cdot)$ satisfy Assumption A1. If the set

$$\{f(x(t),u(t),t) : u(t) \in U\}$$

is convex for each $(x(t),t) \in E^n \times [t_o,\infty)$, then the set $A_{x(t_1)}(t)$ is compact for each $x(t_1) \in E^n$, t_1, and t, $t_o \leq t_1 \leq t < \infty$ [R1] .

It should be recalled that U is compact.

(b) Linear dynamics

Suppose $f(x(t),u(t),t) = C(t)x(t) + h(u(t),t)$, where $C(t)$ is an $n \times n$ matrix, $h(u(t),t)$ is a vector in E^n, and $C(\cdot)$ and $h(\cdot,\cdot)$ are continuous.

Then, the set $A_{x(t_1)}(t)$ is compact and convex for each $x(t_1) \in E^n$, t_1, and t, $t_o \leq t_1 \leq t < \infty$ [N1] .

Furthermore, we need the following property, especially in Chapter II-C, which also follows from Assumption A1.

<u>At-2</u>' (Uniform continuity)

For each $\epsilon>0$, there exists $\delta>0$ such that

$$\left.\begin{array}{l} |t - t'| \leq \delta \\ t_o \leq t_1 \leq t \leq t_2 \\ t_o \leq t_1 \leq t' \leq t_2 \\ t_1 \leq t_2 < \infty \\ x(t_1) \in A_{x_o}(t_1) \end{array}\right\rangle \text{ imply } \left\langle \begin{array}{l} A_{x(t_1)}(t) \subset A_{x(t_1)}(t') + B_\epsilon \\ \\ A_{x(t_1)}(t') \subset A_{x(t_1)}(t) + B_\epsilon \end{array}\right.$$

2. Escapability sets

Now, we shall define the ϵ-escapability set which plays an important role in the following discussions. For each real number $\epsilon \geq 0$ the ϵ-escapability set with respect to $(x(t_1),t_1) \in E^{n+1}$ and $(y(t_2),t_2) \in E^{n+1}$, $t_o \leq t_1 < \infty$, $t_o \leq t_2 < \infty$, denoted symbolically by $A_{y(t_2)}-(A_{x(t_1)}+B_\epsilon)$, is defined to be the component (= the maximal connected subset (see Ref. [K6]))of $A_{y(t_2)} \setminus (A_{x(t_1)} + B_\epsilon)$ which contains $(y(t_2),t_2)$, where

$$A_{y(t_2)} \setminus (A_{x(t_1)} + B_\epsilon)$$
$$= \left\{ z \in E^{n+1} : z \notin A_{x(t_1)} + B_\epsilon \,,\; z \in A_{y(t_2)} \right\} \quad ,$$

and $A_{x(t_1)} + B_\epsilon$ is defined such that its fixed-time cross section at time t, $t_o \leq t_1 \leq t < \infty$, denoted by $(A_{x(t_1)} + B_\epsilon)(t)$, satisfies.

$$(A_{x(t_1)} + B_\epsilon)(t) = A_{x(t_1)}(t) + B_\epsilon \qquad (1.10)$$
$$\text{for all } t,\; t_o \leq t_1 \leq t < \infty \,.$$

If $\epsilon \geq 0$ is fixed and there is no possibility of misunderstanding, we shall express the ϵ-escapability set more succinctly by

$$A_{y(t_2)} - (A_{x(t_1)} + B_\epsilon) = S_{x(t_1),y(t_2)} \tag{1.11}$$

The set $A_{y(t_2)} - (f_{x(t_1)} + B_\epsilon)$ is similarly defined.

D. RULES OF THE GAME

1. ϵ-capture time

Let $u(\cdot)$ ($v(\cdot)$) be a Pursuer's (Evader's) admissible control, and let $x(\cdot)$ ($y(\cdot)$) be the corresponding trajectory with an initial condition $x(t_o)=x_o$ ($y(t_o)=y_o$). Let there be given a non-negative number ϵ.
If there exists a time $\hat{t}$, $t_o < \hat{t} < \infty$ such that

$$\begin{aligned} &\|x(\hat{t}) - y(\hat{t})\| \leq \epsilon \\ \text{and} \quad &\|x(t) - y(t)\| > \epsilon \quad \text{for all } t,\ t_o \leq t < \hat{t} \end{aligned} \tag{1.12}$$

we say that the ϵ-capture occurs at time $\hat{t}$ and $\hat{t} - t_o$ will be called the ϵ-capture time. If there does not exist $\hat{t}$ which satisfies (1.12), we define $\hat{t} = \infty$. The game is considered terminated as soon as the ϵ-capture occurs. We shall assume throughout that $\|x_o - y_o\| > \epsilon$.
Let f_{x_o} and f_{y_o} represent graphs for Pursuer and Evader corresponding to admissible controls $u(\cdot)$ and $v(\cdot)$, respectively. Then, the above definition of $\hat{t}$ is equivalent to

$$\hat{t} = \min(t: \|f_{x_o}(t) - f_{y_o}(t)\| = \epsilon) \tag{1.13}$$

Since the trajectories $x(\cdot)$ and $y(\cdot)$, defined by $x(t)=f_{x_o}(t)$ and $y(t)=f_{y_o}(t)$, $t_o \leq t < \infty$, are continuous, the minimum with respect to t is meaningful.

2. Unbiased game

The rules of the pursuit-evasion game are as follows:

Before starting the game, Pursuer and Evader are informed of

R-1 the dynamics of Pursuer and Evader, and the admissible control sets, $\mathcal{U}$ and $\mathcal{V}$,

R-2 the initial conditions, x_o, y_o and t_o and

R-3 the value of ϵ .

We assume that throughout the game R-1, 2, and 3 are fixed.

In addition to R-1, 2, and 3 above, Pursuer and Evader can observe the states of both players, $x(t)$ and $y(t)$, at each instant of the game, without error. Based upon this information, Pursuer tries to determine a value of $u(t) \epsilon U$, at time t, such that the ϵ-capture will occur within the shortest period of time. On the other hand, Evader selects a value of $v(t) \epsilon V$, at the same time t, so as to escape from the ϵ-capture as long as possible. The rules of the game will be stated more precisely in Chapter II in terms of time-discrete approximations.

This model of the game will be called the unbiased

<u>game</u> G, since the information pattern available for both players at each instant of the game is unbiased.

Since it is difficult to analyze this game any further, we shall circumvent this difficulty by approximating the above time-continuous game by the time-discrete games. Before proceeding to time-discrete approximations, it will be convenient to introduce minorant and majorant games in which the information pattern available for each player is biased in favor of one player or the other.

<u>Remark</u>: It is also possible to consider games where the value of u(t) (v(t)), at each instant of the game, depends not only on the present states but all the past. However, these games are considerably more complicated, and are rarely considered in the literature. We shall not consider these either.

3. <u>Minorant and majorant games</u>

In the <u>minorant game</u> G^-, the game is played just the same way as the unbiased game, except that Pursuer can observe the value of the Evader's control v(t), at each instant of the game, in addition to x(t) and y(t). The information pattern available for Evader is the same as that of the unbiased game.

In the <u>majorant game</u> G^+, the game is played just the same way as the unbiased game, except that Evader can observe the value of the Pursuer's control u(t), at each instant of the game, in addition to x(t) and y(t). The information pattern available for Pursuer is the same as that of the unbiased game.

II. TIME-DISCRETE APPROXIMATIONS AND CONVERGENCE PROBLEMS

As pointed out by W. H. Fleming [F2], [F3], profound difficulties are involved in the precise mathematical formulation of games with a continuum of moves, because of the fact that each player's control is affected by the other player's state continuously.

A possible approach is to replace time-continuous moves by time-discrete moves with time intervals $h>0$, and to show that the values of the approximating discrete games converge to a limit as h tends to zero.

In Section A, the formulation of approximating time-discrete games and some related definitions are introduced.

In Section B, we show that the optimal e-capture times for the discrete games converge to a limit as h goes to zero.

In Section C, we establish a theorem showing that, under certain assumptions, the difference between the optimal e-capture times for the discrete minorant games and the majorant games converges to zero as h goes to zero. This corresponds to the "minimax" theorem in matrix game theory. Only in this case, the unbiased differential game is "determined" in the game-theoretic sence and the optimal pair of strategies can be defined.

Finally, in Section D, the relation between the limit of approximating discrete games and the time-continuous game is clarified. Especially, we show that $\lim_{h \to 0} T_h^- \; (= \lim_{h \to 0} T_h^+)$

coincides with the appropriately defined "value" of the corresponding time-continuous game, provided that the "minimax" theorem given in Section C holds.

A. TIME-DISCRETE APPROXIMATIONS

1. Rules of the game

Since we encounter difficulties in the precise formulation of the time-continuous, pursuit-evasion differential game G, we start instead with a corresponding sequence of discrete games G_h which are defined below.

Let us recall that in the time-continuous game G, the following information is given to Pursuer and Evader, before the game starts:

R-1 the dynamics of Pursuer and Evader, and the admissible control sets $\mathcal{U}$ and $\mathcal{V}$,

R-2 the initial conditions, x_o, y_o, and t_o , and

R-3 the value of ϵ .

At each instant t, $t_o \leq t < \infty$, both players observe the states x(t) and y(t) and decide values of their controls u(t) and v(t), at the same time t, based upon this observation.

Now, corresponding to the time-continuous game G, a time-discrete unbiased game G_h with a sampling time interval $h>0$ is played as follows:

Before starting the game, Pursuer and Evader are told R-1, 2, and 3 above, as well as

R-4 the sampling time interval $h>0$.

We assume that throughout the game R-1, 2, 3, and 4 are fixed.

In addition to R-1, 2, 3, and 4 above Pursuer and Evader can observe the states of both players at time $t_i=t_o+ih$, $i=0,1,2,\ldots$, denoted by x_i and y_i, respectively, without noise. Unlike the time-continuous game G, neither player can observe the states continuously. Now, at each time $t_i=t_o+ih$, $i=0,1,2,\ldots$, based upon this information, Puruser tries to determine his admissible control segment $u_i(\cdot)$ on $[t_i, t_{i+1})$ (see Remark 2 below) such that the e-capture will occur within the shortest period of time. On the other hand, Evader selects his admissible control segment $v_i(\cdot)$ on $[t_i, t_{i+1})$ so as to escape from the e-capture as long as possible.

This model of discrete games will be called the unbiased game since the information pattern available for both players at each time t_i, $i=0,1,2,\ldots$, is unbiased.

Remark 1: W. H. Fleming [F3] discretized both sampling intervals and dynamics; i.e., he approximated differential dynamics by difference dynamics. Here, we discretize only sampling intervals, leaving dynamics of both players unchanged. We shall see that our discretization technique is more convenient for our analysis, especially in connection with sufficient strategies.

Remark 2: Suppose a measurable function $u_i(\cdot)$ from $[t_i, t_{i+1})$ into U, i=0,1,2,... , is given. Then, the trajectory $x(\cdot)$ on $[t_i, t_{i+1}]$ is uniquely determined if the initial condition $x(t_i)=x_i$ is given (see Ref. [Z2] for consistency conventions of dynamic systems).

A measurable function $u_i(\cdot)$ from $[t_i, t_{i+1})$ into U, i=0,1,2,... , will be called a Pursuer's admissible control segment on $[t_i, t_{i+1})$. The set of all Pursuer's admissible control segments on $[t_i, t_{i+1})$ will be denoted by $\mathcal{U}_i$. Similar definition for $v_i(\cdot)$ and $\mathcal{V}_i$ holds for Evader.

In addition to the discrete version of the unbiased game G_h just described, it will be convenient to introduce a time-discrete minorant game G_h^- and a time-discrete majorant game G_h^+ . The discrete minorant game G_h^- with the time interval h>0 is played just the same way as the unbiased game G_h , except that Evader must tell his admissible control segment $v_i(\cdot)$ to Pursuer, at each time t_i, i=0,1,2,... , before Pursuer chooses his admissible control segment $u_i(\cdot)$. Hence, the information pattern available at each time t_i is advantageous to Pursuer.

The discrete majorant game G_h^+ with the time interval h>0 is played just the same way as the unbiased game G_h, except that Pursuer must tell $u_i(\cdot)$ to Evader, at each time t_i, i=0,1,2,... , before Evader chooses $v_i(\cdot)$.

Hence, the information pattern available at each time t_i is advantageous to Evader.

Since both players make use of states $x(t_i)$ and $y(t_i)$, observed at each time t_i, $i=0,1,2,\ldots$, to decide $u_i(\cdot)$ and $v_i(\cdot)$, the games just described are called closed-loop games.

If the rules of the game are modified such that observation of states is not permitted during the game and both players must decide their whole controls $u(\cdot)$ and $v(\cdot)$ at the initial time t_o based upon R-1, 2, 3, and 4, these games are called open-loop games.

Since open-loop games are considered as a special case (h goes to ∞) of closed-loop games, we are mainly interested in closed-loop games in this thesis, and when no confusion results, the words "closed-loop" will be omitted.

2. Optimal ϵ-capture time

Following the rules of the game, let us assume that R-1, 2, 3, and 4 are given. If $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$ are given, a Pursuer's trajectory $x(\cdot)$ and an Evader's trajectory $y(\cdot)$ are uniquely determined (by Assumption A1). Then, the ϵ-capture time denoted by $\hat{t}$, is determined by (1.12). Let m $(=0,1,2,\ldots)$ be defined by

$$t_o + mh < \hat{t} \leq t_o + (m+1)h \tag{2.1}$$

For convenience, we define $m=\infty$ if $\hat{t}=\infty$.

Now, let us decompose $u(\cdot)$ and $v(\cdot)$ by $u_i(\cdot)$ and $v_i(\cdot)$, in the sense that

$$u(t) = u_i(t) \quad t_i \le t < t_{i+1}, \quad i=0,1,2,\ldots \quad \text{and}$$
$$v(t) = v_i(t) \quad t_i \le t < t_{i+1}, \quad i=0,1,2,\ldots \tag{2.2}$$

where $u_i(\cdot)$ and $v_i(\cdot)$ represent Puruser's and Evader's admissible control segments on $[t_i, t_{i+1})$, $i=0,1,2,\ldots$, respectively. It is seen that corresponding to any $u(\cdot) \in \mathcal{U}$, $u_i(\cdot) \in \mathcal{U}_i$, $i=0,1,2,\ldots$, are uniquely determined. Similar fact holds for $v(\cdot)$.

We define the <u>optimal ϵ-capture time</u> for G_h^- by

$$T_h^- = \sup_{v_o(\cdot)} \inf_{u_o(\cdot)} \cdots\cdots\cdots \sup_{v_m(\cdot)} \inf_{u_m(\cdot)} \hat{t} - t_o \tag{2.3}$$

The supremums and infimums are over the sets $\mathcal{V}_i$ and $\mathcal{U}_i$, $i=0,1,2,\ldots,m$, respectively.

It sould be recalled that m is given by (2.1).

T_h^- can be infinity. Conditions for $T_h^- < \infty$, which are called capture conditions, will be given in Chapter III. When no confusion results, we simply say the optimal capture time, instead of the optimal ϵ-capture time.

In what follows, we shall make the following assumption.

<u>A2 (1)</u> There exist $\hat{u}_i^-(\cdot) \in \mathcal{U}_i$ and $\hat{v}_i^-(\cdot) \in \mathcal{V}_i$, $i=0,1,2,\ldots,m$, which attain the minimum and maximum of $\hat{t}$ in (2.3).

Under this assumption, we have

$$T_h^- = \max_{v_o(\cdot)} \min_{u_o(\cdot)} \cdots\cdots\cdots \max_{v_m(\cdot)} \min_{u_m(\cdot)} \hat{t} - t_o \tag{2.4}$$

Let us define $\hat{u}^-(\cdot)$ and $\hat{v}^-(\cdot)$ by

$$\hat{u}^-(t) = \begin{cases} \hat{u}_i^-(t) \ , & t_i \le t < t_{i+1} \ , \quad i=0,1,2,\ldots,m \\ u \in U \ , & t_{i+1} \le t < \infty \end{cases} \tag{2.5}$$

$$\hat{v}^-(t) = \begin{cases} \hat{v}_i^-(t) \ , & t_i \le t < t_{i+1} \ , \quad i=0,1,2,\ldots,m \\ v \in V \ , & t_{i+1} \le t < \infty \end{cases} \tag{2.6}$$

where u and v are arbitrary vectors in U and V, respectively. By (1.3), $\hat{u}(\cdot)$ and $\hat{v}(\cdot)$ are admissible. We shall refer to the pair $(\hat{u}^-(\cdot),\hat{v}^-(\cdot))$ as the <u>optimal pair of controls</u> for G_h^-.

Similarly, we define the optimal ϵ-capture time for G_h^+ by

$$T_h^+ = \inf_{u_o(\cdot)} \ \sup_{v_o(\cdot)} \ \ldots\ldots\ldots\ldots \inf_{u_m(\cdot)} \ \sup_{v_m(\cdot)} \ \hat{t} - t_o \quad (2,7)$$

The infimums and supremums are over the sets $\mathcal{U}_i$ and $\mathcal{V}_i$, $i=0,1,2,\ldots,m$, respectively.

We shall assume

<u>A2 (2)</u> There exist $\hat{u}_i^+(\cdot) \in \mathcal{U}_i$ and $\hat{v}_i^+(\cdot) \in \mathcal{V}_i$, $i=0,1,2,\ldots,m$, which attain the minimum of $\hat{t}$ in (2.7).

The optimal pair of controls $(\hat{u}^+(\cdot),\hat{v}^+(\cdot))$ for G_h^+ is similarly defined.

B. CONVERGENCE PROBLEMS

In this section, we shall show that the optimal ϵ-capture time T_h^- for G_h^- converges to a limit T^- as h goes to zero. Namely, we shall show

$$\lim_{h \to 0} T_h^- = T^-$$

Similarly, we shall show

$$\lim_{h \to 0} T_h^+ = T^+$$

The optimal capture time for G_h^-, defined by (2.3), depends upon the initial conditions x_o, y_o and t_o. Let us denote this dependence explicitly by

$$T_h^- = T_h^-(x_o, y_o, t_o)$$

In general, let $T_h^-(x,y,t)$ represent an optimal capture time for G_h^- with an initial condition (x,y,t) on $E^n \times E^n \times [t_o, \infty)$. Similarly, $T_h^+(x,y,t)$ represents an optimal capture time for G_h^+ with an initial condition (x,y,t) on $E^n \times E^n \times [t_o, \infty)$. It can be seen that for any $(x,y,t) \in E^n \times E^n \times [t_o, \infty)$ and for any $h>0$, $T_h^-(x,y,t)$ and $T_h^+(x,y,t)$ are uniquely determined. For convenience, we define $T_h^-(x,y,t) = T_h^+(x,y,t) = 0$ if $\|x-y\| \le \epsilon$.

Let us define, for each $h>0$, sets G_h^- and G_h^+ by

$$\begin{aligned} G_h^- &= \left\{(x,y,t) \in E^n \times E^n \times [t_o, \infty) : T_h^-(x,y,t) < \infty\right\} \\ G_h^+ &= \left\{(x,y,t) \in E^n \times E^n \times [t_o, \infty) : T_h^+(x,y,t) < \infty\right\} \end{aligned} \tag{2.8}$$

We also define, for each $h>0$, sets $G_{h,m}^-$ and $G_{h,m}^+$, $m=1,2,\ldots$, by

$$\begin{aligned} G_{h,m}^- &= \Big\{ (x,y,t) \in E^n \times E^n \times [t_o, \infty) : \\ &\qquad (m-1)h < T_h^-(x,y,t) \le mh \Big\} \\ G_{h,m}^+ &= \Big\{ (x,y,t) \in E^n \times E^n \times [t_o, \infty) : \\ &\qquad (m-1)h \le T_h^+(x,y,t) \le mh \Big\} \end{aligned} \tag{2.9}$$

With this notation, we have

Lemma 2-1:

(a) If for some positive integer $n>2$ and $h>0$, $G^-_{h,n} \neq \phi$ then, $G^-_{h,m} \neq \phi$ for all $m=1,2,\ldots,n-1$.

(b) If for some positive integer $n>2$ and $h>0$, $G^+_{h,n} \neq \phi$ then, $G^+_{h,m} \neq \phi$ for all $m=1,2,\ldots,n-1$.

Proof:

(1) If $G^-_{h,n} \neq \phi$, there exists $(x,y,t) \in E^n \times E^n \times [t_o,\infty)$ such that

$$(n-1)h < T^-_h(x,y,t) \leq nh$$

Since we can consider discrete games as multistage decision processes, employing the "principle of optimality" by R. Bellman [B1], [B3], we derive the following recurrence relation.

$$T^-_h(x,y,t) = h + \max_{v_1(\cdot)} \min_{u_1(\cdot)} T^-_h(x_1,y_1,t+h) \tag{2.10}$$

where

$$x_1 = x + \int_t^{t+h} f(x(t'),u_1(t'),t')dt'$$

$$y_1 = y + \int_t^{t+h} g(y(t'),v_1(t'),t')dt'$$

$v_1(\cdot)$ = an Evader's admissible control segment on $[t, t+h)$

$u_1(\cdot)$ = a Pursuer's admissible control segment on $[t, t+h)$

Let $\mathcal{V}_1$ = the set of all Evader's admissible control segments on $[t, t+h)$

$\mathcal{U}_1$ = the set of all Pursuer's admissible control segments on $[t, t+h)$

The maximization and minimization in (2.10) are over the sets of $\mathcal{V}_1$ and $\mathcal{U}_1$ such that (dist$(\cdot,\cdot)$ is the Euclidean distance)

$$\text{dist}(f_x^{u_1}(t'),\ f_y^{v_1}(t')) > \epsilon \qquad (2.11)$$

for any t', $t \leq t' < t+h$

Now, by Assumption A2, there exist $u_1^*(\cdot) \in \mathcal{U}_1$ and $v_1^*(\cdot) \in \mathcal{V}_1$ such that

$$T_h^-(x,y,t) = h + T_h^-(x_1^*,y_1^*,t+h) \qquad (2.12)$$

and

$$\text{dist}(f_x^{u_1^*}(t'),\ f_y^{v_1^*}(t')) > \epsilon$$

for any t', $t \leq t' < t+h$,

where

$$x_1^* = x + \int_t^{t+h} f(x(t'),u_1^*(t'),t')dt'$$

$$y_1^* = y + \int_t^{t+h} g(y(t'),v_1^*(t'),t')dt'$$

From (2.12), we have

$$(n-2)h < T_h^-(x_1^*,y_1^*,t+h) \leq (n-1)h$$

Hence,

$$G_{h,n-1}^- \ni (x_1^*,y_1^*,t+h)$$

Similarly, we can show

$$G_{h,m}^- \neq \emptyset \qquad \text{for any } m=1,2,\ldots, n-2$$

(2) Similarly, by the "principle of optimality", we have

$$T_h^+(x,y,t) = h + \min_{u_1(\cdot)} \max_{v_1(\cdot)} T_h^+(x_1,y_1,t+h) \tag{2.13}$$

$$\text{for any } (x,y,t) \in G_{h,n}^+ \ , \ n=2,3,\ldots$$

The minimization and maximization are over the sets $\mathcal{U}_1$ and $\mathcal{V}_1$ such that (2.11) holds.

Proceeding similarly, we obtain (b).

<u>Lemma 2-2</u>:

$$T_h^-(x,y,t) \leq T_h^+(x,y,t)$$

holds for any $(x,y,t) \in E^n \times E^n \times [t_o,\infty)$ and for any $h>0$.

<u>Proof</u>:

The geometrical proof will be given in Chapter III.

<u>Remark 1</u>: Formally, we can prove Lemma 2-1 as follows:

By (2.4), we have

$$\begin{aligned} T_h^-(x,y,t) &= (\max_{v_o(\cdot)} \min_{u_o(\cdot)})(\max_{v_1(\cdot)} \min_{u_1(\cdot)}) \ldots\ldots \\ &\qquad\qquad \ldots\ldots (\max_{v_m(\cdot)} \min_{u_m(\cdot)})\hat{t} - t \\ &\leq (\min_{u_o(\cdot)} \max_{v_o(\cdot)})(\min_{u_1(\cdot)} \max_{v_1(\cdot)}) \ldots\ldots \\ &\qquad\qquad \ldots\ldots (\min_{u_m(\cdot)} \max_{v_m(\cdot)})\hat{t} - t \\ &= T_h^+(x,y,t) \end{aligned}$$

<u>Remark 2</u>: Since information pattern for G_h^- is more advantageously biased to Pursuer than that for G_h^+, Lemma 2-1 is a reasonable result.

