ON SECOND-ORDER NECESSARY CONDITIONS OF OPTIMALITY

by

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I. INTRODUCTION

In the past few years, it has been shown [1,2] that most of the problems of nonlinear programming, the calculus of variations and optimal control can be treated in a unified manner by transcribing these problems into a simple canonical form. Necessary conditions of optimality for this canonical form may then be obtained, and related to the original problems through the structure of each particular problem.

For finite dimensional problems, this canonical form is given as follows.

(1) Basic Problem: Let $f : E^n \to E^1$, $r \to E^n \to E^m$ be continuously differentiable functions, and let Ω be a subset of E^n. Find a vector $\hat{x}$ in E^n such that (i) $\hat{x} \in \Omega$, $r(\hat{x}) = 0$, and (ii) for every x in Ω with $r(x) = 0$, $f(\hat{x}) \leq f(x)$.

Following the convention of nonlinear programming, an $\hat{x}$ satisfying (i) will be called feasible, while an $\hat{x}$ satisfying both (i) and (ii) will be called an optimal solution to the Basic Problem (1).

A similar problem, common in mathematical programming, is perhaps better known.

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(2) <u>Nonlinear Programming Problem</u>: Let $f : E^n \to E^1$, $r : E^n \to E^m$ and $g : E^n \to E^k$ be given functions. Find $\hat{x}$ such that $r(\hat{x}) = 0$, $g(\hat{x}) \leq 0$ and $f(\hat{x}) = \min \{f(x) | r(x) = 0, g(x) \leq 0\}$.

This problem may be put in Basic Problem form by identifying Ω as $\{x | g(x) \leq 0\}$.

As a more interesting example, consider the following discrete optimal control problem: Let $f_i^0 : E^\ell \to E^1$, $f_i : E^n \times E^\ell \to E^n$ $i = 0, 1, \ldots, k-1$, $g : E^n \to E^m$ be given functions and U_i a given set in E^ℓ for $i = 0, 1, \cdots, k-1$. Find a control sequence $(u_0, \cdots, u_{k-1})$ which minimizes $\sum_{i=0}^{k-1} f_i^0(u_i)$ subject to:

(i) $y_{k+1} - y = f_i(y_i, u_i) \qquad i = 0, 1, \cdots, k-1$

(ii) $y_0 = \hat{y}_0$, $g(y_k) = 0$, $u_i \in U_i \qquad i = 0, 1, \cdots, k-1$

To see that this problem may be case in Basic Problem form, let x in $E^{k\ell}$ be given by $x = (u_0, \cdots, u_{k-1})$, $f(x) = \sum_{i=0}^{k-1} f_i^0(u_i)$, $r(x) = g(y_k(x)$, where y_k is given by solving (i) with $y_0 = \hat{y}_0$, and, finally, $\Omega = U_0 \times U_1 \times \cdots \times U_{k-1}$.

The demonstrated generality of the Basic Problem (1) makes it a convenient vehicle for the introduction of second-order conditions of optimality. By a second-order condition of optimality, we mean a condition which augments or replaces the usual first-order conditions, and generally, involves a second derivative of one or more of the cost or constraint functions.

First- and second-order conditions are not independent in that a second-order condition, usually, is only meaningful when a first-order condition is degenerate in some way. To clarify these ideas, we shall state a fundamental first-order necessary condition for the Basic Problem (1). This requires a tractable local representation of the set Ω.

(3) Definition [1]. A convex cone $C(\hat{x},\Omega)$ will be called a conical approximation to the constraint set Ω at the point $\hat{x}$ if for any collection $\{\delta x_1, \cdots, \delta x_k\}$ of linearly independent vectors in $C(\hat{x},\Omega)$ there exists an $\epsilon > 0$ (possibly depending on $\hat{x}$, $\delta x_1, \cdots, \delta x_k$) and a continuous map $\zeta(\cdot)$ from the convex hull of $\{0, \delta x_1, \cdots, \delta x_k\}$ into $\Omega - \hat{x}$, of the form:

(4) $$\zeta(\delta x) = \epsilon\delta x + o(\epsilon\delta x)$$

where $\|o(\delta x)\|/\|\delta x\| \to 0$ as $\|\delta x\| \to 0$.