<u>Lemma 2-3</u>:

(a) If $(x,y,t) \in G^-_{h,1}$, then

$$T^-_h(x,y,t) + t$$

$$= \max_{f_y \in A_y} \min_{t'} (t' : \mathrm{dist}(f_y(t'), A_x(t') + B_e) = 0)$$

(a)' If $(x,y,t) \in G^+_{h,1}$, then

$$T^+_h(x,y,t) + t$$

$$= \min_{f_x \in A_x} \max_{f_y \in A_y} \min_{t'} (t' : \mathrm{dist}(f_y(t'), f_x(t') + B_e) = 0)$$

(b) If $(x,y,t) \in G^-_{h,2}$, then

$$T^-_h(x,y,t) + t$$

$$= \max_{y_1 \in A_y(t+h)} \min_{x_1 \in A_x(t+h)} \max_{f_{y_1} \in A_{y_1}} \min_{t'}$$

$$(t' : \mathrm{dist}(f_{y_1}(t'), A_{x_1}(t') + B_e) = 0)$$

(b)' If $(x,y,t) \in G^+_{h,2}$, then

$$T^+_h(x,y,t) + t$$

$$= \min_{x_1 \in A_x(t+h)} \max_{y_1 \in A_y(t+h)} \min_{f_{x_1} \in A_{x_1}} \max_{f_{y_1} \in A_{y_1}} \min_{t'}$$

$$(t' : \mathrm{dist}(f_{y_1}(t'), f_{x_1}(t') + B_e) = 0)$$

<u>Proof</u>:

(1) Corresponding to any graph f_x and f_y, the e-capture time $\hat{t}$ is given by (see (1.13))

$$\hat{t} = \min_{t'} \, (t' : \mathrm{dist}(f_x(t'), f_y(t')) = \epsilon) \tag{2.14}$$

But, by (2.3) and by Assumption A2, we have

$$T_h^-(x,y,t) + t = \max_{v(\cdot)} \; \min_{u(\cdot)} \; \hat{t} \tag{2.15}$$

for any $(x,y,t) \in G_{h,1}^-$, where the maximization and minimization on the right-hand side are over the sets of all Pursuer's and Evader's admissible control segments on $[t, t+h)$. From relations (2.14) and (2.15), we have,

$$\begin{aligned} &T_h^-(x,y,t) + t \\ &= \max_{v(\cdot)} \; \min_{u(\cdot)} \; \min_{t'} \, (t' : \mathrm{dist}(f_y(t'), f_x(t')) = \epsilon) \end{aligned}$$

Hence, recalling Remark 2 in I-A-1, and $A_x = \bigcup f_x$, we have,

$$\begin{aligned} &T_h^-(x,y,t) + t \\ &= \max_{f_y \in A_y} \; \min_{f_x \in A_x} \; \min_{t'} \, (t' : \mathrm{dist}(f_y(t'), f_x(t')) = \epsilon) \\ &= \max_{f_y \in A_y} \; \min_{t'} \, (t' : \mathrm{dist}(f_y(t'), A_x(t')) = \epsilon) \\ &= \max_{f_y \in A_y} \; \min_{t'} \, (t' : \mathrm{dist}(f_y(t'), A_x(t') + B_\epsilon) = 0) \end{aligned}$$

(2) Other relations follow similarly.

<u>Lemma 2-4</u>:

Suppose $(x,y,t) \in G_{h,m}^-$, $2 \le m$, then

(a) there exists admissible $v_1^*(\cdot)$ such that

$$T_h^-(x,y,t) \le h + T_h^-(x_1,y_1^*,t+h)$$

holds for any admissible $u_1(\cdot)$,

where

$$x_1 = x + \int_t^{t+h} f(x(t'),u_1(t'),t')dt'$$

$$y_1^* = y + \int_t^{t+h} g(y(t'),v_1^*(t'),t')dt'$$

(a)' for each admissible $v_1(\cdot)$, except such $\tilde{v}_1(\cdot)$ as

$$\mathrm{dist}(f_x^{u_1}(t'),f_y^{\tilde{v}_1}(t')) \le \epsilon$$

for some t' , $t \le t' < t+h$, and for some admissible $u_1(\cdot)$, there corresponds admissible $u_1^*(\cdot)$ such that

$$T_h^-(x,y,t) \ge h + T_h^-(x_1^*,y_1,t+h) \quad \text{holds,}$$

where

$$x_1^* = x + \int_t^{t+h} f(x(t'),u_1^*(t'),t')dt'$$

$$y_1 = y + \int_t^{t+h} g(y(t'),v_1(t'),t')dt'$$

Suppose $(x,y,t) \in G_{h,m}^+$, $2 \le m$, then

(b) there exists admissible $u_1^*(\cdot)$ such that

$$T_h^+(x,y,t) \ge h + T_h^+(x_1^*,y_1,t+h)$$

holds, for any admissible $v_1(\cdot)$, except such $\tilde{v}_1(\cdot)$ as

$$\mathrm{dist}(f_x^{u_1^*}(t'),f_y^{\hat{v}_1}(t')) \le \epsilon$$

for some t' , $t \le t' < t+h$,

where x_1^* and y_1 are defined as in (a)' ,

(b)' for each admissible $u_1(\cdot)$, there corresponds admissible $v_1^*(\cdot)$ such that

$$T_h^+(x,y,t) \leq h + T_h^+(x_1,y_1^*,t+h) \qquad \text{holds,}$$

where x_1 and y_1^* are defined as in (a)

Proof:

(a) By (2.10) and Assumption A2, there exists admissible $v_1^*(\cdot)$ such that

$$T_h^-(x,y,t) = h + \min_{u_1(\cdot)} T_h^-(x_1,y_1^*,t+h)$$

and $\qquad \operatorname{dist}(f_x^{u_1}(t'),f_y^{v_1^*}(t')) > \epsilon$

for any t' , $t \leq t' < t+h$, and for any admissible $u_1(\cdot)$

Since

$$T_h^-(x,y,t) \leq h + T_h^-(x_1,y_1^*,t+h)$$

for any admissible $u_1(\cdot)$.

(a)' If Evader uses admissible $\tilde{v}_1(\cdot)$ such that

$$\operatorname{dist}(f_x^{u_1}(t'),f_y^{\tilde{v}_1}(t')) \leq \epsilon$$

holds for some t' , $t \leq t' < t+h$, and for some admissible $u_1(\cdot)$, then the ϵ-capture trivially occurs before time t+h.

If Evader uses admissible $v_1(\cdot)$ other than $\tilde{v}_1(\cdot)$, the ϵ-capture does not occur before time t+h.

In this case

$$T_h^-(x,y,t) = h + \max_{v_1(\cdot)} \min_{u_1(\cdot)} T_h^-(x_1,y_1,t+h)$$
$$\geq h + \min_{u_1(\cdot)} T_h^-(x_1,y_1,t+h)$$

for any admissible $v_1(\cdot)$.

Hence, corresponding to each admissible $v_1(\cdot)$ there exists admissible $u_1^*(\cdot)$ such that

$$T_h^-(x,y,t) \geq h + T_h^-(x_1^*,y_1,t+h)$$

(b) and (b)' can be shown similarly.

Theorem 2-1:

If there exists $\bar{h}>0$ such that $T_h^+(x_o,y_o,t_o) < \infty$ holds for any h, $0<h\leq\bar{h}$, then

(a) $T_h^-(x_o,y_o,t_o)$ converges to a limit as h goes to zero, and

(b) $T_h^+(x_o,y_o,t_o)$ converges to a limit as h goes to zero.

First, we shall give an outline of the proof.

Outline of Proof

(1) We shall show

$$T_h^+(x_o,y_o,t_o) \geq T_{\frac{1}{2}h}^+(x_o,y_o,t_o) \qquad (2.16)'$$

and

$$T_h^-(x_o,y_o,t_o) \leq T_{\frac{1}{2}h}^-(x_o,y_o,t_o) \qquad (2.17)'$$

for any h, $0 < h \leq \bar{h}$.

Then, we see that $\left\{T^{+}_{h/2^k}(x_o,y_o,t_o)\right\}$, k=0,1,2,... , is a non-increasing sequence bounded below (i.e., non-negative by definition), hence Theorem 2-1 (b) holds.

Similarly, $\left\{T^{-}_{h/2^k}(x_o,y_o,t_o)\right\}$ is a non-decreasing sequence bounded above (by Lemma 2-2), hence Theorem 2-1 (a) holds.

(2) First, by applying Lemma 2-3, we shall show

$$T^{+}_{h}(x,y,t) \geq T^{+}_{\frac{1}{2}h}(x,y,t) \tag{2.18}$$

for any $(x,y,t) \in G^{+}_{h,1}$.

(3) Second, by induction method we shall show

$$T^{+}_{h}(x,y,t) \geq T^{+}_{\frac{1}{2}h}(x,y,t) \tag{2.24}$$

for any $(x,y,t) \in G^{+}_{h,m}$, $m=2,3,\ldots,m^{+}$

where

$$(m^{+}-1)h < T^{+}_{h}(x_o,y_o,t_o) \leq m^{+}h$$

The proof involves three steps:

(a) Corresponding to each $x_1 \in A_x(t+h)$, there exists a point $y^{*}_{1}(x_1)$ (here, we express the dependence of y^{*}_{1} on x_1 explicitly), such that

$$T^{+}_{\frac{1}{2}h}(x,y,t) \leq h + T^{+}_{\frac{1}{2}h}(x_1,y^{*}_{1}(x_1),t+h) \tag{2.28}$$

(Existence of such $y^{*}_{1}(x_1)$ comes from Lemma 2-4 (b)').

(b) There exists a point $x^{*}_{1} \in A_x(t+h)$ such that

$$T^{+}_{h}(x,y,t) \geq h + T^{+}_{h}(x^{*}_{1},y^{*}_{1}(x^{*}_{1}),t+h) \tag{2.30}$$

(Existence of such x^{*}_{1} comes from Lemma 2-4 (b)).

(c) Assume (2.24) holds for $m=2,3,\ldots,n<m_o^+$ (induction hypothesis).

Now, corresponding to any $(x,y,t) \in G^+_{h,n+1}$, take x_1^* (by (2.30)) and $y_1^*(x_1^*)$ (by (2.28)).

Then, by (2.30),

$$T_h^+(x_1^*, y_1^*(x_1^*), t+h) \leq nh \quad .$$

Hence, by induction hypothesis (c),

$$T_h^+(x_1^*, y_1^*(x_1^*), t+h) \geq T^+_{\frac{1}{2}h}(x_1^*, y_1^*(x_1^*), t+h) \tag{2.32}$$

Therefore, we have

$$\begin{aligned} T^+_{\frac{1}{2}h}(x,y,t) &\leq h + T^+_{\frac{1}{2}h}(x_1^*, y_1^*(x_1^*), t+h) && \text{(by (2.28))} \\ &\leq h + T_h^+(x_1^*, y_1^*(x_1^*), t+h) && \text{(by (2.32))} \\ &\leq T_h^+(x,y,t) && \text{(by (2.30))} \end{aligned}$$

<u>Remark 1</u>: Heuristically, we can prove (2.16)' as follows:

$$\begin{aligned} T_h^+(x_o,y_o,t_o) &= \min_{u_o(\cdot)} \max_{v_o(\cdot)} \cdots\cdots \min_{u_m(\cdot)} \max_{v_m(\cdot)} \hat{t} - t_o \\ &= \min_{u_o^1(\cdot)} \min_{u_o^2(\cdot)} \max_{v_o^1(\cdot)} \max_{v_o^2(\cdot)} \cdots\cdots \\ &\qquad \cdots \min_{u_m^1(\cdot)} \min_{u_m^2(\cdot)} \max_{v_m^1(\cdot)} \max_{v_m^2(\cdot)} \hat{t} - t_o \\ &\geq \min_{u_o^1(\cdot)} \max_{v_o^1(\cdot)} \min_{u_o^2(\cdot)} \max_{v_o^2(\cdot)} \cdots\cdots \\ &\qquad \cdots \min_{u_m^1(\cdot)} \max_{v_m^1(\cdot)} \min_{u_m^2(\cdot)} \max_{v_m^2(\cdot)} \hat{t} - t_o \\ &= T^+_{\frac{1}{2}h}(x_o,y_o,t_o) \end{aligned}$$

where $u_i(\cdot)$ = a Pursuer's admissible control segment on $[t_i, t_{i+1})$ $i=0,1,2,\ldots,m$

$u_i^1(\cdot)$ = a Pursuer's admissible control segment on $[t_i, t_i+\frac{1}{2}h)$ $i=0,1,2,\ldots,m$

$u_i^2(\cdot)$ = a Pursuer's admissible control segment on $[t_i+\frac{1}{2}h, t_{i+1})$ $i=0,1,2,\ldots,m$

$v_i(\cdot)$, $v_i^1(\cdot)$, and $v_i^2(\cdot)$ are similarly defined.
We can prove (2.17)' similarly.

Remark 2: In G_h^+, Evader knows the Pursuer's control segment on $[t_i, t_i+h)$, at each time t_i, $t_i=t_o+ih$, $i=0,1,2,\ldots$, befofe he decides his control segment on $[t_i, t_i+h)$.
On the other hand, in $G_{\frac{1}{2}h}^+$, Evader knows the Pursuer's control segment on $[t_i, t_i+\frac{1}{2}h)$ at each time t_i, $t_i=t_o+i\frac{1}{2}h$, $i=0,1,2,\ldots$, before he decides his control segment on $[t_i, t_i+\frac{1}{2}h)$.
Hence, the information pattern available for Evader is more advantageously biased in G_h^+ than in $G_{\frac{1}{2}h}^+$.
Hence, (2.16)' is a reasonable result.
Similar facts hold for minorant games.

Proof of Theorem 2-1:

(1) Let $h \to 0$ through a sequence h, $\frac{1}{2}h$, ... , $h/2^k$, ... , and

$$T_h^+(x_o, y_o, t_o) \geq T_{\frac{1}{2}h}^+(x_o, y_o, t_o) \geq \cdots\cdots$$
$$\cdots \geq T_{h/2^k}^+(x_o, y_o, t_o) \geq \cdots\cdots \tag{2.16}$$

and

$$T_h^-(x_o,y_o,t_o) \le T_{\frac{1}{2}h}^-(x_o,y_o,t_o) \le \cdots\cdots$$

$$\cdots \le T_{h/2^k}^-(x_o,y_o,t_o) \le \cdots\cdots \tag{2.17}$$

for any h, $0<h\le\bar{h}$, are shown in what follows.

If (2.16) and (2.17) hold, by Lemma 2-2,

$$T_h^+(x_o,y_o,t_o) \ge T_{h/2^k}^+(x_o,y_o,t_o) \ge T_{h/2^k}^-(x_o,y_o,t_o)$$

hold for any $k=0,1,2,\ldots$.

But, by assumed hypothesis, we have

$$T_h^+(x_o,y_o,t_o) < \infty ,$$

hence, we have

$$T_{h/2^k}^-(x_o,y_o,t_o) < \infty \qquad \text{for any } k=0,1,2,\ldots$$

and

$$T_{h/2^k}^+(x_o,y_o,t_o) < \infty \qquad \text{for any } k=0,1,2,\ldots .$$

Hence, $\left\{T_{h/2^k}^-(x_o,y_o,t_o)\right\}$ is a non-decreasing sequence with $T_{h/2^k}^-(x_o,y_o,t_o)<\infty$ for any $k=0,1,2,\ldots$.

Hence, $T_{h/2^k}^-(x_o,y_o,t_o)$ converges to a limit as k goes to ∞ .

Similarly $\left\{T_{h/2^k}^+(x_o,y_o,t_o)\right\}$ is a non-increasing sequence with $T_{h/2^k}^+(x_o,y_o,t_o)>0$ for any $k=0,1,2,\ldots$.

Hence, $T_{h/2^k}^+(x_o,y_o,t_o)$ converges to a limit as k goes to ∞ .

Since h, $0<h\le\bar{h}$, is arbitrary, we can conclude that Theorem 2-1 holds.

(2) Now, we shall establish (2.16).

Since we assumed $(x_o,y_o,t_o) \in G_h^+$, $0<h\le\bar{h}$, there exists an interger m^+ such that

$$(m^+ - 1)h < T_h^+(x_o, y_o, t_o) \leq m^+h$$

By Lemma 2-1, we see that

$$G_{h,m}^+ \neq \phi \qquad \text{for any } m=1,2,\ldots,m^+-1$$

We now show

$$T_{\frac{1}{2}h}^+(x,y,t) \leq T_h^+(x,y,t) \qquad (2.18)$$

$$\text{for any } (x,y,t) \in G_{h,1}^+ .$$

It is easily seen that if $0<T_h^+(x,y,t)\leq\frac{1}{2}h$ or (and) $0<T_{\frac{1}{2}h}^+(x,y,t)\leq\frac{1}{2}h$ hold, then the inequality (2.18) trivially holds.

Suppose $h<T_{\frac{1}{2}h}^+(x,y,t)$ holds.
Since we assumed $0<T_h^+(x,y,t)\leq h$, there exists a graph $\hat{f}_x$ such that (see Lemma 2-3 (a)' and Assumption A2)

$$T_h^+(x,y,t) + t$$

$$\geq \min_{t'} (t' : \text{dist}(f_y(t'), \hat{f}_x(t') = \epsilon)$$

$$\text{for any } f_y \in A_y$$

Hence,

$$h + t \geq \min_{t'} (t' : \text{dist}(f_y(t'), \hat{f}_x(t')) = \epsilon)$$

$$\text{for any } f_y \in A_y .$$

This means that even in $G_{\frac{1}{2}h}^-$ if Pursuer follows $\hat{f}_x$, the ϵ-capture is guaranteed to occur before time h+t, independent of the Evader's trajectory.

This contradicts assumed hypothesis.

The remaining possibility is

$$\tfrac{1}{2}h < T^{+}_{\frac{1}{2}h}(x,y,t) \leq h \qquad \text{and} \qquad (2.19)$$

$$\tfrac{1}{2}h < T^{+}_{h}(x,y,t) \leq h \quad . \qquad (2.20)$$

Let $\hat{f}_x \in A_x$ be a trajectory which attains the minimum of

$$\max_{f_y \in A_y} \min_{t'} (t' : \mathrm{dist}(f_y(t'), f_x(t') + B_\epsilon) = 0) \ .$$

Then, by Lemma 2-3 (a)', we have

$$T^{+}_{h}(x,y,t) + t$$

$$= \max_{f_y \in A_y} \min_{t'} (t' : \mathrm{dist}(f_y(t'), \hat{f}_x(t') + B_\epsilon) = 0) \qquad (2.21)$$

Since, in view of (2.20), we have

$$T^{+}_{h}(x,y,t) + t$$

$$= \max_{y_1 \in A_y(t+\frac{1}{2}h)} \max_{f_{y_1} \in A_{y_1}} \min_{t'}$$

$$(t' : \mathrm{dist}(f_{y_1}(t'), \hat{f}_{x_1^*}(t') + B_\epsilon) = 0) \qquad (2.22)$$

where $x_1^* = \hat{f}_x(t+\frac{1}{2}h)$ and $\hat{f}_{x_1^*}$ represents a graph emanating from $(x_1^*, t+\frac{1}{2}h)$ such that

$$\hat{f}_{x_1^*}(t') = \hat{f}_x(t') \qquad \text{for any } t',\ t+\tfrac{1}{2}h \leq t' < \infty \ .$$

But, by Lemma 2-3 (b)' and (2.19), we have

$$T^{+}_{\frac{1}{2}h}(x,y,t) + t$$

$$\leq \max_{y_1 \in A_y(t+\frac{1}{2}h)} \max_{f_{y_1} \in A_{y_1}} \min_{t'}$$

$$(t' : \mathrm{dist}(f_{y_1}(t'), f_{x_1}(t') + B_\epsilon) = 0)$$

$$\text{for any } f_{x_1} \in A_{x_1} \text{ and for any } x_1 \in A_x(t+\tfrac{1}{2}h) \tag{2.23}$$

By (2.22) and(2.23), we can obtain (2.18).

(3) Next, we shall show

$$T^+_{\frac{1}{2}h}(x,y,t) \leq T^+_h(x,y,t) \tag{2.24}$$

for any $(x,y,t) \in G^+_{h,m}$, $m=2,3,\ldots ,m^+$, $m^+ \geq 2$.
(if $m^+=1$, (2.18) completes the proof).

First, for any $(x,y,t) \in G^+_{h,m}$, $2 \leq m \leq m^+$, by (2.13),we have

$$T^+_h(x,y,t) = \min_{u_1(\cdot)} \max_{v_1(\cdot)} (h + T^+_h(x_1,y_1,t+h)) \tag{2.25}$$

Second, for any $(x,y,t) \in G^+_{h,m}$, $2 \leq m \leq m^+$, if
$(x,y,t) \in G^+_{\frac{1}{2}h,m'}$, $m'=1,2$, then we trivially have (2.24).
If $(x,y,t) \in G^+_{\frac{1}{2}h,m'}$, $m' \geq 3$, then using the recurrence relation (for the $\frac{1}{2}h$ processes) similar to (2.13) twice, we have

$$T^+_{\frac{1}{2}h}(x,y,t) = \min_{u^1_1(\cdot)} \max_{v^1_1(\cdot)} \min_{u^2_1(\cdot)} \max_{v^2_1(\cdot)}$$

$$(h + T^+_{\frac{1}{2}h}(x_1,y_1,t+h)) \tag{2.26}$$

where $$x_1 = x + \int_t^{t+\frac{1}{2}h} f(x(t'),u^1_1(t'),t')\,dt'$$

$$+ \int_{t+\frac{1}{2}h}^{t+h} f(x(t'),u_1^2(t'),t')dt'$$

$u_1^1(\cdot),u_1^2(\cdot)$ = a Pursuer's admissible control segment on $[t,\ t+\frac{1}{2}h)$ and $[t+\frac{1}{2}h,\ t+h)$, respectively

$\mathcal{U}_1^1,\mathcal{U}_1^2$ = the set of all Pursuer's admissible control segments on $[t,\ t+\frac{1}{2}h)$ and $[t+\frac{1}{2}h,\ t+h)$, respectively

and y_1, $v_1^1(\cdot)$, $v_1^2(\cdot)$, $\mathcal{V}_1^1$, and $\mathcal{V}_1^2$ are similarly defined. The minimization and maximization are over the sets $\mathcal{U}_1^1,\mathcal{U}_1^2$ and $\mathcal{V}_1^1$, $\mathcal{V}_1^2$ such that

$$\text{dist}(f_x^{u_1^1+u_1^2}(t'),\ f_y^{v_1^1+v_1^2}(t')) > \epsilon$$

$$\text{for any } t',\ t \leq t' < t+h \quad .$$

From (2.25), in view of Lemma 2-4 (b), we see that there exists admissible $u_1^*(\cdot)$ such that

$$T_h^+(x,y,t) \geq h + T_h^+(x_1^*,y_1,t+h) \tag{2.27}$$

$$\text{for any admissible } v_1(\cdot)$$

provided that

$$\text{dist}(f_x^{u_1^*}(t'),f_y^{v_1}(t')) > \epsilon \quad \text{for any } t',\ t \leq t' < t+h$$

where

$$x_1^* = x + \int_t^{t+h} f(x(t'),u_1^*(t'),t')dt'$$

From (2.26), in view of Lemma 2-4 (b)', we see that for each admissible $u_1^1(\cdot)$ and $u_1^2(\cdot)$, there correspond admissible $v_1^{1*}(\cdot)$ and $v_1^{2*}(\cdot)$ such that

$$T^{+}_{\frac{1}{2}h}(x,y,t) \leq h + T^{+}_{\frac{1}{2}h}(x_1,y_1^*,t+h) \tag{2.28}$$

where

$$y_1^* = y + \int_t^{t+\frac{1}{2}h} g(y(t'),v_1^{1*}(t'),t')dt' + \int_{t+\frac{1}{2}h}^{t+h} g(y(t'),v_1^{2*}(t'),t')dt'$$

In this case, we have

$$\mathrm{dist}(f_x^{u_1^1+u_1^2}(t'),\ f_y^{v_1^{1*}+v_1^{2*}}(t')) > \epsilon \tag{2.28)"}$$

for any admissible $u_1^1(\cdot)$ and $u_1^2(\cdot)$ and for any t',

$t \leq t' < t+h$.

Otherwise, we clearly have

$$T^{+}_{\frac{1}{2}h}(x,y,t) \leq h$$

which contradicts assumed hypothesis $(x,y,t) \in G^{+}_{\frac{1}{2}h,m'}$, $m' \geq 3$.

Let us suppose

$$T^{+}_{\frac{1}{2}h}(x,y,t) \leq T^{+}_{h}(x,y,t) \tag{2.29}$$

holds for any $(x,y,t) \in G^{+}_{h,m}$, $m=2,3,\ldots,n$, $2 \leq n < m^+$

Let us suppose $(x,y,t) \in G^{+}_{h,n+1}$.

Corresponding to $u_1^*(\cdot)=u_1^{1*}(\cdot)+u_1^{2*}(\cdot)$ (see (2.27)), we construct $v_1^{1*}(\cdot)$ and $v_1^{2*}(\cdot)$ by (2.28). Then we have

$$T^{+}_{\frac{1}{2}h}(x,y,t) \leq h + T^{+}_{\frac{1}{2}h}(x_1^*,y_1^*,t+h) \tag{2.28)'}$$

In view of (2.28)" , we have, (by (2.27))

$$T_h^+(x,y,t) \geq h + T_h^+(x_1^*,y_1^*,t+h) \tag{2.30}$$

Therefore, we have

$$(x_1^*,y_1^*,t+h) \in G_{h,m}^+ , \quad m\leq n \tag{2.31}$$

Hence, by induction hypothesis (2.29), we have

$$T_h^+(x_1^*,y_1^*,t+h) \geq T_{\frac{1}{2}h}^+(x_1^*,y_1^*,t+h) \tag{2.32}$$

Therefore, we have

$$\begin{aligned} T_{\frac{1}{2}h}^+(x,y,t) &\leq h + T_{\frac{1}{2}h}^+(x_1^*,y_1^*,t+h) && \text{(by (2.28)')} \\ &\leq h + T_h^+(x_1^*,y_1^*,t+h) && \text{(by (2.32))} \\ &\leq T_h^+(x,y,t) && \text{(by (2.30))} \qquad (2.33) \end{aligned}$$

By induction, taking $n+1=m^+$, we can conclude that

$$T_h^+(x_o,y_o,t_o) \geq T_{\frac{1}{2}h}^+(x_o,y_o,t_o) \tag{2.34}$$

(4) By almost identical arguments, we see that

$$T_{\frac{1}{2}h}^-(x_o,y_o,t_o) \geq T_h^-(x_o,y_o,t_o) \tag{2.35}$$

From (2.34) and (2.35), using Lemma 2-2, we obtain

$$\begin{aligned} T_h^-(x_o,y_o,t_o) &\leq T_{\frac{1}{2}h}^-(x_o,y_o,t_o) \\ &\leq T_{\frac{1}{2}h}^+(x_o,y_o,t_o) \leq T_h^+(x_o,y_o,t_o) \end{aligned} \tag{2.36}$$

In view of (2.28)" , we have, (by (2.27))

$$T_h^+(x,y,t) \geq h + T_h^+(x_1^*,y_1^*,t+h) \tag{2.30}$$

Therefore, we have

$$(x_1^*,y_1^*,t+h) \in G_{h,m}^+ , \quad m \leq n \tag{2.31}$$

Hence, by induction hypothesis (2.29), we have

$$T_h^+(x_1^*,y_1^*,t+h) \geq T_{\frac{1}{2}h}^+(x_1^*,y_1^*,t+h) \tag{2.32}$$

Therefore, we have

$$
\begin{aligned}
T_{\frac{1}{2}h}^+(x,y,t) &\leq h + T_{\frac{1}{2}h}^+(x_1^*,y_1^*,t+h) && \text{(by (2.28)')} \\
&\leq h + T_h^+(x_1^*,y_1^*,t+h) && \text{(by (2.32))} \\
&\leq T_h^+(x,y,t) && \text{(by (2.30))} \qquad (2.33)
\end{aligned}
$$

By induction, taking $n+1=m^+$, we can conclude that

$$T_h^+(x_o,y_o,t_o) \geq T_{\frac{1}{2}h}^+(x_o,y_o,t_o) \tag{2.34}$$

(4) By almost identical arguments, we see that

$$T_{\frac{1}{2}h}^-(x_o,y_o,t_o) \geq T_h^-(x_o,y_o,t_o) \tag{2.35}$$

From (2.34) and (2.35), using Lemma 2-2, we obtain

$$
\begin{aligned}
T_h^-(x_o,y_o,t_o) &\leq T_{\frac{1}{2}h}^-(x_o,y_o,t_o) \\
&\leq T_{\frac{1}{2}h}^+(x_o,y_o,t_o) \leq T_h^+(x_o,y_o,t_o)
\end{aligned}
\tag{2.36}
$$

By the repetition of same arguments, replacing h by $\frac{1}{2}h$, we obtain

$$T^-_{\frac{1}{2}h}(x_o,y_o,t_o) \le T^-_{\frac{1}{4}h}(x_o,y_o,t_o)$$
$$\le T^+_{\frac{1}{4}h}(x_o,y_o,t_o) \le T^+_{\frac{1}{2}h}(x_o,y_o,t_o)$$

Proceeding similarly, we obtain (2.16) and (2.17).
This completes the proof.

Corollary:

If there exists $h^*>0$ such that $T^-_h(x_o,y_o,t_o)<\infty$ holds for any h, $0<h\le h^*$, then $T^-_h(x_o,y_o,t_o)$ converges to a limit as h goes to zero.

Proof:

By the proof of Theorem 2-1, we know that $\left\{T^-_{h/2^k}(x_o,y_o,t_o)\right\}$, $k=0,1,2,\ldots$, is a non-decreasing sequence for any h, $0<h\le h^*$.

By hypothesis, we have

$$T^-_{h/2^k}(x_o,y_o,t_o) < \infty \quad , \quad k=0,1,2,\ldots .$$

Hence,

$$\lim_{h\to\infty} T^-_{h/2^k}(x_o,y_o,t_o) \qquad \text{exists.}$$

C. MINIMAX THEOREM

In this section, we show a minimax theorem for "closed-loop" games. First, conditions which will be necessary for our theorem are enumerated.

Let there be given an ϵ, $0 \leq \epsilon < \epsilon_o$, (see M-5) and an initial condition $(x_o, y_o, t_o) \in E^n \times E^n \times (-\infty, \infty)$.

Let us define a set $B = \bigsqcup_{t_o \leq t < \infty} A_{x_o}(t) \times A_{y_o}(t) \times t$

Conditions:

M-1 There exists $\bar{h} > 0$, such that for any h, $0 < h \leq \bar{h}$,

$$B \subset G_h^+ \quad .$$

M-2 For any $x \in A_{x_o}(t)$ and $y \in A_{y_o}(t)$, $A_x(t')$ and $A_y(t')$ are compact and convex for all t and t',

$t_o \leq t \leq t' < \infty$ (see At-4 and At-5).

M-3 For any t, $t_o \leq t < \infty$, and for any h, $0 < h \leq \bar{h}$, $T_h^-(x,y,t)$ and $T_h^+(x,y,t)$ are continuous in (x,y) on $A_{x_o}(t) \times A_{y_o}(t)$.

M-4 For any t, $t_o \leq t < \infty$, and for any h, $0 < h \leq \bar{h}$,

(A) $T_h^-(x,y,t)$ is quasi-convex in x on $A_{x_o}(t)$, for each $y \in A_{y_o}(t)$, and

(B) for each $x \in A_{x_o}(t)$, there exists only one point $y^{**} \in A_{y_o}(t)$ such that

$$T_h^+(x,y^{**},t) \geq T_h^+(x,y,t) \qquad \text{for all } y \in A_{y_o}(t)$$

<u>M-4'</u> For any t, $t_o \leq t < \infty$, and for any h, $0 < h \leq \bar{h}$,

(A) $T_h^+(x,y,t)$ is quasi-concave in y on $A_{y_o}(t)$, for each $x \in A_{x_o}(t)$, and

(B) for each $y \in A_{y_o}(t)$, there exists only one point $x^{**} \in A_{x_o}(t)$ such that

$$T_h^-(x^{**},y,t) \leq T_h^-(x,y,t) \quad \text{for all } x \in A_{x_o}(t)$$

<u>M-5</u> There exists $\epsilon_o > 0$ such that $T_{h,\epsilon}^+(x,y,t)$ (see Remark 1) is equicontinuous (see Remark 2) in ϵ on $[0, \epsilon_o]$ for any $(x,y,t) \in B$ and any h, $0 < h \leq \bar{h}$.

<u>Remark 1</u>: For convenience, we shall express, when necessary, dependence of the optimal ϵ-capture time $T_h^+(x,y,t)$ on ϵ explicitly as

$$T_h^+(x,y,t) = T_{h,\epsilon}^+(x,y,t) \qquad (2.37)$$

for any $(x,y,t) \in B$ and h, $0 < h \leq \bar{h}$.

Similar notation will be used for $T_h^-(x,y,t)$.

<u>Remark 2</u>: We say $T_{h,\epsilon}^+(x,y,t)$ is equicontinuous in ϵ on $[0, \epsilon_o]$ for any $(x,y,t) \in B$ and any h, $0 < h \leq \bar{h}$, if for any d>0, there exists e>0 such that

$$\left.\begin{array}{l} |\epsilon_1 - \epsilon_2| \leq e \\ 0 \leq \epsilon_1, \epsilon_2 \leq \epsilon_o \\ 0 < h \leq \bar{h} \\ (x,y,t) \in B \end{array}\right\rangle \text{ imply } \left|T_{h,\epsilon_1}^+(x,y,t) - T_{h,\epsilon_2}^+(x,y,t)\right| \leq d$$

<u>Remark 3</u>: A scalar function $f(\cdot)$ defined on a convex set $X \subset E^n$

is said to be <u>quasi-convex</u> on X if the set

$$\{x : x \in X \ , \ f(x) \le k\}$$

is convex for each scalar k .

A scalar function $f(\cdot)$ defined on a convex set $X \subset E^n$ is said to be <u>quasi-concave</u> on X if the set

$$\{x : x \in X \ , \ f(x) \ge k\}$$

is convex for each scalar k .

<u>Theorem 2-2</u>:

Suppose Conditions M-1, 2, 3, 4 (or 4'), and 5 are satisfied. Then, for each $d>0$, there exists h_1, $0 < h_1 \le \bar{h}$, such that

$$0 < h \le h_1 \quad \text{implies} \quad T_h^+(x_o, y_o, t_o) - T_h^-(x_o, y_o, t_o) \le d \tag{2.38}$$

In what follows, we shall assume Conditions M-1, 2, 3, 4 (or 4'), and 5 are satisfied.

<u>Outline of Proof</u>:

For each $d>0$, we show that there exists $h_1(d)$ which satisfies (2.38).

(1) Construct $h_2(d)$ (by Lemma 2-5), $e_1(d)$ (by (2.58)), $h(e_1(d))$ (by Lemma 2-7), and define

$$h_1(d) = \min(h_2(d),\ h(e_1(d))).$$

(2) Take any h, $0 < h \le h_1(d)$ and consider three cases;

(a) $\|x_o - y_o\| - e \le e_1(d)$:

By definition of $e_1(d)$, (2.38) automaticaly holds.

(b) $(x_o, y_o, t_o) \in G_{h,1}^-$:

By definition of $h_2(d)$, (2.38) automaticaly holds.

(c) $\|x_o - y_o\| - \epsilon > e_1(d)$ and $(x_o, y_o, t_o) \notin G^-_{h,1}$:

By Lemma 2-6, there exist x_1 and y_1 which satisfy

$$T^+_h(x_o, y_o, t_o) - T^-_h(x_o, y_o, t_o)$$
$$\leq T^+_h(x_1, y_1, t_1) - T^-_h(x_1, y_1, t_1)$$

(3) Considering (x_1, y_1, t_1) as a new initial state, repeat (2) above. It should be noted that the constructions of $h_2(d)$, $e_1(d)$, $h(e_1(d))$, and $h_1(d)$ are independent of initial states.

(4) Since there exists an integer $m>0$ which satisfies (2.61), in view of (2.63), after the repetition of at most $m-1$ times of above procedures, we can complete the proof (i.e., the iterative relations always end, see (2.68)).

Lemma 2-5:

For each $d>0$, there exists h_2, $0 < h_2 \leq \bar{h}$, such that

$$0 < h \leq h_2 \quad \text{implies} \quad T^+_{h,\epsilon}(x,y,t) - T^-_{h,\epsilon}(x,y,t) \leq d\ ,$$

for any $(x,y,t) \in B \cap G^-_{h,1}$ and any ϵ, $0 \leq \epsilon < \epsilon_o$.

Proof:

(1) From Assumption A1, we see that corresponding to each h, $0 < h \leq \bar{h}$, there exists a real number d_h which satisfies

$$d_h = \max_{x_1, x_2 \in A_{x(t)}(t+h)} \|x_1 - x_2\|\ ,$$

for any t, $t_o \leq t < \infty$, and for any $x(t) \in A_{x_o}(t)$.

For simplicity, let us denote $x(t) = x$.

(2) For any f_x and $f_x' \epsilon A_x$, we have

$$f_x(t') + B_{(\epsilon+d_h)} \supset f_x'(t') + B_\epsilon \qquad \text{for any } t',\ t \le t' \le t+h$$

Hence,

$$f_x(t') + B_{(\epsilon+d_h)} \supset \bigcup_{f_x' \epsilon A_x} (f_x'(t') + B_\epsilon) = A_x(t') + B_\epsilon \qquad \text{for any } t',\ t \le t' \le t+h$$

This holds for any $f_x \epsilon A_x$ and any $x \epsilon A_{x_o}(t)$.

(3) By Lemma 2-3 (a), we have, for any $(x,y,t) \epsilon B \cap G^-_{h,1}$

$$\begin{aligned} & T^-_{h,\epsilon}(x,y,t) + t \\ &= \max_{f_y \epsilon A_y} \min_{t'} (t' : \operatorname{dist}(f_y(t'), A_x(t') + B_\epsilon) = 0) \\ &\ge \max_{f_y \epsilon A_y} \min_{t'} (t' : \operatorname{dist}(f_y(t'), f_x(t') + B_{(\epsilon+d_h)}) = 0) \end{aligned}$$

$$\text{for any } f_x \subset A_x$$

Hence,

$$\begin{aligned} & T^-_{h,\epsilon}(x,y,t) + t \\ &\ge \min_{f_x \epsilon A_x} \max_{f_y \epsilon A_y} \min_{t'} (t' : \operatorname{dist}(f_y(t'), f_x(t') + B_{(\epsilon+d_h)}) = 0) \\ &= T^+_{h,\epsilon+d_h}(x,y,t) + t \end{aligned} \tag{2.39}$$

(4) By Lemma 2-2 and (2.39), we have

$$T^+_{h,\epsilon+d_h}(x,y,t) \le T_{h,\epsilon}(x,y,t) \le T^+_{h,\epsilon}(x,y,t) \tag{2.40}$$

for any h, $0 < h \le \bar{h}$, any ϵ, $0 \le \epsilon < \epsilon_o$, and any $(x,y,t) \epsilon B \cap G^-_{h,1}$.

(5) By Condition M-5, for any $d>0$, there exists $\bar{d}>0$ such that

$$0 < d_h < \bar{d} \quad \text{implies} \quad T^+_{h,\epsilon}(x,y,t) - T^+_{h,\epsilon+d_h}(x,y,t) \le d$$

for any h, $0 < h \leq \bar{h}$, any ϵ, $0 \leq \epsilon < \epsilon_o$, and any $(x,y,t) \in B \cap G^-_{h,1}$.

By the continuity property of attainability sets (see At-2), for any $\bar{d}>0$, there exists $h_2>0$ such that

$$0 < h \leq h_2 \quad \text{implies} \quad 0 < d_h \leq \bar{d}$$

Hence, for any $d>0$, there exists $h_2>0$ such that

$$0 < h \leq h_2 \quad \text{implies} \quad T^+_{h,\epsilon}(x,y,t) - T^+_{h,\epsilon+d_h}(x,y,t) \leq d$$

But, by (2.40), we see that

$$T^+_{h,\epsilon}(x,y,t) - T^-_{h,\epsilon}(x,y,t) \leq T^+_{h,\epsilon}(x,y,t) - T^+_{h,\epsilon+d_h}(x,y,t)$$

Hence,

$$T^+_{h,\epsilon}(x,y,t) - T^-_{h,\epsilon}(x,y,t) \leq d$$

holds for any h, $0 < h \leq h_2$, any ϵ, $0 \leq \epsilon < \epsilon_o$, and any $(x,y,t) \in B \cap G^-_{h,1}$. This completes the proof.

By Condition M-1, for any $(x,y,t) \in B$, we have $(x,y,t) \in G^+_h$, hence $(x,y,t) \in G^-_h$.

Therefore, there exists a positive integer m such that

$$(m-1)h < T^-_h(x,y,t) \leq mh.$$

Let us consider the case where $m \geq 2$.

In the majorant game, by Lemma 2-4 (b)', corresponding to any Pursuer's admissible $u_1(\cdot)$, Evader can find admissible $v^*_1(\cdot)$ such that

$$T_h^+(x,y,t) \le h + T_h^+(x_1, y_1^*, t_1) \tag{2.41}$$

where

$$t_1 = t + h$$

$$x_1 = x + \int_t^{t_1} f(x(t'), u_1(t'), t')dt'$$

$$y_1^* = y + \int_t^{t_1} g(y(t'), v_1^*(t'), t')dt' .$$

Namely, for any $x_1 \in A_x(t_1)$, there exists $y_1^*(x_1) \in A_y(t_1)$ (here, we express the dependence of $y_1^*(x_1)$ on x_1 explicitly) such that

$$T_h^+(x,y,t) \le h + T_h^+(x_1, y_1^*(x_1), t_1) \tag{2.41)'}$$

holds.

On the other hand, in the minorant game, by Lemma 2-4 (a)', corresponding to Evader's state y_1^* above, Pursuer can find admissible $u_1^*(\cdot)$ such that

$$T_h^-(x,y,t) \ge h + T_h^-(x_1^*, y_1^*, t_1) \tag{2.42}$$

provided

$$\mathrm{dist}(f_x^{u_1^*}(t'),\ f_y^{v_1^*}(t')) > \epsilon \tag{2.43}$$

for any t', $t \le t' < t_1$

where

$$x_1^* = x + \int_t^{t_1} f(x(t'), u_1^*(t'), t')dt' .$$

Namely, corresponding to $y_1^*(x_1)$, there exists $x_1^*(y_1^*(x_1)) \in A_x(t_1)$ (here, we express the dependence of $x_1^*(y_1^*(x_1))$ on $y_1^*(x_1)$ explicitly) such that

$$T_h^-(x,y,t) \geq h + T_h^-(x_1^*(y_1^*(x_1)),y_1^*(x_1),t_1) \qquad (2.42)'$$

holds, if (2.43) is satisfied.
It should be noted that if

$$A_x(t') \cap (A_y(t') + B_\varepsilon) = \emptyset \qquad (2.43)'$$

holds for any t', $t \leq t' < t_1$,

then, (2.43) always holds.

Lemma 2-6:
If $(x,y,t) \in B$ and h, $0 < h \leq \bar{h}$, satisfy $(x,y,t) \in G_{h,m}^-$, $m \geq 2$ and (2.43)'; then there exists $x_1 \in A_x(t_1)$ such that

$$x_1 = x_1^*(y_1^*(x_1))$$

Fixed-point theorem (S. Kakutani [K1], [B4]) is first recalled.
Let C be a non-empty, compact, and convex set in E^n.
If $F(\cdot)$(=general mapping, see Remark 1 after (1.1)) is an upper semi-continuous mapping from C into C and if the set $F(x)$ is convex and non-empty for each x in C, then there exists a point x_o in C such that

$$x_o \in F(x_o)$$

Following C. Berge [B4], we shall use the following terminologies. Let $F(\cdot)$ be a mapping from a topological space X into a topological space Y. Let x be a point of X. we say that $F(\cdot)$ is upper semi-continuous at x_o if for each open set G containing $F(x_o)$, there exists a neighborhood $U(x_o)$ such that

$$x \in U(x_o) \quad \text{implies} \quad F(x) \subset G$$

We say that $F(\cdot)$ is upper semi-continuous in X if it is upper semicontinuous at each point of X and if F(x) is a compact set for each x in X.

We say that $F(\cdot)$ is a closed mapping from X into Y if whenever $x_o \epsilon X$, $y_o \epsilon Y$, $y_o \notin F(x_o)$ there exist two neighborhoods $U(x_o)$ and $V(y_o)$ such that

$$x \in U(x_o) \quad \text{implies} \quad F(x) \cap V(y_o) = \phi$$

Following facts are proven in [B4].

Fact 1: (see [B4])

The graphical representation

$$\sum_{x \epsilon X} F(x) = \{(x,y) : x \epsilon X,\ y \epsilon Y,\ y \epsilon F(x)\} \tag{2.44}$$

of $F(\cdot)$ is closed in X x Y if and only if $F(\cdot)$ is a closed mapping.

Fact 2: (see [B4])

If Y is a compact space, a mapping from X into Y is closed if and only if it is upper semi-continuous.

Proof of Lemma 2-6:

(1) (Sets $Y(x_1)$ and $X(x_1)$)

For convenience, we consider $(x,y,t) \in B \cap G_{h,m}$, $m \geq 2$, and h, $0 < h \leq \bar{h}$, as fixed. However, we shall see the following discussions hold for any such (x,y,t) and h.

For each point $x_1 \in A_x(t_1)$, $t_1 = t+h$, let us define a set $Y(x_1)$ by

$$Y(x_1) = \left\{ y_1 \epsilon A_y(t_1) \ : \ T_h^+(x,y,t) \leq h + T_h^+(x_1,y_1,t_1) \right\} \quad (2.45)$$

By Lemma 2-4 (b)', for each $x_1 \epsilon A_x(t_1)$, $Y(x_1)$ is a non-empty set.

For each $x_1 \epsilon A_x(t_1)$, define a point $y_1^{**}(x_1)$ by

$$T_h^+(x_1,y_1,t_1) \leq T_h^+(x_1,y_1^{**}(x_1),t_1) \quad (2.46)$$

$$\text{for any } y_1 \epsilon A_y(t_1)$$

By Condition M-4 (B), $y_1^{**}(x_1)$ is uniquely determined for each $x_1 \epsilon A_x(t_1)$. It is easy to see $y_1^{**}(x_1) \epsilon Y(x_1)$,

for any $x_1 \epsilon A_x(t_1)$.

For each $x_1 \epsilon A_x(t_1)$, let us define a set $X(x_1)$ by

$$X(x_1) = \left\{ x_1' \epsilon A_x(t_1) \ : \ T_h^-(x,y,t) \geq h + T_h^-(x_1',y_1^{**}(x_1),t_1) \right\} \quad (2.47)$$

(2) ($X(x_1)$ is non-empty, compact, and convex)

In view of (2.43)' and Lemma 2-4 (a)', we see that $X(x_1)$ is non-empty for each $x_1 \epsilon A_x(t_1)$. Since $A_x(t_1)$ is bounded, $X(x_1)$ is bounded for each $x_1 \epsilon A_x(t_1)$. We can also verify that $X(x_1)$ is closed for any $x_1 \epsilon A_x(t_1)$ just the same way as (2.50) to (2.54).

By Condition M-4 (A), $T_h^-(x,y,t)$ is quasi-convex in x on $A_{x_o}(t)$ for each t, $t_o \leq t < \infty$, and $y \epsilon A_{y_o}(t)$. Hence, the set

$$\left\{ x_1' \epsilon A_x(t_1) \ : \ k \geq T_h^-(x_1',y_1^{**}(x_1),t_1) \right\} \quad (2.48)$$

is convex for any k.

Taking $k=T_h^-(x,y,t)-h$, we see that $X(x_1)$ is convex for any $x_1 \in A_x(t_1)$.

Now, $X(\cdot)$ is a mapping from $A_x(t_1)$ into $A_x(t_1)$, which is a non-empty, compact, and convex set in E^n. Furthermore, for each $x_1 \in A_x(t_1)$, the set $X(x_1)$ is non-empty, compact, and convex in $A_x(t_1)$. Remaining task to apply the fixed-point theorem is to show that $X(\cdot)$ is an upper semi-continuous mapping.

(3) ($X(\cdot)$ is upper semi-continuous)

By Fact 1 and Fact 2, $X(\cdot)$ is upper semi-continuous mapping if and only if

$$\sum_{x_1 \in A_x(t_1)} X(x_1) = \left\{(x_1,x_1') : x_1,x_1' \in A_x(t_1), x_1' \in X(x_1)\right\} \tag{2.49}$$

is closed in $A_x(t_1) \times A_x(t_1)$.

Let us take a sequence of points (x_n,x_n'), $n=1,2,\ldots$, such that

$$\begin{aligned} &(x_n,x_n') \longrightarrow (\bar{x},\bar{x}') && \text{and} \\ &x_n' \in X(x_n) && \text{for all } n=1,2,\ldots \end{aligned} \tag{2.50}$$

Suppose $\bar{x}' \notin X(\bar{x})$. Since $A_x(t_1)$ is closed, $\bar{x}' \in A_x(t_1)$. Therefore, our hypothesis implies

$$T_h^-(x,y,t) < h+T_h^-(\bar{x}', y_1^{**}(\bar{x}), t_1) \tag{2.51}$$

But, $x_n' \in X(x_n)$ for all $n=1,2,\ldots$, we have

$$T_h^-(x,y,t) \geq h+T_h^-(x_n', y_1^{**}(x_n), t_1) \tag{2.52}$$

for all $n=1,2,\ldots$.