If the map $\zeta(\cdot)$ is given by $\zeta(\delta x) = \epsilon\delta x$, then $C(\hat{x},\Omega)$ will be called a simple conical approximation.

(5) Fundamental Theorem [1]. If $\hat{x}$ is an optimal solution to the Basic Problem (1), and $C(\hat{x},\Omega)$ is a conical approximation to Ω at $\hat{x}$, then there is a nonzero vector $\psi = (\psi^0, \cdots, \psi^m)$ in E^{m+1}, with $\psi^0 \leq 0$, such that:

(6) $$\langle \psi^0 \nabla f(\hat{x}) + \sum_{i=1}^{m} \psi^i \nabla r^i(\hat{x}), \delta x \rangle \leq 0$$

for all δx in $\overline{C(\hat{x},\Omega)}$, the closure in E^n of $C(\hat{x},\Omega)$.

There are several circumstances under which a condition like (6) may be considered to be degenerate. The first is when the multiplier

ψ^0 must be chosen to be zero, and hence no information about the cost function $f(\cdot)$ enters into the necessary condition (6). This occurs most often when there is only one x in Ω satisfying $r(x) = 0$ and may be avoided by introducing a regularity condition, usually called a constraint qualification [3], on $r(\cdot)$ and Ω. We shall not be concerned with this case.

The second instance for which (6) may be degenerate is when the vectors $\nabla f(\hat{x})$, $\nabla r^1(\hat{x})$, $\cdot\cdot\cdot$, $\nabla r^m(\hat{x})$ are linearly dependent, since one can then always choose a $\psi \neq 0$ which satisfies $\psi^0 \nabla f(\hat{x}) + \sum_{i=1}^{m} \psi^i \nabla r^i(\hat{x}) = 0$, and hence (6), without reference to the optimality of $\hat{x}$. This type of degeneracy does not usually lead to a second-order condition unless it arises from the fact that one or more of the gradients $\nabla f(\hat{x})$, $\nabla r^1(\hat{x})$, $\cdot\cdot\cdot$, $\nabla r^m(\hat{x})$ are the zero vector. We shall call this the <u>zero gradient case</u>.

There is another situation when a second-order condition is meaningful, even though the condition (6) may not be degenerate in the above senses. This occurs when for every δx contained in $C(\hat{x},\Omega)$, we have $\langle \nabla f(\hat{x}), \delta x \rangle = 0$ and $\langle \nabla r^i(\hat{x}), \delta x \rangle = 0$ for $i = 1, \cdot\cdot\cdot, m$. Thus, it is possible to satisfy (6) irrespective of the choice of the vector ψ. Such vectors δx are, in a sense, <u>critical</u> (see (25)) and second-order conditions for this case correspond to examining second-order effects along curves tangent to δx at $\hat{x}$. Of course, any combination of the above three effects may occur simultaneously.

In Section II of this paper, we survey briefly some of the known second-order conditions, corresponding to zero gradient and critical direction type of degeneracies.

The major contributions of this paper are given in Section III--theorem (26) for the critical directions case and theorem (27) for the zero gradient case. Both of these theorems are expressed in terms of local approximations to the set Ω, since, in well-formulated optimization problems, Ω has an interior, which ensures the existence of such approximations. Several ways by which such approximations may be constructed are also given. It is also shown that most second-order necessary conditions are special cases of theorem (26) or theorem (27).

In Section IV, proofs for theorems (26) and (27) are given. These proofs display the fact that some of the techniques useful in first-order theory, in particular the use of fixed-point theorems, can be applied to second-order theory.