Now, by the continuity condition of $T_h^-(\cdot,\cdot,t)$ (see M-3), we have

$$\left|T_h^-(\bar{x}', y_1^{**}(\bar{x}), t_1) - T_h^-(x_n', y_1^{**}(x_n), t_1)\right|$$
$$\leq \left|T_h^-(\bar{x}', y_1^{**}(\bar{x}), t_1) - T_h^-(x_n', y_1^{**}(\bar{x}), t_1)\right|$$
$$+ \left|T_h^-(x_n', y_1^{**}(\bar{x}), t_1) - T_h^-(x_n', y_1^{**}(x_n), t_1)\right| \longrightarrow 0 \tag{2.53}$$

as $x_n' \longrightarrow \bar{x}'$, and $x_n \longrightarrow \bar{x}$, if $y_1^{**}(x_n) \longrightarrow y_1^{**}(\bar{x})$ (i.e., $y_1^{**}(\cdot)$ is continuous.) From the uniqueness condition (M-4) of $y_1^{**}(x_1)$ and the continuity condition (M-3) it can also be verified that $y_1^{**}(\cdot)$ is continuous on $A_x(t_1)$. Since (2.53) contradicts (2.51) and (2.52), we have $\bar{x}' \in X(\bar{x})$. Therefore, the set given by (2.49) is closed in $A_x(t_1) \times A_x(t_1)$. This implies that $X(\cdot)$ is upper semi-continuous.

(4) (Fixed-point theorem)

Invoking the Kakutani's theorem, there exists $x_1 \in A_x(t_1)$ such that $x_1 \in X(x_1)$. In view of (2.45) and (2.47), we have

$$T_h^+(x,y,t) \leq h+T_h^+(x_1,y_1^{**}(x_1),t_1) \quad \text{and} \tag{2.54}$$

$$T_h^-(x,y,t) \geq h+T_h^-(x_1,y_1^{**}(x_1),t_1) \tag{2.55}$$

This completes the proof.

Lemma 2-7:

For any $\epsilon \geq 0$ and for each $e>0$, there exists $h(e)>0$ such that

$$A_x(t') \cap (A_y(t') + B_\epsilon) = \emptyset \tag{2.56}$$

for any t', $t \leq t' < t+h(e)$, and

for any $(x,y,t) \in B$ such that $\|x - y\| - \epsilon > e$

Proof:

Directly follows from the property At-2' of the attainability sets. Namely, for any $e' > 0$ there exists $h_1>0$ and $h_2>0$ such that

$$\left.\begin{array}{l} |t - t'| \leq h_1 \\ t_o \leq t_1 \leq t < \infty \\ t_o \leq t_1 \leq t' < \infty \\ x(t_1) \in A_{x_o}(t_1) \end{array}\right\} \text{ imply } \left\{\begin{array}{l} A_{x(t_1)}(t) \subset A_{x(t_1)}(t') + B_{e'} \quad \text{and} \\ A_{x(t_1)}(t') \subset A_{x(t_1)}(t) + B_{e'} \end{array}\right.$$

$$\left.\begin{array}{l} |t - t'| \leq h_2 \\ t_o \leq t_1 \leq t < \infty \\ t_o \leq t_1 \leq t' < \infty \\ y(t_1) \in A_{y_o}(t_1) \end{array}\right\} \text{ imply } \left\{\begin{array}{l} A_{y(t_1)}(t) \subset A_{y(t_1)}(t') + B_{e'} \quad \text{and} \\ A_{y(t_1)}(t') \subset A_{y(t_1)}(t) + B_{e'} \end{array}\right.$$

Taking $t_1 = t$, $x(t_1) = x$, $y(t_1) = y$, we have

$$\left.\begin{array}{l} |t - t'| \leq \min(h_1,h_2) \\ t_o \leq t \leq t' < \infty \\ x \in A_{x_o}(t) \\ y \in A_{y_o}(t) \end{array}\right\} \text{ imply } \left\{\begin{array}{l} A_x(t') \subset x + B_{e'} \quad \text{and} \\ A_y(t') \subset y + B_{e'} \end{array}\right.$$

If $\|x - y\| > \epsilon + 2e'$, then $A_x(t') \cap (A_y(t') + B_\epsilon) = \emptyset$ for any t', $t \leq t' \leq \min(h_1,h_2)$. Regarding $2e' = e$ and $\min(h_1,h_2) = h(e)$, we complete the proof.

Proof of Theorem 2-2:

We are given an initial condition (x_o,y_o,t_o) and ϵ, $0 \leq \epsilon < \epsilon_o$.

For each $d>0$, we show that there exists h_1 which satisfies (2.38) in Theorem 2-2.

(1) For each $d>0$, construct the following constants:

(a) Construction of h_2:

By Lemma 2-5, for each $d>0$, there exists h_2, $0 < h_2 \leq \bar{h}$, such that

$$0 < h \leq h_2 \quad \text{implies} \quad T_h^+(x,y,t) - T_h^-(x,y,t) \leq d \tag{2.57}$$

for any $(x,y,t) \in B \cap G_{h,1}^-$.

(b) Construction of e_1:

By Assumption M-5, for each $d>0$, there exists $e_1>0$, such that

$$\left.\begin{array}{l} \big|\|x - y\| - \epsilon\big| \leq e_1 \\ 0 \leq \|x - y\| \leq \epsilon_o \\ 0 < h \leq \bar{h} \\ (x,y,t) \in B \end{array}\right\} \text{imply} \quad \begin{array}{c} T_{h,\epsilon}^+(x,y,t) - T_{h,\|x-y\|}^+(x,y,t) \leq d \\ \text{that is} \\ T_{h,e}^+(x,y,t) \leq d \end{array} \tag{2.58}$$

It should be noted that $T_{h,\|x-y\|}^+(x,y,t) \equiv 0$

If (2.58) holds, in view of Lemma 2-2, we have

$$T_{h,\epsilon}^+(x,y,t) - T_{h,e}^-(x,y,t) \leq d \tag{2.58$'$}$$

<u>Remark</u>: If $\epsilon_o \leq \epsilon+e_1$, then replace e_1 by $\epsilon_o - \epsilon$ (>0 by assumption) in the following discussions.

(c) Construction of $h(e_1)$:

By Lemma 2-7, for each $e_1>0$, there exists $h(e_1)>0$ such that

$$A_x(t') \cap (A_y(t') + B_\epsilon) = \emptyset \tag{2.59}$$

for any t', $t \leq t' < t+h(e)$, and

for any $(x,y,t) \in B$ such that $\|x-y\| - \epsilon > e_1$.

(d) Construction of h_1:

We define $h_1 = \min(h_2, h(e_1))$ (2.60)

(2) For any h, $0 < h \leq h_1$, we consider the following three cases:

(a) Suppose $\|x_o - y_o\| - \epsilon \leq e_1$ holds.

Then, by (2.58)', we see that the proof completes.

(b) Suppose $(x_o, y_o, t_o) \in G^-_{h,1}$.

Then, by (2.57), we see that the proof completes.

(c) Suppose $\|x_o - y_o\| - \epsilon > e_1$ and $(x_o, y_o, t_o) \notin G^-_{h,1}$.

First, we note that there exists (see Condition (M-1)) an integer $m \geq 2$ such that

$$(m-1)h < T^-_h(x_o, y_o, t_o) \leq mh \tag{2.61}$$

Now, since $\|x_o - y_o\| - \epsilon > e_1$, by (2.59), we have

$$A_{x_o}(t') \cap (A_{y_o}(t') + B_\epsilon) = \emptyset \tag{2.59'}$$

for any t', $t_o \leq t' < t_o+h$

Hence, the requirements for Lemma 2-6 are satisfied. Therefore, there exist $x_1 \in A_{x_o}(t_1)$, $t_1 = t_o+h$, and $y_1 \in A_{y_o}(t_1)$ such that

$$T^+_h(x_o, y_o, t_o) \leq h+T^+_h(x_1, y_1, t_1) \tag{2.62}$$

and

$$T^-_h(x_o, y_o, t_o) \geq h+T^-_h(x_1, y_1, t_1) \tag{2.63}$$

Hence, we have

$$T_h^+(x_o,y_o,t_o) - T_h^-(x_o,y_o,t_o)$$
$$\leq T_h^+(x_1,y_1,t_1) - T_h^-(x_1,y_1,t_1) \qquad (2.64)$$

(3) Since the constructions of h_2, e_1, $h(e_1)$, and h_1 are independent of initial states, if (x_1,y_1,t_1) above satisfies $\|x_1 - y_1\| - \epsilon \leq e_1$ or $(x_1,y_1,t_1) \in G^-_{h,1}$, similar to (2)(a) and (b), we can complete the proof. If $\|x_1 - y_1\| - \epsilon > e_1$ and $(x_1,y_1,t_1) \in G^-_{h,1}$, similar to (2)(c) above, we can construct $x_2 \in A_{x_1}(t_2)$, $t_2 = t_1+h$, and $y_2 \in A_{y_1}(t_2)$ such that

$$\left.\begin{array}{ll} T_h^+(x_1,y_1,t_1) \leq h+T_h^+(x_2,y_2,t_2) & \text{and} \\ T_h^-(x_1,y_1,t_1) \geq h+T_h^-(x_2,y_2,t_2) & \text{hold} \end{array}\right. \qquad (2.65)$$

and proceed similarly.

(4) Now, after repetition of at most m-1 times of above procedures, we obtain

$$mh \geq T_h^-(x_o,y_o,t_o) \geq h+T_h^-(x_1,y_1,t_1) \geq \;.....$$
$$..... \geq (m-1)h+T_h^-(x_{m-1},y_{m-1},t_{m-1}) \qquad (2.66)$$

Hence,
$$h \geq T_h^-(x_{m-1},y_{m-1},t_{m-1}) > 0$$

Hence, by Lemma 2-5, we get

$$T_h^+(x_{m-1},y_{m-1},t_{m-1}) - T_h^-(x_{m-1},y_{m-1},t_{m-1}) \leq d \qquad (2.67)$$

But, similar to (2.64), we have

$$T_h^+(x_o,y_o,t_o) - T_h^-(x_o,y_o,t_o) \leq \;\cdots\cdots$$

$$\cdots\cdots \leq T_h^+(x_{m-1},y_{m-1},t_{m-1}) - T_h^-(x_{m-1},y_{m-1},t_{m-1}) \tag{2.68}$$

By (2.67) and (2.68), we have

$$T_h^+(x_o,y_o,t_o) - T_h^-(x_o,y_o,t_o) \leq d$$

This completes the proof.

Remark: We prove the case where Assumptions M-1, 2, 3, 4, and 5 are satisfied. If Assumptions M-1, 2, 3, 4', and 5 are satisfied, we can similarly prove the theorem.

D. TIME-CONTINUOUS GAME

In this section, we clarify the relation between the limit of approximating discrete games G_h and the time-continuous game G.

Now, let there be given an initial condition (x_o,y_o,t_o) of the games G and G_h. For each $h>0$, define $t_i=t_o+ih$, $i=0,1,2,\ldots$, as before, and let $\{t_i\}$ represent the set of all such t_i . Let $U_h(\cdot,\cdot,\cdot)$ be a single-valued mapping from $E^n \times E^n \times \{t_i\}$ into the set of all measurable functions.

For a given $h>0$, if

$$\mathcal{U}_i \ni U_h(x(t_i),\; y(t_i),\; t_i) \tag{2.69}$$

for any $i=0,1,2,\ldots$, for any $x(t_i) \in A_{x_o}(t_i)$, and for any $y(t_i) \in A_{y_o}(t_i)$, where $\mathcal{U}_i$ is the set of all admissible controls on $[t_i, t_{i+1})$ (see II-A-1), the function $U_h(\cdot,\cdot,\cdot)$ will be called a <u>Pursuer's admissible strategy</u> for discrete games G_h. For each $h>0$, the set of all such functions will be denoted by $\underline{U}_h$. An Evader's admissible strategy $V_h(\cdot,\cdot,\cdot)$ for discrete games G_h and the set $\underline{V}_h$ are similarly defined.

In this thesis, we define a Pursuer's admissible strategy for the time-continuous game G by a pair

$$(h,\ U_h(\cdot,\cdot,\cdot))$$

Namely, a Pursuer's admissible strategy for the game G consists in choosing a positive number h and a function $U_h(\cdot,\cdot,\cdot)$ in $\underline{U}_h$. An Evader's admissible strategy $(h,\ V_h(\cdot,\cdot,\cdot))$ for the time-continuous game G is similarly defined.

Let us consider the following modified minorant game $G_{h,h'}$. Before the game starts, Pursuer and Evader are informed of R-1, R-2, and R-3 (see II-A-1). In the minorant game G_h^-, both players observe their states with the time interval $h>0$. In $G_{h,h'}^-$, Evader observes states with the time interval $h>0$, whereas, Pursuer observes states with the time interval $h'>0$. Other rules for $G_{h,h'}^-$ are the same as those for G_h^-. Similar to the definition of the ϵ-capture time for G_h^- (see (2.3)), the ϵ-capture time for $G_{h,h'}^-$, $h \geq h' > 0$, is defined by

$$T_{h,h'}^-(x_o,y_o,t_o) = \sup_{v_o(\cdot)}\ (\inf_{u_o^1(\cdot)}\ \inf_{u_o^2(\cdot)}\ \ldots\ldots \inf_{u_o^n(\cdot)})\ \ldots\ldots$$

$$\ldots\ldots \sup_{v_m(\cdot)}\ (\inf_{u_m^1(\cdot)}\ \inf_{u_m^2(\cdot)}\ \ldots\ldots \inf_{u_m^n(\cdot)})\ \hat{t} - t_o \qquad (2.70)$$

where $t_m < \hat{t} \leq t_{m+1}$, $t_m = t_o + mh$, $m=0,1,2,\ldots,\infty$

$u_i^j(\cdot)$ = a Pursuer's admissible control segment on $[t_i+(j-1)h', t_i+jh')$, $i=0,1,\ldots,m$, $j=1,2,\ldots,n-1$

$u_i^n(\cdot)$ = a Pursuer's admissible control segment on $[t_i+(n-1)h', t_{i+1})$, $(n-1)h' < h \leq nh'$

$\mathcal{U}_i^j$ = the set of all admissible $u_i^j(\cdot)$, $i=0,1,\ldots,m$ $j=1,2,\ldots,n$

$v_i(\cdot)$ and $\mathcal{V}_i$, $i=0,1,\ldots,m$, are defined in II-A-2.

The supremums and infimums in (2.70) are over the sets $\mathcal{V}_i$ and $\mathcal{U}_i^j$, $i=0,1,\ldots,m$, $j=1,2,\ldots,n$, respectively.

Now, since

$$u_i^1(\cdot) + u_i^2(\cdot) + \cdots\cdots + u_i^n(\cdot) = u_i(\cdot) \quad \text{for } i=0,1,\ldots,m$$

$$T_{h,h'}^-(x_o,y_o,t_o) = \sup_{v_o(\cdot)} \inf_{u_o(\cdot)} \cdots\cdots \sup_{v_m(\cdot)} \inf_{u_m(\cdot)} \hat{t} - t_o$$
$$= T_h^-(x_o,y_o,t_o) \tag{2.71}$$

The e-capture times $T_{h,h'}^-(x_o,y_o,t_o)$, $h' > h > 0$, and $T_{h,h'}^+(x_o,y_o,t_o)$ for the modified majorant game $G_{h,h'}^+$ are similarly defined and the following relations are similarly verified

$$T_{h,h'}^-(x_o,y_o,t_o) \geq T_h^-(x_o,y_o,t_o) \qquad \text{for } h' > h > 0 \tag{2.72}$$

$$T_{h,h'}^+(x_o,y_o,t_o) = T_h^+(x_o,y_o,t_o) \qquad \text{for } h' \geq h > 0 \tag{2.73}$$

and

$$T_{h,h'}^+(x_o,y_o,t_o) \leq T_h^+(x_o,y_o,t_o) \qquad \text{for } h > h' > 0 \tag{2.74}$$

<u>Remark</u>: In G_h^-, at each time t_i, $i=0,1,2,\ldots$, Pursuer are

told the Evader's control segment $v_i(\cdot)$ on $[t_i, t_{i+1})$ before he decides his control segment $u_i(\cdot)$ on $[t_i, t_{i+1})$.
Namely, at each time t_i, $i=0,1,2,\dots$, Pursuer knows Evader's whole trajectory $y(\cdot)$ on $[t_i, t_{i+1}]$, Therefore, in $G^-_{h,h'}$, $h \geq h' > 0$, even if Pursuer can observe Evader's state $y(t)$, $t_i \leq t \leq t_{i+1}$, with the time interval h', the information pattern available for Pursuer does not change.
Hence, (2.71) is a reasonable result. We can interpret (2.72), (2.73), and (2.74) similarly. With this observation, the following Lemma follows directly. In what follows, we shall assume Assumption A2 (II-A-2) holds.

Lemma 2-8:

(a) In discrete minorant games G^-_h, for each $h>0$, there exists an Evader's admissible strategy $(h, V_h(\cdot,\cdot,\cdot))$ which guarantees that the ϵ-capture does not occur before time $T^-_h(x_o,y_o,t_o)+t_o$ against any Pursuer's admissible strategy $(h', U_{h'}(\cdot,\cdot,\cdot))$, $h'>0$.

(b) In discrete majorant games G^+_h, for each $h>0$, there exists a Pursuer's admissible strategy $(h, U_h(\cdot,\cdot,\cdot))$ which guarantees that the ϵ-capture occurs no later than time $T^+_h(x_o,y_o,t_o)+t_o$ against any Evader's admissible strategy $(h', V_{h'}(\cdot,\cdot,\cdot))$, $h'>0$.

Proof:

We shall prove (a).

(1) For a given $h>0$, suppose $(x_o,y_o,t_o) \in G^-_{h,1}$. Then, by the proof of Lemma 2-3 (a), there exists admissible $v^*_1(\cdot)$

such that if Evader uses $v_1^*(\cdot)$, $T_h^-(x_o,y_o,t_o)+t_o = \min_{u_1(\cdot)} \hat{t}$ holds.
Take $V_h^*(\cdot,\cdot,\cdot) \in V_h$ such that

$$v_1^*(\cdot) = V_h^*(x_o,y_o,t_o) \qquad (2.75)$$

Then, the strategy $(h,\ V_h^*(\cdot,\cdot,\cdot))$ gurantees that the ϵ-capture does not occur before time $T_h^-(x_o,y_o,t_o)+t_o$ against any $(h,\ U_h(\cdot,\cdot,\cdot))$. But by (2.71) and (2.72), the strategy $(h,\ V_h^*(\cdot,\cdot,\cdot))$ guarantees that the ϵ-capture does not occur before time $T_h^-(x_o,y_o,t_o)+t_o$ against any Pursuer's strategy $(h',\ U_{h'}(\cdot,\cdot,\cdot))$, $h'>0$.

(2) Suppose $(x_o,y_o,t_o) \in G_{h,m}^-$, $m\geq 2$.
By Lemma 2-4 (a), there exists admissible $v_1^*(\cdot)$ such that the resulting state y_1^* satisfies

$$T_h^-(x_o,y_o,t_o) \leq h+T_h^-(x_1,y_1^*,t_1) \ ; \quad t_1=t_o+h$$

for any admissible $u_1(\cdot)$.
If $(x_1,y_1,t_1) \in G_{h,n}^-$, $n\geq 2$, there exists admissible $v_2^*(\cdot)$ such that the resulting state y_2^* satisfies

$$T_h^-(x_1,y_1,t_o+h) \leq h+T_h^-(x_2,y_2^*,t_2), \quad t_2=t_o+2h \ .$$

If $(x_1,y_1,t_1) \in G_{h,1}^-$ proceed as in (1).
Thus, we can construct admissible control segments $v_1^*(\cdot),v_2^*(\cdot)\ldots$.
Take $V_h^*(\cdot,\cdot,\cdot)$ such that

$$v_i^*(\cdot) = V_h^*(x_i,y_i,t_i) \ , \ t_i=t_o+ih,\ i=0,1,2,\ldots \qquad (2.76)$$

Then, the strategy $(h,\ V_h^*(\cdot,\cdot,\cdot))$ guarantees that the ϵ-capture

does not occur before time $T_h^-(x_o,y_o,t_o)+t_o$ against any $(h, U_h(\cdot,\cdot,\cdot))$, hence, against any $(h', U_{h'}(\cdot,\cdot,\cdot))$, $h'>0$.
(3) (b) will be shown similarly.

Now, the relation between the time-continuous game G and the approximating discrete games G_h is not known, in general. In this thesis, we define a value of the time-continuous game G as follows:
If there is a real number T such that, for any $e>0$, Evader has an admissible strategy $(h, V_h(\cdot,\cdot,\cdot))$, $h>0$, which yields a payoff (i.e., an e-capture time) of at least T-e against any Pursuer's admissible strategy $(h', U_{h'}(\cdot,\cdot,\cdot))$, $h'>0$; and Pursuer has an admissible strategy $(h, U_h(\cdot,\cdot,\cdot))$, $h>0$, which prevents yielding a payoff (i.e., an e-capture time) of more than T+e against any Evader's strategy $(h', V_{h'}(\cdot,\cdot,\cdot))$, $h'>0$, then T will be called the <u>value</u> of the time-continuous game G. The game G which has the value is called <u>determined</u>. Furthermore, if the game has the value T, any Evader's admissible strategy $(h, V_h(\cdot,\cdot,\cdot))$, $h>0$, which yields an e-capture time of at least T-e, $e\geq 0$, against any Pursuer's admissible strategy $(h', U_{h'}(\cdot,\cdot,\cdot))$, $h'>0$, will be called an Evader's <u>e-effective strategy</u>. A Pursuer's e-effective strategy is similarly defined. It is easy to see that the game has the value, if and only if, for any $e>0$, there exist e-effective strategies for both players.

<u>Remark</u>: The value of discrete games G_h, $h>0$, and Pursuer's and Evader's e-effective strategies for G_h are similarly defined.

Theorem 2-3:

Suppose Condition M-1, 2, 3, 4 (or 4'), and 5 of Theorem 2-2 are satisfied.

Then,

$$\hat{T} = \lim_{h \to 0} T_h^- = \lim_{h \to 0} T_h^+ \tag{2.77}$$

is the value of the time-continuous game G.

Proof:

(1) By Theorem 2-1, Condition M-1 guarantees that $\{T_h^-\}$ is a non-decreasing sequence and that $\lim_{h \to 0} T_h^- = T^-$.

Since, for any e>0, there exists $h^*>0$ such that $0 < h \leq h^*$ implies $T^- - e \leq T_h^-$. (2.78)

Similarly, there exists $h^{**} > 0$ such that $0 < h \leq h^{**}$ implies $T_h^+ \leq T^+ + e$. (2.79)

Let $\min(h^*, h^{**}) = h_o$.

(2) By Lemma 2-8 (a), for each h>0, there is an Evader's admissible strategy $(h, V_h(\cdot,\cdot,\cdot))$ which yields an ϵ-capture time of at least T_h^- against any Pursuer's admissible strategy $(h', U_{h'}(\cdot,\cdot,\cdot))$, $h'> 0$.

Hence, for any e>0, Evader has an admissible strategy $(h, V_h(\cdot,\cdot,\cdot))$, $0 < h \leq h_o$, which yields an ϵ-capture time of at least $T^- - e$ against any Pursuer's admissible strategy $(h', U_{h'}(\cdot,\cdot,\cdot))$, $h' > 0$.

Similarly, Pursuer has an admissible strategy $(h, U_h(\cdot,\cdot,\cdot))$, $0 < h \leq h_o$, which prevents yielding an ϵ-capture time of more than $T^+ + e$ against any Evader's strategy.

(3) But, Condition M-1, 2, 3, 4 (or 4'), and 5 guarantees that $T^- = T^+ \quad (= \hat{T})$.

Hence, $\hat{T}$ is the value of the time-continuous game G.

Corollary:

The time-continuous game G has the value if and only if

$$\hat{T} = \lim_{h \to 0} T_h^- = \lim_{h \to 0} T_h^+ \qquad \text{exists.} \qquad (2.80)$$

Proof:

(1) If (2.80) holds, then, by Theorem 2-3, we see that $\hat{T}(= \lim_{h \to 0} T_h^- = \lim_{h \to 0} T_h^+)$ is the value of G.

(2) If (2.80) does not hold.

Suppose $T^- = \lim_{h \to 0} T_h^-$ and $T^+ = \lim_{h \to 0} T_h^+$ exist, but $T^- < T^+$.

($T^- > T^+$ is not possible by Lemma 2-2).

Then, for $e>0$ sufficiently small, say $\frac{1}{2}(T^+ - T^-) > e$, we see that at least one player fails to have the desired strategy.

Suppose T_h^- and T_h^+ exists for any $h>0$, then by the proof of Theorem 2-1, T^- and T^+ must exist.

Suppose T_h^- or T_h^+ diverges to infinity, since the value of the game should be finite, it can be seen that at least one player fails to have the desired strategy.

III. CAPTURE CONDITIONS

In the preceding chapter, some aspects of convergence problems were studied and the relation between the time-continuous game and discrete approximating games was clarified. In this case, we made the essential assumption that capture occurs within a finite period of time.