II. A BRIEF SURVEY OF SOME SECOND-ORDER CONDITIONS

Since our interest is in the Basic Problem (1), or the closely related Nonlinear Programming Problem (2), we shall not cover any special results from the calculus of variations [4,5] or optimal control. [6,7]. Nor shall we be concerned with sufficiency conditions either, because in many cases the required strengthening of the necessary conditions may be obvious, or because our local approximation to the set Ω may not be sufficiently rich to describe completely the nature of Ω in the vicinity of $\hat{x}$.

In the following, it is understood that any derivatives used are assumed to exist, with $\langle \nabla f(\hat{x}), \delta x\rangle$ or $\frac{\partial f}{\partial x}(\hat{x})(\delta x)$ representing the first differential of $f(\cdot)$ at $\hat{x}$, $\frac{1}{2}\langle \delta x, \frac{\partial^2 f}{\partial x^2}(\hat{x})\delta x\rangle$ or $\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\hat{x})(\delta x)$ representing the second differential of $f(\cdot)$ at x, etc. Vectors in E^{m+1} will be understood to have components numbered from 0, 1, $\cdot\cdot\cdot$, m.

Perhaps the simplest second-order condition arises when the gradient of the cost $f(\cdot)$ at the optimal solution is zero and we do not choose to isolate the equality constraints for special attention.

(7) <u>Definition</u> [6]. Let Ω be an arbitrary set. The <u>sequential tangent cone</u> $STC(\hat{x},\Omega)$ of Ω at $\hat{x} \in \Omega$ is the set of all δx such that there is a sequence $\{x_i\}_{i=1}^{\infty}$ in Ω, and a sequence $\{d_i\}_{i=1}^{\infty}$ of positive scalars, such that

(i) $x_i \to \hat{x}$,

(ii) $(x_i - \hat{x})/d_i \to \delta x$.

(8) <u>Theorem</u> [6]. If $\hat{x}$ is an optimal solution to the Basic Problem (1) and $\nabla f(\hat{x}) = 0$, then $\langle \delta x, \frac{\partial^2 f}{\partial x^2}(\hat{x})\delta x\rangle \geq 0$ for all δx in $STC(\hat{x},\Omega')$, where $\Omega' = \{x | r(x) = 0, x \in \Omega\}$.

This is seen to be a simple but general result for one of the degenerate cases mentioned, however its application depends on our having a characterization for $STC(\hat{x},\Omega')$. In some cases, we may represent $STC(\hat{x},\Omega')$ as the intersection of $STC(\hat{x},\Omega)$ and $STC(\hat{x},\{x|r(x) = 0\})$, and this facilitates matters. However, this is not true in general.

For the critical vector case, we are able to obtain second-order conditions without requiring the gradient of the cost to be zero.

(9) Theorem: Let $\hat{x}$ be an optimal solution to the Basic Problem (1), and let $x : E^1 \to E^n$ be any twice continuously differentiable function such that $x(0) = \hat{x}$ and $x(\theta)$ is feasible for all $\theta \in [0,\bar{\theta}]$, with $\bar{\theta} > 0$. If $df(x(0))/d\theta = 0$, then $d^2f(x(0))/d\theta^2 \geq 0$.

In general, without making additional assumptions about $r(\cdot)$ and Ω, the conditions of theorems (8) and (9) cannot be decomposed into more structured forms.

One approach to a more structured condition is that followed by Dubovickii and Miljutin [8,9]. Essentially, for a fixed $\delta\tilde{x}$ satisfying $\langle \nabla f(\hat{x}), \delta\tilde{x} \rangle = 0$ and $\langle \nabla r^i(\hat{x}), \delta\tilde{x} \rangle = 0$ for $i = 1, \cdots, m$ (i.e., $\delta\tilde{x}$ is critical) they consider the following sets.