In this chapter, capture and escape conditions are derived in terms of attainability sets and escapability sets. It will be seen that the introduction of these sets is convenient in giving a common framework with which we can treat capture and escape conditions. Namely, by using these sets, the relations between minorant and majorant, open-loop and closed-loop, capture and escape conditions become transparent.

General conditions obtained in this chapter will be applied to some invidual cases in the next chapter.

In Section A, necessary and sufficient conditions for capture and escape are derived. Some duality relations are studied. These conditions are in terms of trajectories or graphs and difficult to apply. They are almost in the nature of definitions. Hence, in Section B, we proceed to consider some sufficient (possiblly not necessary) conditions for capture and escape in terms of escapability sets. These results are still difficult to apply but they are important in preparing the background to the concept of sufficient strategies to be introduced in the next chapter. With the concept of sufficient strategies, we obtain conditions which are easily verified. The resultsof this chapter, though not easily verified,are nevertheless interesting. For example, the results of N. N. Krasovskiy et al. [K8] can be shown by the results here to be faulty at least in part. Finally algorithms for constructing capture and escape strategies are given.

A. GENERAL CAPTURE CONDITIONS

In this section, conditions under which the ε-capture is guaranteed within a finite period of time are derived, for both closed-loop and open-loop, minorant and majorant discrete games. Escape conditions are also derived. Some duality relations between them are derived.

1. Capture and escape conditions for minorant games

As in the preceding chapters, let us assume that the game starts at time t_o, with the Pursuer's initial state $x_o \in E^n$ and the Evader's initial state $y_o \in E^n$. Let there be given a time T, $t_o < T < \infty$. Let $h>0$ be a discretization interval. Let $t_i = t_o+ih$, $i=0,1,2,\ldots$, $t_o \leq t_i \leq T$, and Pursuer's and Evader's states at time t_i are denoted by x_i and y_i, respectively.

Now, we shall introduce some auxiliary notation. For each t_i, $t_o \leq t_i \leq T$, we shall define a subset $A^*_{x_i}$ of the Pursuer's attainability set A_{x_i} by (see (1.7))

$$A^*_{x_i} = \Big\{ (x(t),t) \in E^{n+1} : t \in [t_i,\ t_{i+1}] \ ,$$
$$(x(t),t) \text{ is attainable from } (x_i,t_i) \Big\} \qquad (3.1)$$

A subset $A^*_{y_i}$ of A_{y_i} for Evader is similarly defined.
We shall also define a graph $f^*_{x_i}$ by (see (1.4))

$$f^*_{x_i} = \Big\{ (x(t),t) \in E^{n+1} : t \in [t_i,\ t_{i+1}] \ , \qquad (3.2)$$
$$\left.\begin{array}{l} x(\cdot) \text{ is a Pursuer's trajectory on } t_i,t_{i+1} \\ \text{corresponding to some admissible } u(\cdot) \\ \text{and } x(t_i)=x_i \end{array}\right\}$$

A subset $f^*_{y_i}$ of E^{n+1} for Evader is similarly defined.
Similar to the definition of $A_{x(t_i)}+B_\epsilon$, we shall define an $(n+1)$ dementional pipe $(f^*_{x_i}+B_\epsilon)$, such that its fixed-time cross section at time t, denoted by $(f^*_{x_i}+B_\epsilon)(t)$, satisfies

$$(f^*_{x_i}+B_\epsilon)(t) = f^*_{x_i}(t) + B_\epsilon \quad \text{for all } t,\ t_i \le t \le t_{i+1}$$
$$= \emptyset \quad \text{otherwise.}$$

A set $A^*_{x_i}+B_\epsilon$, defined on $[t_i,t_{i+1}]$, is similarly defined.

Let Graph $(x_o,x_1,\ldots,x_j)$ or simply $(x_o,x_1,\ldots,x_j)$, $x_{i+1} \in A_{x_i}(t_{i+1})$, $i=0,1,\ldots,j-1$, $t_o \le t_o+jh \le T$, represent an $(n+1)$ dementional graph connecting points (x_i,t_i) to (x_{i+1},t_{i+1}), $0 \le i \le j-1$, successively using some admissible control segments $u_i(\cdot)$.
Namely, $(x_o,x_1,\ldots,x_j)$ is a graph $f^u_{x_o}$, defined on $[t_o,t_j]$. such that $f^u_{x_o}(t_i)=x_i$, $i=0,1,\ldots,j$ (3.3)
Since $x_{i+1} \in A_{x_i}(t_{i+1})$, $i=0,1,\ldots,j-1$, there exists admissible $u(\cdot)$ which satisfies (3.3).
An Evader's graph $(y_o,y_1,\ldots,y_j)$ is similarly defined.

With this notation, we shall give conditions under which the ϵ-capture is guaranteed to occur no later than time T.

<u>Theorem 3-1</u>: (Minorant closed-loop capture)

For any $h>0$, $\epsilon \geq 0$, and T, $t_o < T < \infty$, $t_o+T_h^- \leq T$ holds if and only if Condition 3-1 is satisfied.

<u>Condition 3-1</u>:

Corresponding to any Evader's graph $(y_o, y_1, \ldots, y_n)$, $t_o+(n-1)h < T \leq t_o+nh$, following the rules of the game (see II-A-1), Pursuer can find a graph $(x_o, x_1, \ldots, x_{n-1})$ such that

$$(f^*_{y_i} \cap (A^*_{x_i} + B_\epsilon))(t) \neq \phi \quad \text{holds for some } i,\ 0 \leq i \leq n-1, \text{ and for some } t,\ t_o \leq t \leq T \tag{3.4}$$

where
$$f^*_{y_i} \ni (y_{i+1}, t_{i+1}) \quad \text{for all } i,\ 0 \leq i \leq n-1 .$$

<u>Proof</u>:

(1) Suppose Condition 3-1 holds. Then, for some i, $0 \leq i \leq n-1$, there exists a point (a, t), such that $(a, t) \in f^*_{y_i} \cap (A^*_{x_i}+B_\epsilon)$, $t_o \leq t \leq T$.

Hence, $(a, t) \in f^*_{y_i}$ and $(a, t) \in (A^*_{x_i}+B_\epsilon)$.

But, since any point in $A^*_{x_i}$ is attainable from (x_i, t_i), there exists an admissible control segment $u^*_i(\cdot)$ such that the corresponding graph $f^{u^*_i}_{x_i}$ $(= \{(x(t),t) \in E^{n+1} : t \in [t_i, t_{i+1}]$, $x(\cdot)$ is a Pursuer's trajectory on $[t_i, t_{i+1}]$ corresponding to $u^*_i(\cdot)$ and $x(t_i) = x_i\})$ satisfies

$$\| f^{u^*_i}_{x_i}(t) - a \| \leq \epsilon$$

Since (x_i, t_i) is attainable from (x_o, t_o), $(f_{x_i}^{u_i^*}(t), t)$ is attainable from (x_o, t_o).

Hence, the e-capture occurs at time $t \leq T$.

But, by the definition of optimality (see (1.13) and (2.3)), we have $t_o + T_h^- \leq t$. Hence, $t_o + T_h^- \leq T$.

(2) Suppose Condition 3-1 does not hold.

Then, corresponding to some Evader's graph $(\bar{y}_o, \bar{y}_1, \dots, \bar{y}_n)$, Pursuer can not find a graph $(x_o, x_1, \dots, x_{n-1})$ such that

$$(f_{y_i}^* \cap (A_{x_i}^* + B_\epsilon))(t) \neq \phi \quad \text{for some } i,\ 0 \leq i \leq n-1, \text{ and for some } t,\ t_o \leq t \leq T.$$

Namely, so long as Evader follows the graph $(\bar{y}_o, \bar{y}_1, \dots, \bar{y}_n)$,

$$(f_{\bar{y}_i}^* \cap (A_{x_i}^* + B_\epsilon))(t) = \phi \quad \text{for any } i,\ 0 \leq i \leq n-1, \text{ and for any } t,\ t_o \leq t \leq T.$$

Hence, for any $(\bar{y}(t), t) \in f_{\bar{y}_i}^*$ and for any $(x(t), t) \in A_{x_i}^*$, $0 \leq i \leq n-1$,

$$\|\bar{y}(t) - x(t)\| > \epsilon \quad , \quad t_o \leq t \leq T .$$

Therefore, Evader is not captured before or at time T.

<u>Theorem 3-1'</u>: (Minorant closed-loop escape)

For any $h>0$, $\epsilon \geq 0$, and T, $t_o < T < \infty$, $t_o + T_h^- > T$ holds if and only if Condition 3-1' is satisfied.

<u>Condition 3-1'</u>:

Corresponding to any Pursuer's graph $(x_o, x_1, \dots, x_{n-1})$, $t_o + (n-1)h < T \leq t_o + nh$, following the rules of the game, Evader can find a graph $(y_o, y_1, \dots, y_n)$ such that

$$((f^*_{y_i} + B_\varepsilon) \cap A^*_{x_i})(t)=\phi \quad \text{for any } i,\ 0 \le i \le n-1, \text{ and for any } t,\ t_o \le t \le T. \tag{3.5}$$

where

$$f^*_{y_i} \ni (y_{i+1}, t_{i+1}) \quad \text{for all } i,\ 0 \le i \le n-1 \ .$$

Proof:

Directly follows from Theorem 3-1, noticing that

$$((f^*_{y_i} + B_\varepsilon) \cap A^*_{x_i})(t)=\phi \Leftrightarrow (f^*_{y_i} \cap (A^*_{x_i} + B_\varepsilon))(t)=\phi \quad \text{for any } t,\ t_o \le t \le T.$$

In the minorant open-loop game, Evader must tell his control admissible $v(\cdot)$ to Pursuer before the game starts. Therefore, the minorant open-loop game is regarded as a special case of minorant closed-loop games with $h \to \infty$. Hence, the next corollaries follow directly from the above theorems. For convenience, we shall denote the ε-capture time for the minorant open-loop game by T^-_∞ .

Corollary 3-1: (Minorant open-loop capture)

For any $\varepsilon \ge 0$ and T, $t_o < T < \infty$, $t_o + T^-_\infty \le T$ holds if and only if Condition 3-1-0 is satisfied.

Condition 3-1-0:

For any Evader's graph $f_{y_o} \in A_{y_o}$

$$(f_{y_o} \cap (A_{x_o} + B_\varepsilon))(t) \ne \phi \quad \text{holds for some } t,\ t_o \le t \le T. \tag{3.6}$$

Corollary 3-1': (Minorant open-loop escape)

For any $\epsilon \geq 0$, and T, $t_o < T < \infty$, $t_o + T_{\infty}^{-} > T$ holds if and only if Condition 3-1'-0 is satisfied.

Condition 3-1'-0:

There exists an Evader's graph $f_{y_o} \in A_{y_o}$ such that

$$(f_{y_o} \cap (A_{x_o} + B_e))(t) = \emptyset \quad \text{for any } t,\ t_o \leq t \leq T \tag{3.7}$$

Remark: Since $\bigsqcup A_{x_i}^{*} \subset A_{x_o}$ always holds, if Condition 3-1 holds, Condition 3-1-0 always holds (see Theorem 3-1 and Corollary 3-1).

Hence, if minorant closed-loop capture occurs for some h>0, minorant open-loop capture always occurs.

Hence, if T_h^- exists for some h>0, then T_{∞}^- also exists and $T_{\infty}^- \leq T_h^-$ holds. This is a reasonable result, because in the minorant open-loop game, the information pattern available for both players is biased most advantageously to Pursuer. Similarly, we see that if minorant open-loop escape is possible, then, minorant closed-loop escape is also possible for any h>0.

2. Capture and escape conditions for majorant games

Next, we shall consider capture and escape conditions for majorant games.

Theorem 3-2: (Majorant closed-loop capture)

For any $h>0$, $\epsilon \geq 0$, and T, $t_o < T < \infty$, $t_o+T_h^+ \leq T$ holds if and only if Condition 3-2 is satisfied.

Condition 3-2:

Following the rules of the game, Pursuer can find a graph $(x_o, x_1, \ldots, x_n)$, $t_o+(n-1)h < T \leq t_o+nh$, such that

$$(f^*_{y_i} \cap (f^*_{x_i} + B_\epsilon))(t) \neq \emptyset \tag{3.8}$$

holds for any Evader's graph $(y_o, y_1, \ldots, y_n)$, for some i, $0 \leq i \leq n-1$, and for some t, $t_o \leq t \leq T$

where

$$f^*_{x_i} \ni (x_{i+1}, t_{i+1}), f^*_{y_i} \ni (y_{i+1}, t_{i+1}) \quad \text{for all } i,\ 0 \leq i \leq n-1.$$

Proof:

(1) Suppose Condition 3-3 holds. Then, for some i, $0 \leq i \leq n-1$, there exists a Pursuer's admissible control segment $u_i^*(\cdot)$ such that the corresponding graph $f_{x_i}^{u_i^*}$ satisfies

$$f_{x_i}^{u_i^*} = f^*_{x_i} \quad \text{and}$$

$$(f^*_{y_i} \cap (f_{x_i}^{u_i^*} + B_\epsilon))(t) \neq \emptyset \quad \text{for some } t,\ t_o \leq t \leq T, \text{ and for any } f^*_{y_i} \in A^*_{y_i}.$$

Let

$$f^*_{y_i} \cap (f_{x_i}^{u_i^*} + B_\epsilon) \ni (a, t).$$

Then,

$$\| f_{x_i}^{u_i^*}(t) - a \| \leq \epsilon \quad , \quad t_o \leq t \leq T .$$

Since $(a, t) \in f^*_{y_i}$, the ϵ-capture occurs at time $t \leq T$. This holds for each $f^*_{y_i} \in A^*_{y_i}$. By the definition of optimality, we have $t_o + T^+_h \leq t$.

Hence, $t_o + T^+_h \leq T$.

(2) Suppose Condition 3-3 does not hold. Then, corresponding to some Evader's graph $(\bar{y}_o, \bar{y}_1, \ldots, \bar{y}_n)$, Pursuer can not find a graph $(x_o, x_1, \ldots, x_n)$ such that

$$(f^*_{\bar{y}_i} \cap (f^*_{x_i} + B_\epsilon))(t) \neq \emptyset \quad \text{for some } i,\ 0 \leq i \leq n-1, \text{ and for some } t,\ t_o \leq t \leq T.$$

Hence, for any $(\bar{y}(t), t) \in f^*_{\bar{y}_i}$ and for any $(x(t), t) \in f^*_{x_i}$,

$$\|\bar{y}(t) - x(t)\| > \epsilon \qquad t_o \leq t \leq T \quad .$$

Therefore, Evader is not captured before or at the time T.

<u>Theorem 3-2'</u>: (Majorant closed-loop escape)

For any $h>0$, $\epsilon \geq 0$, and T, $t_o < T < \infty$, $t_o + T^+_h > T$ holds if and only if Condition 3-2' is satisfied.

<u>Condition 3-2'</u>:

Corresponding to any Pursuer's graph $(x_o, x_1, \ldots, x_n)$, $t_o + (n-1)h < T \leq t_o + nh$, following the rules of the game, Evader can find a graph $(y_o, y_1, \ldots, y_n)$ such that

$$((f^*_{y_i} + B_\epsilon) \cap f^*_{x_i})(t) = \emptyset \quad \text{for any } i,\ 0 \leq i \leq n-1, \text{ and any } t,\ t_o \leq t \leq T. \tag{3.9}$$

where

$$f^*_{x_i} \ni (x_{i+1}, t_{i+1}),\ f^*_{y_i} \ni (y_{i+1}, t_{i+1}) \quad \text{for all } i,\ 0 \leq i \leq n-1 \quad .$$

<u>Proof</u>:

Directly follows from Theorem 3-2.

Similar to minorant games, the following corollaries follow from the above theorems. For convenience, we shall denote the ϵ-capture time for the majorant open-loop game by T_{∞}^{+} .

<u>Corollary 3-2</u>: (Majorant open-loop capture)

For any $\epsilon \geq 0$ and T, $t_o < T < \infty$, $t_o + T_{\infty}^{+} \leq T$ holds if and only if Condition 3-2-0 is satisfied.

<u>Condition 3-2-0</u>:

There exists a Pursuer's graph $f_{x_o} \epsilon A_{x_o}$ such that

$$(f_{y_o} \cap (f_{x_o} + B_{\epsilon}))(t) \neq \phi \qquad (3.10)$$

holds for some t, $t_o \leq t \leq T$, for any Evader's graph $f_{y_o} \epsilon A_{y_o}$.

<u>Corollary 3-2'</u>: (Majorant open-loop escape)

For any $\epsilon \geq 0$, and T, $t_o < T < \infty$, $t_o + T_{\infty}^{+} > T$ holds if and only if Condition 3-2'-0 is satisfied.

<u>Condition 3-2'-0</u>:

Corresponding to any Pursuer's graph $f_{x_o} \epsilon A_{x_o}$, there exists an Evader's graph $f_{y_o} \epsilon A_{y_o}$ such that

$$((f_{y_o} + B_{\epsilon}) \cap f_{x_o})(t) = \phi \quad \text{for any } t,\ t_o \leq t \leq T . \qquad (3.11)$$

<u>Remark 1</u>: Since $\bigsqcup A_{y_i}^{*} \subset A_{y_o}$ always holds, formally, if Condition 3-2-0 holds, Condition 3-2 always holds (see Theorem

3-2 and Corollary 3-2).

Hence, if T^+_∞ exists, then, T^+_h also exists for any h>0 and $T^+_\infty \geq T^+_h$ holds.

This is a reasonable result, because in the majorant open-loop game, the information pattern available for both players is biased most advantageously to Evader.

Similarly, we see that if majorant closed-loop escape is possible for some h>0, then, majorant open-loop escape also is possible.

<u>Remark 2</u>: Next, we shall compare the rules of the game for the minorant game and the majorant game. In the majorant (closed-loop) game, at each time t_i, i=0,1,2,... , based upon observation of the Evader's state y_i, Pursuer chooses $f^*_{x_i} \subset A^*_{x_i}$, trying to realize,

$$((f^*_{x_i} + B_e) \cap f^*_{y_i})(t) \neq \emptyset \quad \text{for any } f^*_{y_i} \subset A^*_{y_i} \text{ and for some } t,\ t_o \leq t \leq T, \tag{3.12}$$

$f^*_{x_i}$ will be told to Evader, before he chooses $v_i(\cdot)$.

On the other hand, in the minorant (closed-loop) game, at each time t_i, i=0,1,2,... , based upon observation of the Evader's state y_i, Pursuer chooses $A^*_{x_i}$ (namely he chooses x_i), trying to realize,

$$((A^*_{x_i} + B_e) \cap f^*_{y_i})(t) \neq \emptyset \quad \text{for any } f^*_{y_i} \subset A^*_{y_i} \text{ and for some } t,\ t_o \leq t \leq T \tag{3.13}$$

$A^*_{x_i}$ will be told to Evader before he chooses $v_i(\cdot)$.
Now, we see that if $f^*_{x_i}$ is replaced by $A^*_{x_i}$, the rules for the majorant game coincide with those for the minorant game.
Since, $f^*_{x_i} \subset A^*_{x_i}$ for any $i=0,1,2,\ldots$, we see that if Pursuer can find $f^*_{x_i}$, at some stage i, $i=0,1,2,\ldots$, which satisfies (3.12), then (3.13) also holds no later than stage i.
Hence, if T^+_h exists for some $h>0$, then T^-_h exists for the same h and

$$T^-_h \leq T^+_h \qquad \text{holds.}$$

This is true for any $\epsilon \geq 0$ and any initial condition.

Combining the results in above Remarks, we see that

$$T^-_\infty \leq T^-_h \leq T^+_h \leq T^+_\infty \tag{3.14}$$

holds for any $h>0$, $\epsilon \geq 0$, and initial condition.

Similarly, it is easy to see that if Condition 3-1' is satisfied for some $h>0$, then Condition 3-2' is satisfied for the same h.
Hence, if minorant closed-loop escape is possible for some $h>0$, then majorant closed-loop escape is also possible for the same h.

Remark 3: We now compare minorant open-loop capture conditions (MIOC) and majorant open-loop escape conditions (MJOE).
From Condition 3-1-0, MIOC is;

Corresponding to any $f_{y_o} \in A_{y_o}$, there exists $f_{x_o} \in A_{x_o}$ such that

$$(f_{y_o} \cap (f_{x_o} + B_\epsilon))(t) \neq \emptyset \quad \text{for some } t,\ t_o \leq t \leq T . \tag{3.15}$$

From Condition 3-2-0, MJOE is;

Corresponding to any $f_{x_o} \in A_{x_o}$, there exists $f_{y_o} \in A_{y_o}$ such that

$$(f_{x_o} \cap (f_{y_o} + B_\epsilon))(t) = \emptyset \quad \text{for any } t,\ t_o \leq t \leq T . \tag{3.16}$$

Now, a <u>dual relation</u> is found between MJOE and MIOC. Namely, if we replace f_{y_o} in MIOC by f_{x_o}, some by all, and $\neq$ by $=$, we obtain MJOE.

Similar duality relations hold between MIOE (minorant open-loop escape conditions) and MJOC (majorant open-loop capture conditions). Similarly, the same duality relations can be found in closed-loop conditions.

B. SUFFICIENT CAPTURE CONDITIONS

In this section, sufficient conditions for capture which will play an important role in the construction of sufficient strategies in the next chapter are derived.

The main notion used in this section is escapability sets introduced in I-C-2.

<u>Theorem 3-3</u>: (Sufficient conditions for minorant closed-loop capture)

For any $h>0$, $\epsilon \geq 0$, and T, $t_o < T < \infty$, $t_o + T_h^- \leq T$ holds if Condition 3-3 is satisfied.

<u>Condition 3-3</u>:

There exists a time t_h^-, $t_o < t_h^- \leq T$, such that

(a) $S_{x_o,y_o}(t_h^-) = \emptyset$ and

(b) for each $t_i = t_o+ih$, $i=0,1,\ldots,j-1$, $t_o \leq t_j < t_h^- \leq t_{j+1}$,

if $x_i \in A_{x_o}(t_i)$ and $y_i \in A_{y_o}(t_i)$ satisfy

$$S_{x_i,y_i}(t_h^-) = \emptyset \quad \text{and}$$

$$t_i^* = \inf(t : S_{x_i,y_i}(t) = \emptyset) > t_{i+1}$$

then, for each $y_{i+1} \in A_{y_i}(t_{i+1})$,

either there exists a point $x_{i+1} \in A_{x_i}(t_{i+1})$ such that

$$S_{x_{i+1},y_{i+1}}(t_h^-) = \emptyset$$

or, the ϵ-capture occurs between time t_i and t_{i+1}.

<u>Proof</u>:

We shall show that if Condition 3-3 holds, there exists a pursuit algorithm which guarantees the ϵ-capture at some time $t \leq t_h^-$. If so, $t_o+T_h^- \leq t \leq t_h^- \leq T$. Hence, $t_o+T_h^- \leq T$. Let us recall that, before starting the game, Pursuer and Evader are informed of R-1, R-2, R-3, and R-4 (see II-A-1). In addition to these, we shall assume that

<u>R-5</u> the time T, $t_o < T < \infty$, are to be given.

Now, we shall give a pursuit procedure. This procedure will be called the minorant capture algorithm (MIC).

<u>MIC (1)</u> Calculate attainability sets A_{x_o} and A_{y_o} for both players using R-1 and R-2. By A_{x_o}, A_{y_o}, and R-3, calculate

the ϵ-escapability set S_{x_o,y_o} .
Calculate $t_o^* = \min(t : S_{x_o,y_o}(t) = \phi)$ (see remark below).
For a chosen h, by R-4, calculate $t_1 = t_o + h$.
If $t_o^* \le t_1$, go to MIC (2). If $t_o^* > t_1$, go to MIC (3).

<u>MIC (2)</u> If $t_o^* \le t_1$

the ϵ-capture occurs no later than time $t_o^* (\le t_h^-)$ and the game ends.

We shall prove this:

First, we show that, for any choice of admissible control segment $v_o(\cdot)$, there corresponds a time t^* , $t_o < t^* \le t_o^*$, such that

$$y(t^* ; v_o(\cdot)) \in A_{x_o}(t^*) + B_\epsilon$$

where $y(t^* ; v_o(\cdot))$ is the Evader's state at time t^* , corresponding to $v_o(\cdot)$, showing dependence of the state on $v_o(\cdot)$ explicitly.

Suppose, for some admissible $v_o^*(\cdot)$, there does not exist t^* , $t_o < t^* \le t_o^*$, such that

$$y(t^* ; v_o^*(\cdot)) \in A_{x_o}(t^*) + B_\epsilon \qquad (3.17)$$

Then, the graph defined by

$$f_{y_o}^{v_o^*} = \{(y(t ; v_o^*(\cdot)), t) : t \in [t_o, t_o^*]\}$$

satisfies

(a) $f_{y_o}^{v_o^*} \cap (A_{x_o} + B_\epsilon) = \phi$ (By (3.17))

(b) $f_{y_o}^{v_o^*}$ is connected

(c) $f_{y_o}^{v_o^*} \ni (y_o,\ t_o)$

Hence, $f_{y_o}^{v_o^*} \subset S_{x_o,y_o}$ (see I-C-2) .
Therefore, $S_{x_o,y_o}(t_o^*) \neq \emptyset$, which contradicts assumed hypothesis.