(10) $C_0(\delta\tilde{x}) = \{\delta x \mid$ there exists an $\epsilon_0 > 0$ and a function $o:[0,\epsilon_0] \to E^n$, with $\lim_{\epsilon \to 0} \frac{\|o(\epsilon)\|}{\epsilon} = 0$, such that $r(\hat{x} + \epsilon\delta\tilde{x} + o(\epsilon^2)) = 0$ for all $\epsilon \in [0,\epsilon_0]\}$.

(11) $C_1(\delta\tilde{x}) = \{\delta x \mid$ there exists an $\epsilon_0' > 0$ such that

$$\hat{x} + \epsilon\delta\tilde{x} + \epsilon^2\delta x \in \Omega \text{ for all } \epsilon \in [0,\epsilon_0']\} .$$

(12) $C_2(\delta\tilde{x}) = \{\delta x \mid$ there exists an $\epsilon_0'' > 0$ such that

$$f(\hat{x} + \epsilon\delta\tilde{x} + \epsilon^2\delta x) < f(\hat{x}) \text{ for all } \epsilon \in (0,\epsilon_0'']\}.$$

(13) Theorem [9]: If $\hat{x}$ is an optimal solution to the Basic Problem (1), then $C_0(\delta\tilde{x}) \cap \overset{\circ}{C}_1(\delta\tilde{x}) \cap \overset{\circ}{C}_2(\delta\tilde{x}) = \phi$.†

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†The interior of a set C is denoted by $\overset{\circ}{C}$.

Whenever $C_0(\delta\tilde{x})$ is a linear manifold, and $C_1(\delta\tilde{x})$, $C_2(\delta\tilde{x})$ are convex cones (possibly translated) with nonempty interiors, the condition $C_0(\delta\tilde{x}) \cap \overset{\circ}{C}_1(\delta\tilde{x}) \cap \overset{\circ}{C}_2(\delta\tilde{x}) = \Phi$ guarantees the existence of affine functionals, $c_0(\cdot), c_1(\cdot), c_2(\cdot)$, not all zero, with $c_0(\cdot)$ vanishing on $C_0(\delta\tilde{x})$ and $c_i(\cdot)$ nonnegative on $C_i(\delta\tilde{x})$ for $i = 1, 2$, such that $c_0(x) + c_1(x) + c_2(x) = 0$ for all x in E^n [8]. When specialized to the Nonlinear Programming Problem (2) <u>with rather restrictive assumptions</u>, this gives a result similar to theorem (15) below.

Finally, McCormick [10] has observed that in some cases the first-order necessary conditions for the Nonlinear Programming Problem (2) display a multiplier vector which can also be used in a second-order condition.

(14) <u>Definition</u> [10]: Consider the Nonlinear Programming Problem (2). The <u>second-order constraint qualification</u> is said to be satisfied at $\hat{x}$ if for each δx such that $\langle \nabla r^i(\hat{x}), \delta x \rangle = 0$ for $i = 1, \cdots, m$, and $\langle \nabla g^i(\hat{x}), \delta x \rangle = 0$ for $i \in I(\hat{x}) \triangleq \{i \mid g^i(\hat{x}) = 0\}$, there is a twice continuously differentiable function $x: E^1 \to E^n$ and a $\bar{\theta} > 0$ such that:

(i) $x(0) = \hat{x}$, $\quad dx(0)/d\theta = \delta x$;

(ii) for all $\theta \in [0, \bar{\theta}]$, $x(\theta)$ is feasible, and moreover, $g^i(x(\theta)) = 0$ for $i \in I(\hat{x})$.