Since any point in A_{x_o} is attainable from (x_o,y_o), the e-capture occurs at time t^* (see Corollary 3-1).
Hence, $t^* \leq t_o^* \leq t_h^-$, we complete the proof.

<u>Remark</u>: If $\min(t : S_{x_o,y_o}(t) = \emptyset)$ does not exist, taking

$$t_o^* = \inf(t : S_{x_o,y_o}(t) = \emptyset),$$

we can assert that if $t_o^* \leq t_1$, then the e-capture occurs no later than time t_o^*+e for any $e>0$. The proof is almost identical.
For simplicity, in what follows, we shall assume $\min(t : S_{x_i,y_i}(t) = \emptyset)$, $i=0,1,2,\ldots$, exists without loss of generality.

<u>MIC (3)</u> If $t_o^* > t_1$, for each $y_1 \in A_{y_o}(t_1)$, either Pursuer can find $x_1 \in A_{x_o}(t_1)$ such that

$$S_{x_1,y_1}(t_h^-) = \emptyset$$

and go to MIC (2)' (if $t_i^* \leq t_{i+1}$) or (3)' (if $t_i^* > t_{i+1}$) with $i=1$, or, the e-capture occurs between time t_o and t_1, and the game ends.

We shall prove this:
In the minorant game, Evader must specify $y_1 \in A_{y_o}(t_1)$ before Pursuer chooses $x_1 \in A_{x_o}(t_1)$.
Hence, Condition 3-3 implies that corresponding to any choice

of $y_1 \in A_{y_o}(t_1)$, Pursuer can find $x_1 \in A_o(t_1)$ such that

$$S_{x_1,y_1}(t_h^-) = \emptyset$$

or, the e-capture occurs between time t_o and t_1 and the game ends. (If for some $y_1 \in A_{y_o}(t_1)$, Pursuer can not find such $x_1 \in A_{x_o}(t_1)$, we do not know whether the e-capture occurs within a finite period of time).

<u>MIC (2)'</u> If $t_i^* = \min(t : S_{x_i,y_i}(t) = \emptyset) \leq t_{i+1}$, the e-capture occurs no later than time t_i^* and the game ends.

The proof is the same as in MIC (2).

<u>MIC (3)'</u> If $t_i^* > t_{i+1}$, for each $y_{i+1} \in A_{y_i}(t_{i+1})$, either Pursuer can find $x_{i+1} \in A_{x_i}(t_{i+1})$ such that

$$S_{x_{i+1},y_{i+1}}(t_h^-) = \emptyset$$

and go to MIC (2)' or (3)' with $i \rightarrow i+1$,
or , the e-capture occurs between time t_i and t_{i+1} , and the game ends.

Now, we shall prove that the game always ends.
With the repetition of MIC (3)', suppose we arrive at time t_{j-1}, with $t_j < t_h^- \leq t_{j+1}$.
Suppose $t_{j-1}^* = \min(t : S_{x_{j-1},y_{j-1}}(t) = \emptyset) \leq t_j$,
then, by MIC (2)', the e-capture occurs no later than time $t_{j-1}^* \leq t_j < t_h^- \leq T$.
Suppose $t_{j-1}^* > t_j$.
Then, by Condition 3-3 (b), we see that for each $y_j \in A_{y_{j-1}}(t_j)$

either there exists $x_j \in A_{x_{j-1}}(t_j)$ such that

$$S_{x_j,y_j}(t_h^-) = \emptyset$$

or, the ϵ-capture occurs between time t_{j-1} and t_j (in this case, since $t_j < t_h^-$, the game ends no later than time t_h^- , as required).

Since $t_j^* = \min(t : S_{x_j,y_j}(t) = \emptyset)$,
we have $t_j^* \leq t_h^-$, hence, $t_j < t_j^* \leq t_{j+1}$.
By MIC (2)', the ϵ-capture occurs no later than time $t_j^* \leq t_h^- \leq T$. This completes the proof.

The following follows directly.

Corollary 3-3: (Sufficient conditions for minorant open-loop capture)

For any $\epsilon \geq 0$ and T, $t_o < T < \infty$, $t_o + T_\infty^- \leq T$ holds if Condition 3-3-0 is satisfied.

Condition 3-3-0:

There exists a time t_∞^- , $t_o < t_\infty^- \leq T < \infty$, such that

$$S_{x_o,y_o}(t_\infty^-) = \emptyset$$

Remark: It is interesting to note that Condition 3-3-0 is not, in general, a necessary condition for minorant open-loop capture.

Next, we shall give a sufficient condition for capture which is convenient in constructing pursuit algorithms for majorant discrete games. This will also be used to construct pursuit algorithms for time-continuous games.

In what follows, we use the property At-2' for the Pursuer's attainability set. By this property, we see that corresponding to any $\epsilon > 0$, there exists a positive real number h_o such that for each h, $0 < h \leq h_o$,

$$A^*_{x_i} \subset f^*_{x_i} + B_\epsilon \qquad (3.18)$$

for some $f^*_{x_i} \subset A^*_{x_i}$, for any $x_i \in A_{x_o}(t_i)$, and for any i, $0 \leq i \leq n-1$,

where

$$t_o + (n-1)h < T \leq t_o + nh \ .$$

It should be recalled that $f^*_{x_i}$ and $A^*_{x_i}$ are defined on $[t_i, t_i+h]$, $t_i = t_o + ih$ . .

If $\epsilon=0$, a necessary and sufficient condition for the existence of $h_o > 0$ which satisfies (3.18) is $A^*_{x_i} = f^*_{x_i}$ for any $x_i \in A_{x_o}(t_i)$ and i, $0 \leq i \leq n-1$.

This is a restrictive requirement.

We shall denote the escapability set with $\epsilon=0$ by $S^o_{x_i,y_i}$, namely,

$$A_{y_i} - A_{x_i} = S^o_{x_i,y_i} \qquad \text{(see (1.11))} \qquad (3.19)$$

for any $y_i \in A_{y_o}(t_i)$ and $x_i \in A_{x_o}(t_i)$.

<u>Theorem 3-4</u>: (Sufficient conditions for majorant closed-loop capture)

For any h, $0 < h \leq h_o$, $\epsilon \geq 0$, and T, $t_o < T < \infty$, $t_o + T^+_h \leq T$ holds if Condition 3-4 is satisfied.

Condition 3-4:

There exists a time t_h^+ , $t_o < t_h^+ \leq T$, such that

(a) $S^o_{x_o,y_o}(t_h^+) = \phi$ and

(b) for each $t_i = t_o+ih$, $i=0,1,\ldots,j-1$, $t_o \leq t_j < t_h^+ \leq t_{j+1}$,

if $x_i \in A_{x_o}(t_i)$ and $y_i \in A_{y_o}(t_i)$ satisfy

$$S^o_{x_i,y_i}(t_h^+) = \phi \quad \text{and}$$

$$t_i^{**} = \inf(t : S^o_{x_i,y_i}(t) = \phi) > t_{i+1}$$

then, "there exists a point $x_{i+1} \in A_{x_i}(t_{i+1})$ such that

$$S^o_{x_{i+1},y_{i+1}}(t_h^+) = \phi \quad \text{for all } y_{i+1} \in A_{y_i}(t_{i+1}) \text{ ".}$$

Proof:

We shall show that if Condition 3-4 holds, there exists a pursuit algorithm which guarantees the ϵ-capture at some time $t \leq t_h^+ \leq T$.

If so, $t_o+T_h^+ \leq t \leq t_h^+ \leq T$. Hence, $t_o+T_h^- \leq T$.

Now, we shall give a pursuit procedure. This procedure will be called the majorant capture algorithm (MJC).

MJC (1) Calculate A_{x_o} , A_{y_o}, $S^o_{x_o,y_o}$, h_o by (3.18), and $t_o^{**}=\min(t : S^o_{x_o,y_o}(t)=\phi)$.

If $t_o^{**} \leq t_1$, go to MJC (2). If $t_o^{**} > t_1$, go to MJC (3).

MJC (2) If $t_o^{**} \leq t_1$,

the ϵ-capture occurs no later than time t_o^{**} and the game ends.

We shall prove this:

First, by (3.18), for each h, $0 < h \leq h_o$, we have

$$f^*_{x_o} + B_\epsilon \supset A^*_{x_o} \qquad \text{for some} \quad f^*_{x_o} \in A^*_{x_o}$$

Hence, we have

$$S^o_{x_o,y_o} = A^*_{y_o} - A^*_{x_o} \supset A^*_{y_o} - (f^*_{x_o} + B_\epsilon) \tag{3.20}$$

$$\text{for some} \quad f^*_{x_o} \in A^*_{x_o}$$

Let $\bar{t}^{**}_o = \min(t : A^*_{y_o} - (f^*_{x_o} + B_\epsilon) = \phi)$

Then, by (3.20), we have $\bar{t}^{**}_o \leq t^{**}_o$.

Hence, corresponding to any admissible control segment $v_o(\cdot)$, there exists t^* , $t_o < t^* \leq \bar{t}^{**}_o$ such that (see the proof of MIC (2), Theorem 3-3)

$$y(t^* ; v_o(\cdot)) \in f^*_{x_o}(t^*) + B_\epsilon \tag{3.21}$$

$$\text{for some} \quad f^*_{x_o} \in A^*_{x_o}$$

Hence, similar to the proof of MIC (2) of Theorem 3-3, we see that the ϵ-capture occurs no later than t^* .

But $t^* \leq \bar{t}^{**}_o \leq t^{**}_o$, we can conclude that the ϵ-capture occurs no later than time t^{**}_o .

<u>Remark</u>: If t^{**}_o does not exist, we replace

$$t^{**}_o = \inf(t : S^o_{x_o,y_o}(t) = \phi)$$

and treat the problem similar to Remark after MIC (2).

Similar fact holds for $\bar{t}^{**}_o$.

<u>MJC (3)</u> If $t^{**}_o > t_1$, find $x_1 \in A_{x_o}(t_1)$ such that

$$S^o_{x_1,y_1}(t_h^+) = \phi \qquad \text{for all} \quad y_1 \in A_{y_o}(t_1) \quad ,$$

and go to MJC (2)' (if $t_i^{**} \leq t_{i+1}$) or (3)' (if $t_i^{**} > t_{i+1}$) with i=1.

In the majorant game, Pursuer must choose $x_1 \in A_{x_o}(t_1)$ before Evader chooses $y_1 \in A_{y_o}(t_1)$.

Hence, Condition 3-4 implies that there exists $x_1 \in A_{x_o}(t_1)$ such that, so long as Pursuer specifies x_1, $S^o_{x_1,y_1}(t_h^+) = \phi$ holds independent of the choice of y_1.

<u>MJC (2)'</u> If $t_i^{**} = \min(t : S^o_{x_i,y_i}(t) = \phi) \leq t_{i+1}$, the ϵ-capture occurs no later than time t_i^{**} and the game ends.

The proof is the same as in MJC (2).

<u>MJC (3)'</u> If $t_i^{**} > t_{i+1}$, find $x_{i+1} \in A_{x_i}(t_{i+1})$ such that $S^o_{x_{i+1},y_{i+1}}(t_h^+) = \phi$ for all $y_{i+1} \in A_{y_i}(t_{i+1})$ and go to MJC (2)' or (3)' with $i \to i+1$.

Similar to Theorem 3-3, we can prove that the game always ends. This completes the proof.

<u>Remark</u>: Similar to Condition 3-3, the statement in " " of Condition 3-4 (b) can be weakened as follows;

(b) "there exists a point $x_{i+1} \in A_{x_i}(t_{i+1})$ such that any

$y_{i+1} \in A_{y_i}(t_{i+1})$ satisfies

either $S^o_{x_{i+1},y_{i+1}}(t_h^+) = \phi$

or, $\mathrm{dist}(\mathrm{Graph}\ (x_i,x_{i+1})(t'),\ \mathrm{Graph}\ (y_i,y_{i+1})(t')) \leq \epsilon$

for some t', $t_i \leq t' < t_{i+1}$ "

where Graph (x_i, x_{i+1}) is defined in III-A-1, and Graph $(x_i, x_{i+1})(t')$ represents a fixed-time cross section at time t'.

The following follows directly.

Corollary 3-4: (Sufficient conditions for majorant open-loop capture)

For some $\epsilon \geq 0$, suppose there exists h_o such that $h_o \geq T$. Then $t_o + T^+_\infty \leq T$ holds if Condition 3-4-0 is satisfied.

Condition 3-4-0:

There exists a time t^+_∞, $t_o < t^+_\infty \leq T < \infty$, such that

$$S^o_{x_o, y_o}(t^+_\infty) = \phi$$

Remark: Since the majorant pursuit algorithm (MJC) given above is independent of the Evader's observation interval (see (2.73) and Remark after (2.74)) we see that this pursuit algorithm for Pursuer remains valid even for the time-continuous game. This will be discussed in the next chapter in detail.

IV. SUFFICIENT STRATEGIES

Based upon general capture conditions derived in Chapter III, we shall consider a new and important class of strategies. These strategies are called sufficient strategies. Existing papers are mainly concerned with saddle-point "optimal" strategies which require "continuous" observation of the states of both players. From the point of view of applications, "continuous" observation and feed-back are sometimes undesirable, because they are difficult to realize. The sufficient pursuit strategy introduced here is essentially a discrete version of continuous controls, and guarantees Pursuer e-capture against any Evader's strategy within some given finite period of time, requiring only discrete observation on the part of Pursuer. The geometrical approach taken here is straightforward, yet rigorous. Furthermore, it provides new insight and interpretation for existing results which were obtained by heuristic optimization techniques.

In Section A, the concept of sufficient strategies is introduced and their relationship to "optimal" strategies of the saddle-point type is discussed.

In Section B, some algorithms for constructing sufficient pursuit strategies are derived. They are applied to simple examples.

In Section C, existence theorems for such strategies are derived. The relationship to capture conditions is explained.

In Section D, these results are applied to some specific problems. The concept of lower dimensional capture is introduced and briefly explained. Although this is an interesting generalization of the capture problem, our results in this regard are still preliminary and somewhat fragmentary.

A. MOTIVATIONS AND DEFINITION OF SUFFICIENT STRATEGIES

Suppose a time-continuous game G has a value $\hat{T}$. Namely, for any $e > 0$, Pursuer and Evader have e-effective strategies (see II-D). In this case, by an appropriate choice of strategies, Pursuer guarantees himself an e-capture time of at most $\hat{T}+e$, and Evader can prevent Pursuer from yielding more than $\hat{T}-e$. This holds for any $e > 0$, no matter how small. However, this "equilibrium" situation is realized if and only if the minimax relation (2.80) holds.

On the other hand, saddle-point "optimal" strategies have been studied by several authors using existing optimization techniques, such as the calculus of variations and dynamic programming [B6] , [B7] , [H3] , [K5] . Saddle-point strategies are described as follows:

"Analogous to (2.69), we define a mapping $U_o(\cdot,\cdot,\cdot)$ by

$$u(t) = U_o(x(t),y(t),t) \quad , \; t_o \leq t < \infty$$

where $u(\cdot)$ is a Pursuer's admissible control. The set of all such mappings will be denoted by $\underline{U}_o$. A mapping $V_o(\cdot,\cdot,\cdot)$ and the set $\underline{V}_o$ for Evader are similarly defined. For given $U_o(\cdot,\cdot,\cdot) \in \underline{U}_o$ and $V_o(\cdot,\cdot,\cdot) \in \underline{V}_o$, the ϵ-capture time can be formally determined. This will be denoted by

$$T(U_o(\cdot,\cdot,\cdot),\ V_o(\cdot,\cdot,\cdot)) .$$

If there exist $U_o^*(\cdot,\cdot,\cdot) \in \underline{U}_o$ and $V_o^*(\cdot,\cdot,\cdot) \in \underline{V}_o$ such that

$$T(U_o(\cdot,\cdot,\cdot),\ V_o^*(\cdot,\cdot,\cdot)) \geq T(U_o^*(\cdot,\cdot,\cdot),\ V_o^*(\cdot,\cdot,\cdot)$$

$$\geq T(U_o^*(\cdot,\cdot,\cdot),\ V_o(\cdot,\cdot,\cdot)) \tag{4.1}$$

the pair $(U^*(\cdot,\cdot,\cdot),\ V^*(\cdot,\cdot,\cdot))$ is called the optimal pair of strategies and the corresponding controls $u^*(\cdot)$ and $v^*(\cdot)$ are called optimal controls."

It is seen that "optimal" strategies of this type formally correspond to our e-effective strategy with e=0. In this chapter, instead of formulating differential games as above and trying to obtain "optimal" pair of strategies, we introduce the concept of sufficient strategies, which, we hope, will be convenient in circumventing the following difficulties which are inherent in "optimal" strategies.

(a) It is known that saddle-point "optimal" strategies exist if and only if the following minimax relation holds,

$$\min_{\substack{U_o(\cdot,\cdot,\cdot)\\ \in \underline{U}_o}} \ \max_{\substack{V_o(\cdot,\cdot,\cdot)\\ \in \underline{V}_o}} T(U_o(\cdot,\cdot,\cdot),\ V_o(\cdot,\cdot,\cdot))$$

$$= \max_{\substack{V_o(\cdot,\cdot,\cdot)\\ \in \underline{V}_o}} \ \min_{\substack{U_o(\cdot,\cdot,\cdot)\\ \in \underline{U}_o}} T(U_o(\cdot,\cdot,\cdot),\ V_o(\cdot,\cdot,\cdot)) \tag{4.2}$$

Conditions under which this relation holds are not known, in general.

(b) Even if (a) holds, in order to realize "optimal" controls, continuous observation of states is necessary. From the point of view of applications, this is undesirable.

(c) The applicability of existing optimization techniques for finding above "optimal" strategies is limited to only simple problems, mainly because of the fact that domains of regularity (in which partial derivatives of the capture time are continuous) are usually difficult to find.

Now, we shall introduce the concept of sufficient pursuit strategies.

From the point of view of applications, it is sometimes desirable to find an algorithm which guarantees Pursuer ϵ-capture, $\epsilon > 0$, against any Evader's strategy, within a given finite period of time T^*.

Thus, with a given initial condition (x_o, y_o, t_o), an <u>(ϵ, T^*) sufficient pursuit strategy</u>, $\epsilon > 0$, $0 < T^* < \infty$, is defined to be any Pursuer's strategy $(h, U_h(\cdot,\cdot,\cdot))$, $h > 0$, which guarantees the ϵ-capture no later than time $t_o + T^*$, against any Evader's strategy $(h', V_{h'}(\cdot,\cdot,\cdot))$, $h' > 0$. Similarly, an <u>(ϵ, T^*) sufficient evasion strategy</u>, $\epsilon > 0$, $0 < T^* < \infty$, is defined to be any Evader's strategy $(h, V_h(\cdot,\cdot,\cdot))$, $h > 0$, which guarantees the ϵ-capture not to occur before or at time $t_o + T^*$ against any Pursuer's strategy $(h', U_{h'}(\cdot,\cdot,\cdot))$, $h' > 0$.

In this chapter, we are mainly concerned with sufficient pursuit strategies.

Remark 1: The concept of (ϵ, T^*) sufficient pursuit strategies is defined such that it is possible (and natural) that there exist "better" strategies. For instance, if Pursuer takes a smaller sampling interval h', $0 < h' < h$, there may exist $(h', U_{h'}(\cdot,\cdot,\cdot))$, $0 < h' < h$, such that the ϵ'-capture, $0 < \epsilon' < \epsilon$, is guaranteed to occur no later than time $t_o + T^{*'}$, $0 < T^{*'} < T^*$, against any Evader's strategy.
Moreover, suppose $(h, U_h(\cdot,\cdot,\cdot))$ is an (ϵ, T^*) sufficient pursuit strategy. Even for the same sampling interval h, there may exist $(h, U'_h(\cdot,\cdot,\cdot))$ which is an $(\epsilon', T^{*'})$ sufficient pursuit strategy, with $0 < \epsilon' < \epsilon$, $0 < T^{*'} < T^*$.

Remark 2: Further modification is to change the sampling interval as the game proceeds. Heuristically, as the game proceeds and the states x(t) and y(t) for both players get nearer, more frequent observations are required.
This will be commented later by an example (Example 4-2).

B. CONSTRUCTIVE ALGORITHMS FOR SUFFICIENT PURSUIT STRATEGIES

The following theorem follows directly from Theorem 3-4.

Theorem 4-1:
For each $\epsilon > 0$, and T^*, $0 < T^* < \infty$, an (ϵ, T^*) sufficient

pursuit strategy exists for each h, $0 < h \leq h_o$,(where h_o is given by (3.18) with $T = T^*+t_o$) if there exists t_h^+, $t_o < t_h^+ \leq t_o+T^*$, which satisfies Condition 3-4. Furthermore, an algorithm for constructing the (ϵ, T^*) sufficient prusuit strategy, for a given h, $0 < h \leq h_o$, is given by MJC.

Remark 1: Suppose we are given $\epsilon > 0$ and T^* , $0 < T^* < \infty$. Then, by (3.18), we can calculate h_o . Let us take some h , $0 < h \leq h_o$. With this h, we examine whether there exists t_h^+ , $t_h^+ \leq t_o+T^*$, which satisfies Condition 3-4. If there is, an (ϵ, T^*) sufficient pursuit strategy exists for that sampling interval $h > 0$, and it can be constructed by MJC.

Remark 2: Suppose $\epsilon = 0$. Then, majorant capture with a time interval $h > 0$ is possible only under restrictive situations such as;

Example 4-1:

Both players move on a line;

Pursuer's dynamics is given by

$$\frac{dx(t)}{dt} = u(t) , \quad x(t) \in E^1 , \; x(t_o)=0 \; ; |u(t)| \leq 2 , \; t_o=0$$

Evader's dynamics is given by

$$\frac{dy(t)}{dt} = v(t) , \quad y(t) \in E^1 , \; y(t_o)=3 \; ; |v(t)| \leq 1$$

In this case, if Pursuer uses strategy $(h, U_h(\cdot,\cdot,\cdot))$ such that the value of $U_h(x(t_i),y(t_i),t_i)$ at time t, $0 \leq t < \infty$, is 2 for any $x(t_i) \in A_{x_o}(t_i)$ and $y(t_i) \in A_{y_o}(t_i)$,(and $h > 0$ is arbitrary), then the 0-capture ($\epsilon = 0$) occurs no later than time 3.

In general, the 0-capture ($\epsilon = 0$) does not occur for discrete majorant games ($h > 0$).
Therefore, in general, we can not construct $(0, T^*)$ sufficient strategies by Theorem 4-1. Hence, we exclude the case $\epsilon = 0$ from Theorem 4-1.
In Theorem 4-1, we treat the problem in terms of escapability sets. The following modified version of Theorem 4-1 is of use in applications.
For a given $\epsilon > 0$ and $0 < T^* < \infty$, a positive real number h^* is determined such that, for any h, $0 < h \leq h^*$,

$$\sup_{x_{i+1},x'_{i+1} \in A_{x_i}(t_i+h)} \|x_{i+1} - x'_{i+1}\| \leq \epsilon \qquad (4.3)$$

for any $x_i \in A_{x_o}(t_i)$ and for any i, $0 \leq i \leq n-1$
with $(n-1)h < T^* \leq nh$.
By At-2', it is seen that such h^* exists.

Theorem 4-2:
For each $\epsilon > 0$ and T^* , $0 < T^* < \infty$, an (ϵ, T^*) sufficient pursuit strategy exists for each h, $0 < h \leq h^*$, if there exists t_h^* , $t_o < t_h^* \leq t_o+T^*$, which satisfies Condition 4-2.
Furthermore, an algorithm for constructing the (ϵ, T^*) suffi-

cient pursuit strategy, for a given h, $0 < h \leq h^*$, is given by SCA (1), (2), (2)' , (3), and (3)' below.

Condition 4-2:

(a) $A_{x_o}(t_h^*) \supset A_{y_o}(t_h^*)$ and

(b) for each $t_i = t_o+ih$, $i=0,1,\ldots,j-1$, $t_o \leq t_j < t_h^* \leq t_{j+1}$, if $x_i \in A_{x_o}(t_i)$ and $y_i \in A_{y_o}(t_i)$ satisfy

$$A_{x_i}(t_h^*) \supset A_{y_i}(t_h^*) \quad \text{and}$$

$$t_i^{\#} = \inf(t : A_{x_i}(t) \supset A_{y_i}(t)) > t_{i+1} ,$$

then, there exists a point $x_{i+1} \in A_{x_i}(t_{i+1})$ such that

$$A_{x_{i+1}}(t_h^*) \supset A_{y_{i+1}}(t_h^*) \quad \text{for all } y_{i+1} \in A_{y_i}(t_{i+1}).$$

Proof:

We shall show that if Condition 4-2 holds, there exists a pursuit algorithm which guarantees the ϵ-capture at some time $t \leq t_h^* \leq t_o+T^*$.

The following pursuit procedure will be called the sufficient capture algorithm (SCA).

SCA (1) Calculate A_{x_o}, A_{y_o}, and $t_o^{\#}=\min(t : A_{x_o}(t) \supset A_{y_o}(t))$. If $t_o^{\#} \leq t_1$, go to SCA (2). If $t_o^{\#} > t_1$, go to SCA (3).

Remark: We assume $\min(t : A_{x_o}(t) \supset A_{y_o}(t))$ exists. If it does not exist, we replace

$$t_o^{\#} = \inf(t : A_{x_o}(t) \supset A_{y_o}(t))$$

and treat the problem just the same as in MIC and MJC.

SCA (2) If $t_o^{\#} \leq t_1$, the ϵ-capture occurs no later than time $t_o^{\#}$ and the game ends.

The proof follows from MJC (2) or;

Pursuer takes any admissible $u_o(\cdot)$, then $f_{x_o}^{u_o}(t_o^{\#}) \in A_{x_o}(t_o^{\#})$.

But, $f_{y_o}^{v_o}(t_o^{\#}) \in A_{y_o}(t_o^{\#}) \subset A_{x_o}(t_o^{\#})$ for any admissible $v_o(\cdot)$.

Hence,

$$\| f_{x_o}^{u_o}(t_o^{\#}) - f_{y_o}^{v_o}(t_o^{\#}) \| \leq \epsilon$$

holds for any admissible $u_o(\cdot)$ and $v_o(\cdot)$.

Hence, for any admissible $u_o(\cdot)$, taking $u_o(\cdot) = U_h(x_o, y_o, t_o)$, $(h, U_h(\cdot,\cdot,\cdot))$ is an (ϵ, T^*) sufficient pursuit strategy.

SCA (3) If $t_o^{\#} > t_1$, find $x_1 \in A_{x_o}(t_1)$ such that

$$A_{x_1}(t_h^*) \supset A_{y_1}(t_h^*) \quad \text{for all } y_1 \in A_{y_o}(t_1) \tag{4.4}$$

If $t_1^{\#} = \min(t : A_{x_1}(t) \supset A_{y_1}(t)) \leq t_2$, go to SCA (2)' with i=1.

If $t_1^{\#} > t_2$, go to SCA (2)' with i=1.

Remark: To find $x_1 \in A_{x_o}(t_1)$ which satisfies (4.4) is equivalent to find $x_1 \in A_{x_o}(t_1)$ such that

$$A_{x_1}(t_h^*) \supset A_{y_o}(t_h^*) \tag{4.5}$$

This fact will be used later.

SCA (2)' If $t_i^{\#} = \min(t : A_{x_i}(t) \supset A_{y_i}(t)) \leq t_{i+1}$ the ϵ-capture occurs no later than time $t_i^{\#}$ and the game ends.

The proof is the same as in SCA (2).

SCA (3)' If $t_i^{\#} > t_{i+1}$, find $x_{i+1} \in A_{x_i}(t_i)$ such that

$$A_{x_{i+1}}(t_h^*) \supset A_{y_{i+1}}(t_h^*) \quad \text{for all } y_{i+1} \in A_{y_i}(t_{i+1})$$

If $t_{i+1}^{\#} \leq t_{i+2}$, go to SCA (2)' with $i \to i+1$.
If $t_{i+1}^{\#} > t_{i+2}$, go to SCA (3)' with $i \to i+1$.

The fact that the game always ends is shown similar to Theorem 3-4.

Remark: Condition 4-2 (b) is weakened just the same way as in Remark after MJC (3)'. The same fact holds for Condition 4-3, 4-4, and 4-4'.

With these results for closed-loop games, we shall examine Example 4-2, which was presented by J. H. Eaton [E1] as an improper example for the open-loop game.

Example 4-2:

Pursuer's dynamics;

$$\frac{dx(t)}{dt} = \begin{bmatrix} 0 & -\pi \\ \pi & 0 \end{bmatrix} x(t) + u(t) \quad ,$$

with $x(t) \in E^2$, $x(t_o) = x_o = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $|u(t)| \leq 2$, $t_o = 0$

Evader's dynamics;

$$\frac{dy(t)}{dt} = \begin{bmatrix} 0 & -\pi \\ \pi & 0 \end{bmatrix} (y(t) - y_o) + v(t) \quad ,$$

with $y(t) \in E^2$, $y(t_o) = y_o = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $|v(t)| \leq 1$

Suppose we specify $\epsilon = 1$, and $T^* = 2$, then by (4.3) $h^* = 0.25$.

It is easy to see

$$A_{x_o}(t) = \{x : \|x\| \leq 2t\}$$

$$A_{y_o}(t) = \{y : \|y-y_o\| \leq t\}$$

Take $h = h^* = 0.25$ and $t_h^* = t_o + T^* = 2$.

and we examine if Condition 4-2 is satisfied.

Condition 4-2: (a)

$$A_{x_o}(t_h^*) = A_{x_o}(2) = \{x : \|x\| \leq 4\}$$

$$A_{y_o}(t_h^*) = A_{y_o}(2) = \left\{y : \left\|y - \begin{bmatrix}2\\0\end{bmatrix}\right\| \leq 2\right\}$$

Hence, $A_{x_o}(t_h^*) \supset A_{y_o}(t_h^*)$

Condition 4-2: (b)

SCA (1) $t_o^\# = \min(t : A_{x_o}(t) \supset A_{y_o}(t)) = 2$

$t_1 = t_o + h = 0.25$

Since $t_o^\# = 2 > 0.25 = t_1$ go to SCA (3).

SCA (3) Want to find $x_1 \in A_{x_o}(0.25) = \{x : \|x\| \leq 0.5\}$

such that

$$A_{x_1}(2) \supset A_{y_o}(2) \tag{4.6}$$

It is easy to see that the only point that satisfies (4.6) is

$$x_1^* = \frac{0.5}{\sqrt{2}} \begin{bmatrix}1\\1\end{bmatrix}$$

Now, we shall calculate

$$t_1^\# = \min(t : A_{x_1^*}(t) \supset A_{y_1}(t))$$

If Evader chooses

$$y_1 \in A_o(0.25) = \left\{ y : \left\| y - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\| \leq 0.25 \right\}$$

other than

$$y_1^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 0.25 \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

then, we have

$$\min(t : A_{x_1^*}(t) \supset A_{y_1}(t)) < \min(t : A_{x_1^*}(t) \supset A_{y_1^*}(t))$$

$$\text{for any } y_1 \in A_{x_o}(0.25) \ , \quad y_1 \neq y_1^*$$

Namely, if Evader chooses $y_1 \in A_{x_o}(0.25)$ other than y_1^* , the game ends earlier than time $t_o^{\#} = T^* = 2$.

Although, we are only concerned with pursuit strategies here, and the value of y_1 is to be observed during the game, anticipating the worst case, let us suppose that Evader chooses y_1^* .

Then, $t_1^{\#} = 2$

$t_2 = t_o + 2h = 0.5$.

Since $t_1^{\#} > t_2$, go to SCA (3)' with i=1 .

<u>SCA (3)'</u> Want to find $x_2 \in A_{x_1}(0.5) = \left\{ x : \left\| x - \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \right\| \leq 0.5 \right\}$ such that

$$A_{x_2}(2) \supset A_{y_1^*}(2) \tag{4.7}$$

It is easy to see the only point that satisfies (4.7) is

$$x_2^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Similar to SCA (3), we get

$$y_2^* = \begin{bmatrix}2\\0\end{bmatrix} + \begin{bmatrix}0\\0.5\end{bmatrix} \qquad \text{and} \qquad t_2^\# = 2$$

Repeating SCA (3)' until we arrive at x_7^* , y_7^* ,

with

$$\begin{aligned}
x_o &= \begin{bmatrix}0\\0\end{bmatrix} & y_o &= \begin{bmatrix}2\\0\end{bmatrix}\\
x_1^* &= \frac{0.5}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} & y_1^* &= \begin{bmatrix}2\\0\end{bmatrix} + \frac{0.25}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\\
x_2^* &= \begin{bmatrix}0\\1\end{bmatrix} & y_2^* &= \begin{bmatrix}2\\0\end{bmatrix} + \begin{bmatrix}0\\0.5\end{bmatrix}\\
x_3^* &= \frac{1.5}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} & y_3^* &= \begin{bmatrix}2\\0\end{bmatrix} + \frac{0.75}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}\\
x_4^* &= \begin{bmatrix}-2\\0\end{bmatrix} & y_4^* &= \begin{bmatrix}2\\0\end{bmatrix} + \begin{bmatrix}-1\\0\end{bmatrix}\\
x_5^* &= \frac{2.5}{\sqrt{2}}\begin{bmatrix}-1\\-1\end{bmatrix} & y_5^* &= \begin{bmatrix}2\\0\end{bmatrix} + \frac{1.25}{\sqrt{2}}\begin{bmatrix}-1\\-1\end{bmatrix}\\
x_6^* &= \begin{bmatrix}0\\-3\end{bmatrix} & y_6^* &= \begin{bmatrix}2\\0\end{bmatrix} + \begin{bmatrix}0\\-1.5\end{bmatrix}\\
x_7^* &= \frac{3.5}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} & y_7^* &= \begin{bmatrix}2\\0\end{bmatrix} + \frac{1.75}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}
\end{aligned}$$

and $t_i^\# = 2$, $i=0,1,\ldots,7$.

Hence, going to SCA (2)' with $i=7$, we can conclude that the ϵ-capture occurs no later than time $t_7^\# = 2$ and the game ends. Hence, we see that the $(1, 2) = (\epsilon, T^*)$ sufficient pursuit strategy exists.

<u>Remark 1</u>: The attainability sets for linear systems are compact and convex (see At-4 and At-5, I-C-1). Hence, some improvements of the above algorithm SCA are possible. Especially, in the above example, since attainability sets for both players are always 2-dimentional circles, SCA is improved as follows;

In SCA (2)' with i=7, denoting the center of $A_{x_7^*}(t_7^\#)$ by $x^\#$, if Pursuer chooses admissible $u_7^*(\cdot)$ such that

$$f_{x_7^*}^{u_7^*}(t_7^\#) = x^\# \in A_{x_7^*}(t_7^\#)$$

then, for any $x \in A_{x_7^*}(t_7^\#)$

$$\|x - x^\#\| \leq \tfrac{1}{2}\epsilon$$

Hence,

$$\|f_{x_7^*}^{u_7^*}(t_7^\#) - f_{y_7^*}^{v_7}(t_7^\#)\| \leq \tfrac{1}{2}\epsilon$$

for any admissible $v_7(\cdot)$.

Therefore, we see that the (0.5, 2) = ($\frac{1}{2}\epsilon$, T^*) sufficient pursuit strategy exists. This is an improvement of (1, 2) strategy.

Remark 2: Another improvement is as follows:
For given ϵ=1 and T^*=2, it is not necessary for Pursuer to observe the Evader's state at the end of every incremental interval h.
Actually, it is easy to see, instead of observing the Evader's state at times t_o, $t_o + h$, $t_o + 2h, \ldots, t_o + 7h$, Pursuer can construct the same pursuit strategy observing the Evader's state only at times t_o, $t_o + 4h$, $t_o + 6h$, $t_o + 7h$.
Heuristically, this is because, at the beginning of the game, Pursuer has a greater flexibility, and as the both players' states get nearer, Pursuer needs more frequent observations of the Evader's state.

Remark 3: It is interesting to note that the admissible trajectories connecting the points x_o, $x_1^*, \ldots, x_7^*$ and the points y_o, $y_1^*, \ldots, y_7^*$ (it is easy to see that such trajectories are unique) coinside, in this case, with the solutions for the open-loop game obtained by the formal application of the maximum principle [E1].
The geometrical approach used here provides simple insight and a rigorous treatment of "closed-loop" games.

C. EXISTENCE OF SUFFICIENT PURSUIT STRATEGIES FOR LINEAR SYSTEMS

Although Theorem 4-1 and 4-2 are of use of construct an (ϵ, T^*) sufficient pursuit algorithm, following theorems for linear systems are more convenient in verifying, a priori, whether an (ϵ, T^*) strategy exists. These theorems are geometric in nature and provide simple interpretation for capture conditions. Some examples given in the next chapter subsume the results previously obtained by applying classical optimization techniques.

Let Pursuer's and Evader's dynamics are described by the linear differential equations

$$\frac{dx(t)}{dt} = A_p(t)x(t) + B_p(t)u(t) \qquad x(t_o)=x_o \quad (4.8)$$

$$\frac{dy(t)}{dt} = A_e(t)y(t) + B_e(t)v(t) \qquad y(t_o)=y_o \quad (4.9)$$

where $A_p(t)$ and $A_e(t)$ are n x n continuous, time-varying matrices defined on $[t_o, \infty)$ and $B_p(t)$ and $B_e(t)$ are n x m continuous, time-varying matrices defined on $[t_o, \infty)$.
In this case, it is easy to see that the following property concerning attainability sets holds.

<u>At-6</u> For any t, $t_o < t < \infty$, and for any $x(t)$, $x'(t) \in A_{x_o}(t)$ there exists a vector $z \in E^n$ such that

$$A_{x(t)}(t') = A_{x'(t)}(t')+z \qquad \text{for any } t', \ t \leq t' < \infty \ .$$

The same property holds for Evader's attainability sets.

Other conditions and the rules of the game are the same as before.

Now, we are given $\epsilon > 0$ and T^*, $t_o < T^* < \infty$. A positive real number h^* is determined such that, for any h, $0 < h \leq h^*$,

$$\max_{x,x' \in A_{x(t)}(t+h)} \|x - x'\| \leq \epsilon \qquad \text{holds} \qquad (4.3)'$$

for any $x(t) \in A_{x_o}(t)$, and for any t, $t_o \leq t \leq T^*-h$.
This is a modified condition of (4.3). The existence of such h^* is easily verified.

<u>Theorem 4-3</u>: (Linear systems)
For each $\epsilon > 0$ and T^*, $0 < T^* < \infty$, an (ϵ, T^*) sufficient pursuit strategy exists for each h, $0 < h \leq h^*$, if there exists t_h^*, $t_o < t_h^* \leq t_o+T^*$, which satisfies Condition 4-3.

Condition 4-3:

(a) $A_{x_o}(t_h^*) \supset A_{y_o}(t_h^*)$ and

(b) for each $i=0,1,\ldots,j-1$, $t_o+jh < t_h^* \leq t_o+(j+1)h$, corresponding to any $x_{i+1} \in A_{x_o}(t_{i+1})$ and $y_i \in A_{y_o}(t_i)$ there exists a vector $z \in E^n$ such that

$$A_{x_{i+1}}(t_h^*) \supset A_{y_i}(t_h^*) + z \qquad \text{holds.}$$

Proof:

The proof follows from Theorem 4-2 as follows:

Suppose Condition 4-3 is satisfied. Then, by the property At-6 and At-5 (convexity), it is easy to see that exists $x_{i+1}^* \in A_{x_o}(t_{i+1})$, $i=0,1,\ldots,j-1$, such that

$$A_{x_{i+1}^*}(t_h^*) \supset A_{y_i}(t_h^*)$$

Since $A_{y_i}(t_h^*) \supset A_{y_{i+1}}(t_h^*)$ for any $y_{i+1} \in A_{y_i}(t_{i+1})$, we have $A_{x_{i+1}^*}(t_h^*) \supset A_{y_{i+1}}(t_h^*)$ for any $y_{i+1} \in A_{y_i}(t_{i+1})$. Hence, Condition 4-2 (b) is satisfied.

If the dynamics of both players are linear and time-invariant, attainability sets satisfy the following property.

At-7 For any t and t', $t_o < t \leq t+t' < \infty$, for any t^*, $t \leq t^* < \infty$, and for any $x(t) \in A_{x_o}(t)$ and $x(t+t') \in A_{x_o}(t+t')$, there exists a vector $z \in E^n$ such that

$$A_{x(t)}(t^*) = A_{x(t+t')}(t^*+t')+z \qquad .$$

The same property holds for Evader's attainability sets.

Hence, the following Corollary follows from Theorem 4-3.

Corollary: (Linear, time-invariant systems)

For each $\epsilon > 0$ and T^*, $0 < T^* < \infty$, an (ϵ, T^*) sufficient pursuit strategy exists for each h, $0 < h \leq h^*$, if there exists t_h^*, $t_o < t_h^* \leq t_o+T^*$, which satisfies Condition 4-3'.

Condition 4-3':

(a) $A_{x_o}(t_h^*) \supset A_{y_o}(t_h^*)$ and

(b) for each i, $i=0,1,\ldots,j-1$, $t_o+jh < t_h^* \leq t_o+(j+1)h$,

there exists a vector $z \in E^n$ such that

$$A_{x_o}(t_h^* - (i+1)h) \supset A_{y_o}(t_h^* - ih)+z$$

Proof:

Directly follows from Theorem 4-3, using At-7.

Example 4-3:

We shall apply above Corollary to Example 4-2. Taking $\epsilon=1$ and $T^*=2$, we get $h^*=0.25$.

Let us take $t_h^*=2$, $h=0.25$, and $t_o=0$ as before. Then $j=7$.

$$A_{x_o}(t_h^* - (i+1)h) = \{x : \|x\| \leq 2(t_h^* - (i+1)h)\}$$

$$A_{y_o}(t_h^* - ih) = \left\{ y : \left\|y-\begin{bmatrix}2\\0\end{bmatrix}\right\| \leq (t_h^* - ih)\right\}$$

Taking $z=\begin{bmatrix}2\\0\end{bmatrix}$ for any $i=0,1,\ldots,6$, we see that

$$A_{x_o}(t_h^* - (i+1)h) \supset A_{y_o}(t_h^* - ih) + \begin{bmatrix}2\\0\end{bmatrix}$$

for any $i=0,1,\ldots,6$.

Hence, the (1, 2)=(ϵ, T^*) sufficient pursuit strategy exists.

Now, from the point of view of applications, "continuous" observation of the Evader's state, i.e., $h \rightarrow 0$, is practically undesirable.

However, it is interesting to investigate conditions under which, corresponding to any $\epsilon > 0$, no matter how small, and for a given T^*, $0 < T^* < \infty$, there exists h^* such that for each h, $0 < h \leq h^*$, an (ϵ, T^*) sufficient pursuit strategy exists. Since if they are satisfied, for any $\epsilon > 0$ no matter how small, Pursuer can find his strategy (h, $U_h(\cdot,\cdot,\cdot)$), $h>0$, which guarantees himself ϵ-capture no later than time t_o+T^*. This conditions coincide with "capture conditions for the differential game" in the existing literatures. This will be explained by examples in the next section.

It should be noted, that the greater the "accuracy" of capture ($\epsilon \rightarrow 0$) desired, the finer the sampling interval required ($h \rightarrow 0$), except for special cases such as in Example 4-1. Practically, Pursuer should find some compromising values of $\epsilon > 0$ and $h > 0$.

The following theorem is of use to establish capture conditions.

<u>Theorem 4-4</u>: (Linear systems)

For each $\epsilon > 0$ and T^*, $0 < T^* < \infty$, if there exists t^*, $t_o < t^* \leq t_o+T^*$, which satisfies Condition 4-4, then there exists h^{**} such that for each h, $0 < h \leq h^{**}$, an (ϵ, T^*) sufficient pursuit strategy exists.

Condition 4-4:

(a) $A_{x_o}(t^*) \supset A_{y_o}(t^*)$,

(b) corresponding to any $x(t) \in A_{x_o}(t)$ and $y(t) \in A_{y_o}(t)$, $t_o \leq t \leq t^*-h^*$, there exists a vector $z \in E^n$ such that

$$\mathring{A}_{x(t)}(t^*) \supset A_{y(t)}(t^*) + z$$

where h^* is given by (4.3)' and $\mathring{A}_{x(t)}(t^*)$ represents the interior of $A_{x(t)}(t^*)$, and

(c) there exists a positive number $h^\# > 0$, independent of t, and corresponding to each t, $t_o \leq t \leq t^*-h^*-h^\#$, there exists a vector $z' \in E^n$ such that

$$A_{x(t+h^\#)}(t^*) \supset A_{y(t)}(t^*) + z'$$

Proof:

Taking $h^{**}=\min(h^*, h^\#)$, the above follows directly from Theorem 4-3.

Corollary: (Linear time-invariant systems)

If the systems dynamics is time-invariant, Condition 4-4 is replaced by Condition 4-4'.

Condition 4-4':

(a) $A_{x_o}(t^*) \supset A_{y_o}(t^*)$,

(b) for any t, $t_o+h^* \leq t \leq t^*$, there exists a vector $z \in E^n$ such that

$$\mathring{A}_{x_o}(t) \supset A_{y_o}(t) + z$$

(c) there exists a positive number $h^\# > 0$, independent of t, and corresponding to each t, $t_o+h^*+h^\# \leq t \leq t^*$, there exists a vector $z' \epsilon E^n$ such that

$$A_{x_o}(t - h^\#) \supset A_{y_o}(t) + z'$$

D. SOME EXAMPLES

We shall apply the results obtained in Section C to some specific problems. Since, the evaluation of attainability sets is, in general, not too easy, one must exert one's ingenuity in applying our results to individual cases.

1. Lower dimensional capture

In I-D-1, the ϵ-capture time $\hat{t}$ was defined by

$$\|x(\hat{t}) - y(\hat{t})\| \leq \epsilon \qquad \text{and}$$
$$\|x(t) - y(t)\| > \epsilon \qquad \text{for all } t,\ t_o \leq t < \hat{t}$$

However, in actual problems, we often encounter the cases where the above definition is too restrictive. A weaker version of the ϵ-capture time $\hat{t}^*$ will be defined by

$$\|x^{(i)}(\hat{t}^*) - y^{(i)}(\hat{t}^*)\| \leq \epsilon \quad \text{and}$$
$$\|x^{(i)}(t) - y^{(i)}(t)\| > \epsilon \qquad \text{for all } t,\ t_o \leq t < \hat{t}^* \tag{4.10}$$

where $x^{(i)}(t)$ and $y^{(i)}(t)$ represent the $i^{\underline{th}}$, $1 \leq i \leq n$, components of Pursuer's and Evader's states x(t) and y(t), respectively, at time t, $t_o \leq t < \infty$.

Example 4-4: (see Ref. [P1])

Pursuer's dynamics is given by

$$\ddot{x}(t) + a\dot{x}(t) = cu(t) \qquad \begin{array}{l} |u(t)| \leq 1 \\ a,c > 0 \end{array} \tag{4.11}$$

Evader's dynamics is given by

$$\ddot{y}(t) + b\dot{y}(t) = dv(t) \qquad \begin{array}{l} |v(t)| \leq 1 \\ b,c > 0 \end{array} \tag{4.12}$$

Equivalently,

Pursuer's dynamics:

$$\frac{d}{dt}\begin{bmatrix} x^{(1)}(t) \\ x^{(2)}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix}}_{A_p} \begin{bmatrix} x^{(1)}(t) \\ x^{(2)}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ c \end{bmatrix}}_{b_p} u(t) \tag{4.13}$$

Evader's dynamics:

$$\frac{d}{dt}\begin{bmatrix} y^{(1)}(t) \\ y^{(2)}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix}}_{A_e} \begin{bmatrix} y^{(1)}(t) \\ y^{(2)}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ d \end{bmatrix}}_{b_e} v(t) \tag{4.14}$$

Let us assume for simplicity

$$\begin{bmatrix} x^{(1)}(t_o) \\ x^{(2)}(t_o) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y^{(1)}(t_o) \\ y^{(2)}(t_o) \end{bmatrix} = \begin{bmatrix} y_o^{(1)} \\ 0 \end{bmatrix} \neq 0$$

Now, let us suppose we are interested in capture of the first component of the state. Namely, the ε-capture time $\hat{t}^*$ is defined by

$$\|x^{(1)}(\hat{t}^*) - y^{(1)}(\hat{t}^*)\| \leq \epsilon \quad \text{and}$$
$$\|x^{(1)}(t) - y^{(1)}(t)\| > \epsilon \quad \text{for all } t,\ t_o \leq t < \hat{t}^* \tag{4.15}$$

Then, the projections of the fixed-time cross sections of attainability sets $A_{x_o}(t)$ and $A_{y_o}(t)$ on the first coordinate, denoted by $A_{x_o}^{(1)}$ and $A_{y_o}^{(1)}$, respectively, are

$$A_{x_o}^{(1)}(t) = \left\{ x^{(1)}(t) \ :\ |x^{(1)}(t)| \leq \int_{t_o}^{t} \frac{c}{a}\,(1-e^{-a(t-s)})\,ds \right\} \tag{4.16}$$

$$A_{y_o}^{(1)}(t) = \left\{ y^{(1)}(t) \ :\ |y^{(1)}(t)-y_o^{(1)}| \leq \int_{t_o}^{t} \frac{d}{b}(1-e^{-b(t-s)})\,ds \right\} \tag{4.17}$$

From (4.16) and (4.17), we can easily derive a sufficient condition for Condition 4-4'.