(15) <u>Theorem</u> [10]: If $\hat{x}$ is an optimal solution to the Nonlinear Programming Problem (2), and the first-order† [10] and second-order

†The first-order constraint qualification is a statement of the Kuhn-Tucker constraint qualification [11] for a constraint set defined by both equalities and inequalities.

constraint qualifications are satisfied, then there exists multipliers $\psi^1, \cdots, \psi^m$ and $u^1, \cdots, u^k$ with $u^i \leq$ for $i = 1, \cdots, k$ such that:

(i) $-\nabla f(\hat{x}) + \sum_{i=1}^{m} \psi^i \nabla r^i(\hat{x}) + \sum_{i=1}^{k} u^i \nabla g^i(\hat{x}) = 0$;

(ii) $\langle u, g(\hat{x}) \rangle = 0$,

and

(iii) for every $\tilde{\delta x}$ such that $\langle \nabla r^i(\hat{x}), \tilde{\delta x} \rangle = 0$ for $i = 1, 2, \cdots, m$, and $\langle \nabla q^i(\hat{x}, \tilde{\delta x} \rangle = 0$ for $i \in I(\hat{x})$;

$$\langle \tilde{\delta x} \left(- \frac{\partial^2 f}{\partial x^2} (\hat{x}) + \sum_{i=1}^{m} \psi^i \frac{\partial^2 r^i}{\partial x^2} (\hat{x}) + \sum_{i=1}^{k} u^i \frac{\partial^2 g^i}{\partial x^2} (\hat{x}) \right) \tilde{\delta x} \rangle \leq 0 .$$

Conditions (i) and (ii) above represent the first-order necessary conditions for the Nonlinear Programming Problem (2), with the first-order constraint qualification ensuring a nonzero cost multiplier, -1, in (i). Choosing $\tilde{\delta x}$ such that $\langle \nabla r^i(\hat{x}), \tilde{\delta x} \rangle = 0$ for $i = 1, \cdots, m$ and $\langle \nabla q^i(\hat{x}), \tilde{\delta x} \rangle = 0$ for $i \in I(\hat{x})$, we see that (i) and (ii) imply $\langle \nabla f(\hat{x}), \tilde{\delta x} \rangle = 0$, i.e., $\tilde{\delta x}$ is critical. The second-order constraint qualification then leads to the third condition.

It is also clear, however, that $\langle \nabla f(\hat{x}), \tilde{\delta x} \rangle = 0$ also follows from the optimality of $\hat{x}$ and (14), since if $\langle \nabla f(\hat{x}), \tilde{\delta x} \rangle \neq 0$, then either $\langle \nabla f(\hat{x}), \tilde{\delta x}, \rangle < 0$ or $\langle \nabla f(\hat{x}), -\tilde{\delta x} \rangle < 0$, and (14) leads to a contradiction of optimality. Thus, it is apparent that the first-order constraint qualification may be removed to obtain a slightly weaker theorem. In addition, one would expect to have a condition in terms of curves $x(\theta)$ that are feasible for $\theta \in [0, \bar{\theta}]$, rather than in terms of feasible curves which satisfy the rather demanding condition: $g^i(x(\theta)) = 0$ for $i \in I(\hat{x})$.

Our task in the next section will be to obtain optimality conditions without explicit assumptions relating $r(\cdot)$ and Ω i.e., without constraint qualifications.

III. SECOND-ORDER NECESSARY CONDITIONS

We have seen that the Fundamental Theorem (5), which gives first-order necessary conditions of optimality, relies on a local approximation to the set Ω. While this approximation can also be used for some second-order conditions, it is convenient to introduce a new local approximation.

(16) <u>Definition</u>: A pair $\{C(\hat{x},\delta\tilde{x},\Omega),\ \delta x^*\}$ will be called a <u>$\delta\tilde{x}$-directed conical approximation</u> to Ω at $\hat{x}$ if $C(\hat{x},\delta\tilde{x},\Omega)$ is a convex cone, and for any collection $\{\delta x_1, \cdot \cdot \cdot, \delta x_k\}$ of vectors in $C(\hat{x},\delta\tilde{x},\Omega)$, any $k - 1$ of which are linearly independent, there is an $\epsilon_0 > 0$ and a continuous map $\zeta_{\delta\tilde{x}}(\cdot,\cdot)$, (possibly depending on $\hat{x}$, $\delta\tilde{x}$, δx^*, $\delta x_1, \cdot \cdot \cdot, \delta x_k$) from $[0,\epsilon_0] \times co\{\delta x_1, \cdot \cdot \cdot, \delta x_k\}$ into $\Omega - \hat{x}$, of the form:

(17) $$\zeta_{\delta\tilde{x}}(\epsilon,\delta x) = \epsilon\delta\tilde{x} + \frac{\epsilon^2}{2}(\delta x^* + \delta x) + o(\epsilon^2(\delta x^* + \delta x)), \text{ where}$$

$$\|o(\delta x')\|/\|\delta x'\| \to 0 \text{ as } \|\delta x'\| \to 0.$$

We shall refer to $\{C(\hat{x}, \delta\tilde{x}, \Omega),\ \delta x^*\}$ simply as a directed conical approximation when $\delta\tilde{x}$ is clear from the context. The special cases which arise when $C(\hat{x}, \delta\tilde{x}, \Omega) = \{0\}$, or $\delta x^* = 0$, or even $\delta\tilde{x} = 0$, (or any combination of these) are not excluded from consideration.

There may, of course, be many directed conical approximations for a single $\delta\tilde{x}$, as well as useful relations between the conical approximation defined in (3), and the directed conical approximations defined above. Thus, if $\{C(\hat{x}, \delta\tilde{x}, \Omega), \delta x^*\}$ is a directed conical approximation, the ray $\{\delta x | \delta x = \lambda\delta\tilde{x}, \lambda \geq 0\}$ may be regarded as a trivial conical approximation with map $\zeta(\delta x) = \zeta(\lambda\delta\tilde{x}) = \epsilon\lambda\delta\tilde{x} + \frac{\epsilon^2}{2}\lambda^2(\delta x^* + \delta\bar{x}) + o(\epsilon^2\lambda^2(\delta x^* + \delta\bar{x}))$, where $\delta\bar{x}$ is any vector in $C(\hat{x}, \delta\tilde{x}, \Omega)$ and $o(\cdot)$ is given by (17). Conversely, we may often obtain directed conical approximations from conical approximations, the most important case being the following one.

(18) Lemma: If $C(\hat{x},\Omega)$ is a simple conical approximation to Ω at $\hat{x}$ and $\delta\tilde{x}$ is any vector in $C(\hat{x},\Omega)$, then $\{RC(\delta\tilde{x},C(\hat{x},\Omega)),0\}$ is a $\delta\tilde{x}$-directed conical approximation, (where for any set S and $x \in S$, we define $RC(x,S) = \{\delta x |$ there exists a $\bar{\lambda} > 0$ such that $x + \lambda\delta x \in S$ for $0 \leq \lambda \leq \bar{\lambda}\}$).

Note that if $\delta\tilde{x} \in C(\hat{x},\Omega)$, then $C(\hat{x},\Omega) \subset RC(\delta\tilde{x},C(\hat{x},\Omega))$, with strict inclusion whenever $-\delta\tilde{x} \notin C(\hat{x},\Omega)$.

We digress to indicate several important cases for which simple conical approximations may be constructed. (For $\Omega = \{x|g^i(x) \leq 0, i = 1,\cdot\cdot\cdot, k\}$ and $x \in \Omega$, the index set $I(x) \triangleq \{i|i \in \{1,\cdot\cdot\cdot, k\}$ and $g^i(x) = 0\}$.)

(19) Lemma: Suppose $\Omega = \{x|g^i(x) \leq 0$ for $i = 1, \cdot\cdot\cdot, k\}$ and $x \in \Omega$. If the set

(20) $IC(\hat{x},\Omega) \triangleq \{\delta x | \langle\nabla g^i(\hat{x}),\delta x\rangle < 0, \ i \in I(\hat{x})\}$

is not empty, then it is a simple conical approximation to Ω at $\hat{x}$.