For example, it is easily verified that if

$$c \geq d \quad \text{and} \quad \frac{c}{a} > \frac{d}{b} \qquad \text{hold,} \tag{4.18}$$

$$\frac{c}{a}\,(1 - e^{-a(t-s)}) > \frac{d}{b}\,(1 - e^{-b(t-s)}) \tag{4.19}$$

holds for any $t-s > 0$.

For any $\epsilon > 0$, following (4.3)', we take $h^* > 0$ such that

$$\epsilon \geq 2 \int_{t'}^{t'+h^*} \frac{c}{a}\,(1 - e^{-a(t'+h^*-s)})\,ds$$
$$= 2\frac{c}{a}\,(h^* - \frac{1}{a}\,(1 - e^{-ah^*}))$$

It is interesting to note that for any $\epsilon > 0$ there exists such $h^* > 0$ and $\epsilon \to 0$ implies $h^* \to 0$.

We shall examine Condition 4-4': Let (4.18) be satisfied.

(a) It is easily verified that there exists t^*, $t_o < t^* < \infty$, such that

$$A_{x_o}^{(1)}(t^*) \supset A_{y_o}^{(1)}(t^*)$$

(b) Taking $z = -y_o^{(1)}$, (4.19) implies

$$\mathring{A}_{x_o}^{(1)}(t) \supset A_{y_o}^{(1)}(t) + z \quad \text{for any } t,\ t_o+h^* \leq t \leq t^*$$

(c) Since,

$$f_1(t) = \int_{t_o}^{t} \frac{c}{a}\,(1 - e^{-a(t-s)})\,ds \quad \text{and} \quad (4.20)$$

$$f_2(t) = \int_{t_o}^{t} \frac{d}{b}\,(1 - e^{-b(t-s)})\,ds \qquad (4.21)$$

are strictly increasing and $f_1(t) > f_2(t)$ (by (4.19)) for any $t > t_o$, it is easily verified that there exists $h^\# > 0$ such that

$$f_1(t - h^\#) \geq f_2(t) \qquad (4.21)'$$

for any t, $t_o+h^*+h^\# \leq t \leq t^*$.

Therefore, by Corollary (p. 113), we see that the ϵ-capture occurs no later than some finite time t^*.

But, the above discussions hold for any $\epsilon > 0$, no matter how small. Therefore, (4.18) is a sufficient condition for 1-dimensional capture.

We can generalize conditions for 1-dimensional capture to the following case [H2].

Let the Pursuer's dynamics be given by

$$\frac{dx(t)}{dt} = A_p x(t) + b_p u(t) \qquad x(t_o) = x_o \tag{4.22}$$

$$\frac{dy(t)}{dt} = A_e y(t) + b_e v(t) \qquad y(t_o) = y_o \tag{4.23}$$

where A_p and A_e are n x n matrices and b_p and b_e are n vectors, and $x(t) \in E^n$, $y(t) \in E^n$, and $|u(t)| \leq 1$, $|v(t)| \leq 1$, as before. Let us suppose, for simplicity, all the eigenvalues of A_p and A_e are negative. Similar to (4.19), we can verify that Condition 4-4' is satisfied if

$$|X_p^{(1)}(t, t_o)b_p| > |X_e^{(1)}(t, t_o)b_e| \tag{4.24}$$

$$\text{for any } t, \ t_o < t < \infty$$

2. Energy constraint capture

In this thesis, we assume that the admissible controls $u(\cdot)$ and $v(\cdot)$ are constrained by

$$\begin{aligned} u(t) &\in U \qquad \text{for a.e. } t \in [t_o, \infty) \\ v(t) &\in V \qquad \text{for a.e. } t \in [t_o, \infty) \end{aligned} \tag{4.25}$$

where U and V are given compact subsets of E^n. This assumption is not vital in our discussions and most of the results obtained remain valid even if we replace it by some other constraints. Let us consider the cases where admissible controls $u(\cdot)$ are constrained in the following form:

$$\|u(\cdot)\|_p = \left[\sum_{i=1}^{m} \int_{t_o}^{t^*} |u^{(i)}(t)|^p \, dt\right]^{1/p} \leq k_p \tag{4.26}$$

where k_p and p are given positive numbers, $1 \leq p < \infty$, and $u^{(i)}(t)$, $1 \leq i \leq m$, represents the $i^{\underline{th}}$ component of u(t).
We consider the analogous constraint for $v(\cdot)$.
For convenience, if $p=\infty$,

$$\|u(\cdot)\|_{\infty} = \max_{1\leq i\leq m} \operatorname*{ess\,sup}_{t_o\leq t\leq t^*} |u^{(i)}(t)| \leq k_p \quad (4.27)$$

which is equivalent to

$$|u^{(i)}(t)| \leq k_p \qquad \text{for any } i=1,2,\ldots,m \text{ and } t,\ t_o\leq t\leq t^*$$

This is the "amplitude" constraint.
It should be noted that bounded controls can always be brought into this form by appropriate transformation.
Taking p=2, we obtain the "energy" constraint, and taking p=1, we obtain the "area" constraint.
Now, we shall apply the results obtained in the preceding section to games with these constraints.

Example 4-5: (see Refs. [H3] and [K8])
Dynamics for Pursuer and Evader are given by (4.8) and (4.9).
Since the purpose of this example is to demonstrate the applicability of the results previouly obtained, for simplicity, we shall assume that u(t) and v(t) are scalars, both dynamics are asymptotically stable and totally controllable.
Denoting $B_p(t)$ and $B_e(t)$ by $b_p(t)$ and $b_e(t)$ (both n-dimentional vectors), respectively, the solution of (4.8) and (4.9) are

$$x(t) = X_p(t, t_o)x(t_o) + \int_{t_o}^{t} X_p(t,s)b_p(s)u(s)ds \quad (4.28)$$

and

$$y(t) = X_e(t,\ t_o)y(t_o) + \int_{t_o}^{t} X_e(t,s)b_e(s)v(s)ds \quad (4.29)$$

Equivalently, we have

$$x(t) = X_p(t,\ t_o)x(t_o) + \int_{t_o}^{t} h_p(t,s)u(s)ds \quad (4.30)$$

$$y(t) = X_e(t,\ t_o)y(t_o) + \int_{t_o}^{t} h_e(t,s)v(s)ds \quad (4.31)$$

where

$$h_p(t,s) = X_p(t,s)b_p(s) \qquad \text{and}$$

$$h_e(t,s) = X_e(t,s)b_e(s) \quad .$$

Let us consider the case where both controls are constrained, by "energy", namely, by the form,

$$\int_{t}^{t^*} |u(s)|^2 ds \le (k_p(t))^2 \qquad \text{and} \qquad (4.32)$$

$$\int_{t}^{t^*} |v(s)|^2 ds \le (k_e(t))^2 \qquad (4.33)$$

where $k_p(\cdot)$ and $k_e(\cdot)$ are continuous functions from $[t_o, \infty)$ into E^1 and $k_p(t) > 0$ and $k_e(t) > 0$ for any t, $t_o \le t < \infty$, and t^*, $t_o \le t \le t^* < \infty$, is determined such that Condition 4-4 (a) is satisfied. In this case, it is known [K9] that the boundary of the sets

$$R_p(t^*,\ t) = A_{x(t)}(t^*) - X_p(t^*,\ t)x(t) \qquad (4.34)$$

$$\text{where } x(t) \in A_{x_o}(t), \text{ and } t_o \le t < t^*$$

represents a hyperellipsoid, centered on the origin.

More precisely, the boundary of the set (4.34) is given by

$$< x(t^*),\ P_p^{-1}(t^*,\ t)x(t^*) > = k_p^2(t) \qquad (4.35)$$

where

$$P_p(t^*,\ t) = \int_t^{t^*} h_p(t^*,\ s)h_p'(t^*,\ s)ds \qquad (4.36)$$

$<\cdot,\cdot>$ represents an inner product,

and the superscript $'$ represents the conjugate transpose.

Since we assumed that both dynamics are totally controllable, it is known that $P_p(t^*,\ t)$ is positive definite, hence $P_p^{-1}(t^*,\ t)$ exists for all t, $t_o \leq t < t^*$.

Similarly, the boundary of the set

$$R_e(t^*,\ t) = A_{y(t)}(t^*) - X_e(t^*,\ t)y(t) \qquad (4.37)$$

where $y(t) \in A_{y_o}^{-}(t)$, and $t_o \leq t < t^*$

is given by

$$< y(t^*),\ P_e^{-1}(t^*,\ t)y(t^*) > = k_e^2(t) \qquad (4.38)$$

where

$$P_e(t^*,\ t) = \int_t^{t^*} h_e(t^*,\ s)h_e'(t^*,\ s)ds \qquad (4.39)$$

From (4.35) and (4.38), we see that

$$\overset{o}{R}_p(t^*,\ t) \supset R_e(t^*,\ t) \qquad (4.39)'$$

holds for all t, $t_o \leq t < t^*$ if and only if

$$< x, k_e^{-2}(t)P_e^{-1}(t^*, t)x > - < x, k_p^{-2}(t)P_p^{-1}(t^*, t)x > \;> 0 \qquad (4.40)$$

$$\text{for all } x \in R_p(t^*, t) \quad x \neq 0$$
$$\text{for all } t, \; t_o \leq t < t^*.$$

But, if $k_e^{-2}(t)P_e^{-1}(t^*, t) - k_p^{-2}(t)P_p^{-1}(t^*, t)$ is positive definite for all t, $t_o \leq t < t^*$, then (4.40) holds.
Therefore, we can conclude that if the following condition

$$k_p^2(t)P_p(t^*, t) - k_e^2(t)P_e(t^*, t) \text{ is positive definite for all } t, \; t_o \leq t < t^*, \qquad (4.41)$$

is satisfied, then (4.39)' holds.
Hence, by Theorem 4-4, similar to previous Example, we can verify that (4.41) is a sufficient condition for "energy" constraint capture.

Now, we can generalize the above results as follows:
Suppose, both player's controls are constrained by

$$\left[\int_t^{t^*} |u(t)|^p \, dt\right]^{1/p} \leq k_p(t) \qquad (4.42)$$

and

$$\left[\int_t^{t^*} |v(t)|^{p'} \, dt\right]^{1/p'} \leq k_e(t) \qquad (4.43)$$

where $1 \leq p \leq \infty$ and $1 \leq p' \leq \infty$, $t_o \leq t < t^*$, and t^* is some finite time which satisfies Condition 4-4 (a).
Let q and q' are defined by

$$1/p + 1/q = 1 \tag{4.44}$$

$$1/p' + 1/q' = 1 \tag{4.45}$$

Then, it can be seen that [K9] if

$$k_e(t) \left[\int_t^{t^*} \left|< w,\ h_e(t^*,\ s) >\right|^{q'} ds \right]^{1/q'}$$

$$< k_p(t) \left[\int_t^{t^*} \left|< w,\ h_p(t^*,\ s) >\right|^{q} ds \right]^{1/q} . \tag{4.46}$$

for any vector $w \in E^n$; $w \neq 0$, and for all t, $t_o \leq t < t^*$, then

$$\overset{o}{R}_p(t^*,\ t) \supset R_e(t^*,\ t) \qquad \text{for all } t,\ t_o \leq t < t^* \tag{4.47}$$

If p=2, q=2, p'=2, and q'=2, we have

$$k_e^2(t) \int_t^{t^*} \left|< w,\ h_e(t^*,\ s) >\right|^2 ds$$

$$< k_p^2(t) \int_t^{t^*} \left|< w,\ h_p(t^*,\ s) >\right|^2 ds \tag{4.48}$$

for any vector $w \in E^n$, $w \neq 0$, and for all t, $t_o \leq t < t^*$

But,

$$\int_t^{t^*} \left|< w,\ h_e(t^*,\ s) >\right|^2 ds = < w,\ P_e(t^*,\ t)w > \quad \text{and}$$

$$\int_t^{t^*} \left|< w,\ h_p(t^*,\ s) >\right|^2 ds = < w,\ P_p(t^*,\ t)w >$$

we see that (4.48) becomes

$$< w,\ k_e^2(t)P_e(t^*,\ t)w > \ < \ < w,\ k_p^2(t)P_p(t^*,\ t)w > \quad (4.49)$$

for any $w \in E^n$, $w \neq 0$, and
for any t, $t_o \leq t < t^*$

which is equivalent to (4.41).

If $p=\infty$, $q=1$, $p'=\infty$, and $q'=1$, we have

$$k_e(t) \int_t^{t^*} |< w,\ h_e(t^*,\ s) >|\ ds$$

$$< k_p(t) \int_t^{t^*} |< w,\ h_p(t^*,\ s) >|\ ds \quad (4.50)$$

for any vector $w \in E^n$, $w \neq 0$, and
for any t, $t_o \leq t < t^*$.

If

$$k_e(t)\ |< w,\ h_e(t^*,\ s) >| < k_p(t)\ |< w,\ h_p(t^*,\ s) >| \quad (4.51)$$

for any vector $w \in E^n$, $w \neq 0$, and
for any s, $t_o \leq t \leq s < t^*$

then, (4.50) holds.

If we take $k_e(t)=k_p(t)=1$ and consider 1-dimensional capture as in Example 4-4, we get

$$\left|h_e^{(1)}(t^*,\ s)\right| < \left|h_p^{(1)}(t^*,\ s)\right| \quad (4.52)$$

for any s, $t_o \leq s < t^*$

which coincides with (4.24).

If $p=2$, $q=2$, $p'=\infty$, and $q'=1$, we have

$$k_e^2(t)(t^*-t) < w,\ P_e(t^*,\ t)w > \ < \ k_p^2(t) < w,\ P_p(t^*,\ t)w >.$$

for any vector $w \in E^n$, $w \neq 0$, and
for any t, $t_o \leq t < t^*$

(4.53)

Analogous to (4.41) or (4.48), we see that if

$$k_p^2(t)P_p(t^*, t) - (t^*-t)k_e^2(t)P_e(t^*, t) \tag{4.54}$$

for all t, $t_o \leq t < t^*$

is positive definite, then (4.53) holds.

Hence, (4.54) is a sufficient condition for (4.47).

V. CONCLUSIONS

In this thesis, an important class of "closed-loop" differential games arising from the study of pursuit-evasion games is studied by means of discrete-time approximations. Since we encounter profound difficulties in the precise formulation of games with a continuum of moves, such as the closed-loop, pursuit-evasion game studied here, we approximated them by a series of time-discrete games with sampling time intervals $h>0$.

We showed in Chapter II that the values of approximating discrete games converge to a limit as h tends to zero (Theorem 2-1) and the limit coincides with the appropriately defined value of the time-continuous game if the "minimax" theorem holds (Theorem 2-3). Necessary and sufficient conditions under which the minimax relation holds are not known. Theorem 2-2 gives a set of sufficient conditions for insuring the validity of the minimax relation.

Now, if we try to find saddle-point "optimal" strategies, the verification of the minimax theorem becomes crucial. Moreover, although it has been conjectured that "optimal" strategies are closely related to the solution of the modified Hamilton-Jacobi-Bellman partial differential equation, precise relation between them is, in general, open. Furthermore, this approach is restricted by many technical difficulties, above all by the fact that the domains of regularity are, in general, difficult to obtain. Thirdly, it has been pointed out in Chapter IV that continuous observation of states, which is undesirable from the point of view of applications, is indispensable for constructing these "optimal" strategies.

In Chapter IV, we introduced the concept of "sufficient" strategies, which, we hope, effectively circumvent the difficulties inherent in "optimal" strategies. A sufficient pursuit strategy guarantees Pursuer capture with an "accuracy" $\epsilon>0$, within a finite period of time. This requires neither continuous observation of states nor the verification of the "minimax" theorem. Existence theorems and constructive algorithms for such strategies are derived and applied to some simple examples in Chapter IV. It is hoped that the method developed here will provide new insight and interpretation for problems in this field.

REFERENCES

A1 M. Athan, P. L. Falb, and R. T. Lacoss, "Time-, fuel-, and energy-optimal control of nonlinear norm-invariant systems," IEEE Trans. on Automatic Control, vol. AC-8, pp. 196-202, 1963.

B1 R. Bellman, "Functional equations in the theory of dynamic programming III," Rendiconti del Circolo Matematico, vol. 5, pp. 297-319, 1956.

B2 R. Bellman, "Functional equations in the theory of dynamic programming VI, A direct convergence proof," Annals of Mathematics, vol. 65, pp. 215-223, 1957.

B3 R. Bellman, Adaptive Control Processes: A Guided Tour. Princeton, N. J.: Princeton University Press, 1961.

B4 C. Berge, Topological Spaces (Translation). New York: MacMillan, 1963.

B5 C. Berge and A. Ghouila-Houri, Programming, Games and Transportation Networks (Translation). New York: Wiley, 1965.

B6 L. D. Berkovitz, "A variational approach to differential games," in Advances in Game Theory, Annals of Math. Study, No. 52, Princeton, N. J.: Princeton University Press, 1964

B7 L. D. Berkovitz, "Necessary conditions for optimal strategies in a class of differential games and control problems," J. SIAM on Control, vol. 5, pp. 1-24, 1967.

B8 L. D. Berkovitz and M. Dresher, "A multimove infinite game with linear payoff," Pacific J. of Math., vol. 10, pp. 743-765, 1960.

B9 L. D. Berkovitz and W. H. Fleming, "On differential games with integral payoff," in Contributions to the Theory of Games III, Annals of Math. Study, No. 39, Princeton, N. J.: Princeton University Press, 1957.

B10 D. Blackwell, "An analog of the minimax theorem for vector payoffs," Pacific J. of Math., vol. 6, pp. 1-8, 1956.

B11 D. Blackwell, "On the functional equation of dynamic programming," J. Math. Anals. Appl., vol. 2, pp. 273-276, 1961.

B12 D. Blackwell, "Discrete dynamic programming," Ann. Math. Statist., vol. 33, pp. 719-726, 1962.

B13 D. Blackwell, "Discounted dynamic programming," Ann. Math. Statist., vol. 36, pp. 226-235, 1965.

C1 E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations. New York: McGraw-Hill, 1955.

D1 S. E. Dreyfus, Dynamic Programming and the Calculus of Variations. New York: Academic Press, 1965.

D2 S. E. Dreyfus, "A geometric derivation of transversality and corner conditions of the calculus of variations via dynamic programming," RAND Corporation Research Memorandum RM-5093-PR, 1966.

D3 S. E. Dreyfus, "On the dynamic programming treatment of discrete-time variational problems," RAND Corporation Research Memorandum RM-5159-PR, 1966.

E1 J. H. Eaton, "Improper solutions under existence assumptions: an example," IEEE Trans. on Automatic Control, vol. AC-10, pp. 186-189, 1965.

E2 J. H. Eaton and L. A. Zadeh, "Optimal pursuit strategies in discrete-state probabilistic systems," J. Basic Engineering Ser. D, vol. 84, pp. 23-29, 1962.

F1 K. Fan, "Minimax theorems," Proc. N. A. S., vol. 39, pp. 42-47, 1953.

F2 W. H. Fleming, "A note on differential games of prescribed duration," in Contributions to the Theory of Games III.

F3 W. H. Fleming, "The convergence problem for differential games," J. Math. Anal. Appl., vol. 3, pp. 102-116, 1961.

F4 W. H. Fleming,"The convergence problem for differential games, II," in Advances in Game Theory.

F5 W. H. Fleming, "The Cauchy problem for degenerate parabolic equations," J. Math. and Mech., vol. 13, pp. 987-1008, 1964.

G1 M. Yu. Gadzhiev, "Application of the theory of games to some problems of automatic control I," Automation and Remote Control, vol. 23, pp. 957-971, 1963.

G2 M. Yu. Gadzhiev, "Application of the theory of games to some problems of automatic control II," Automation and Remote Control, vol. 23, pp. 1074-1083, 1963.

G3 D. Gale, The Theory of Linear Economic Models. New York: McGraw-Hill, 1960.

G4 V. B. Gindes, "Optimal control in a linear system under conflict," J. SIAM on Control, vol. 5, pp. 163-169, 1967.

G5 V. P. Grishin, "A minimax problem in the theory of analytical design of control systems," Automation and Remote Control, vol. 25, pp. 779-789, 1964.

H1 H. Halkin, "Mathematical foundation of system optimization," in Topics in Optimization, G. Leitmann, Ed. New York: Academic Press, 1967.

H2 Y. C. Ho and S. Baron, "A minimal time intercept problem," IEEE Trans. on Automatic Control, vol. AC-10, p. 200, 1965.

H3 Y. C. Ho, A. E. Bryson, and S. Baron, "Differential games and optimal pursuit-evasion strategies," IEEE Trans. on Automatic Control, vol. AC-10, pp. 385-389, 1965.

I1 R. Isaacs, Differential Games. New York: Wiley, 1965.

J1 J. P. Jacob, "Games, pursuit-evasion and rendezvous problems," Ph.D. dissertation, Univ. of California, Berkeley, 1966.

K1 S. Kakutani, "A generalization of Brouwer's fixed point theorem," Duke Math. Journal, vol. 8, pp. 457-459, 1941.

K2 R. E. Kalman, Y. C. Ho and K. S. Narendra, "Controllability of linear dynamical systems," in Contributions to Differential Equations, vol. 1, pp. 189-213, 1963.

K3 S. Karlin, Mathematical Methods and Theory in Games, Programming, and Economics, Vol. I and Vol. II. Mass.: Addison-Wesley, 1959.

K4 D. L. Kelendzeridze, "Theory of an optimal pursuit strategy," Soviet Mathmatics, vol. 2, pp. 654-656, 1961.

K5 D. L. Kelendzeridze, "On a problem of optimum tracking," Automation and Remote Control, vol. 23, pp. 942-947, 1963.

K6 J. L. Kelley, General Topology. Princeton, N. J.: D. Van Nostrand, 1955.

K7 Yu. A. Kochetkov, "Pontryagin maximum principle applied to investigation of minimax problems in control processes," Engineering Cybernetics, pp. 9-18, 1965.

K8 N. N. Krasovskiy, Yu. M. Repin and V. Ye. Tret'yakov, "Theory of games in control systems," Engineering Cybernetics, pp. 1-11, 1965.

K9 E. Kreindler, "Contributions to the theory of time-optimal control," J. of the Franklin Institute, vol. 275, pp. 314-344, 1963.

L1 J. L. LeMay, "Recoverable and reachable zones for control systems with linear plants and bounded controller outputs," IEEE Trans. on Automatic Control, vol. AC-9, pp. 346-354, 1964.

N1 L. W. Neustadt, "The existence of optimal controls in the absence of convexity conditions," J. Math. Anal. Appl., vol. 7, pp. 110-117, 1963.

N2 L. W. Neustadt, "Optimization, a moment problem, and non-linear programming," J. SIAM on Control, vol. 2, pp. 33-53, 1964.

P1 L. S. Pontryagin, "On some differential games," Soviet Mathematics, vol. 5, pp. 712-716, 1964.

P2 L. S. Pontryagin, "On some differential games," J. SIAM Control, vol. 3, pp. 49-52, 1965.

R1 E. Roxin, "The existence of optimal controls," Mich. Math. J., vol. 9, pp. 109-119, 1962.

R2 E. Roxin, "A geometric interpretation of Pontryagin's maximum principle," in Nonlinear Differential Equations and Nonlinear Mechanics, J. P. LaSalle and S. Lefshetz, Ed. New York: Academic Press, 1963.

R3 E. Roxin, "Stability in general control systems," J. Differential Equations, vol. 1, pp. 115-150, 1965.

S1 G. Sansome and R. Conti, Nonlinear Differential Equations (Translation). New York: MacMillan, 1964.

S2 H. E. Scarf, "On differential games with survival payoff," in Contributions to the Theory of Games III.

S3 E. N. Simakova, "Differential games," Automation and Remote Control, vol. 27, pp. 1980-1996, 1966.

V1 P. P. Varaiya, "On the trajectories of a differential system," Univ. of California ERL Memorandum, ERL-M178, 1966.

V2 P. P. Varaiya, "On the existence of solutions to a differential game," J. SIAM on Control, vol. 5, pp. 153-162, 1967.

W1 J. Warga, "Minimax problems and unilateral curves in the calculus of variations," J. SIAM on Control, vol. 3, pp. 91-105, 1965.

W2 E. Wong and M. Zakai, "On convergence of the solutions of differential equations involving Brownian motion," Univ. of California ERL Report, No. 65-5, 1965.

Z1 L. A. Zadeh, "Optimality and non-scalar-valued performance criteria," IEEE Trans. on Automatic Control, vol. AC-8, pp. 59-60, 1963.

Z2 L. A. Zadeh and C. A. Desoer, Linear System Theory. New York: McGraw-Hill, 1963.