(21) Lemma: If $\hat{x}$ is contained in Ω and Ω^* is any set containing $\hat{x}$ such that $\Omega \cap \Omega^*$ is convex, then $RC(\hat{x}, \Omega \cap \Omega^*)$ is a simple conical approximation to Ω at $\hat{x}$.

(22) Lemma: If $C(\hat{x},\Omega)$ is a conical approximation with nonempty interior $\overset{\circ}{C}(\hat{x},\Omega)$, then $\overset{\circ}{C}(\hat{x},\Omega)$ is a simple conical approximation to Ω at $\hat{x}$.

Whenever Ω has the description given in lemma (19), it is consistent with our idea of a well-formulated problem that Ω will have an interior, and hence $IC(\hat{x},\Omega)$ (20) will be nonempty. Now, let $\delta\tilde{x}$ be an arbitrary vector contained in $\overline{IC(\hat{x},\Omega)}$, and let the index set $\tilde{I}(\hat{x},\delta\tilde{x})$ be defined by:

$$\tilde{I}(\hat{x},\delta\tilde{x}) = \{i \in I(\hat{x}) \,|\, \langle\nabla g^i(\hat{x}),\delta\tilde{x}\rangle = 0\} . \tag{23}$$

(24) Lemma: Suppose $\Omega = \{x|g^i(x) \leq 0,\ i = 1,\cdot\cdot\cdot, k\}$ and $\hat{x}$ belongs to Ω. If $IC(\hat{x},\Omega) \neq \Phi$, then for any $\delta\tilde{x}$ contained in $\overline{IC(\hat{x},\Omega)}$, there exists a δx^* in E^n such that $\langle\delta\tilde{x}, \frac{\partial^2 g^i}{\partial x^2}(\hat{x})\delta\tilde{x}\rangle + \langle\nabla g^i(\hat{x}),\delta x^*\rangle \leq 0$ for all $i \in \tilde{I}(\hat{x},\delta\tilde{x})$, and the pair $\{\{\delta x|\langle\nabla g^i(\hat{x}),\delta x\rangle < 0,\ i \in \tilde{I}(\hat{x},\delta\tilde{x})\},\delta x^*\}$ is a $\delta\tilde{x}$-directed conical approximation. (If $\tilde{I}(\hat{x},\delta\tilde{x}) = \Phi$, then the pair $\{E^n,\delta x^*\}$ is a $\delta\tilde{x}$-directed conical approximation for any δx^* in E^n.)

To illustrate the usefulness of lemma (24), and to see that there are situations when δx^* must be nonzero if one wishes to obtain a directed conical approximation, let $x = (y,z)$, $\Omega = \{x|g(x) \triangleq \frac{1}{2}\{(y - 1)^2 + z^2 - 1\} < 0\}$, and let $\hat{x} = (0,0)$. With $\delta\tilde{x} = (0,1)$ there is no cone C such that $\{C,(0, 0)\}$ is a $\delta\tilde{x}$-directed conical approximation, however $\{IC(\hat{x},\Omega),(1,0)\}$ is a $(0,1)$-directed conical approximation.

We now isolate those vectors $\delta\tilde{x}$ for which, in the context of the Basic Problem (1), a $\delta\tilde{x}$-directed conical approximation will lead to a very general second-order necessary condition of optimality.

(25) Definition: A vector $\delta\tilde{x}$ is said to be a critical direction for the Basic Problem (1) if $\langle \nabla f(\hat{x}), \delta\tilde{x}\rangle \leq 0$ and $\langle \nabla r^i(\hat{x}), \delta\tilde{x}\rangle = 0$ for $i = 1, \cdots, m$.

(26) Theorem: Suppose $\hat{x}$ is an optimal solution to the Basic Problem (1) and $\delta\tilde{x}$ is a critical direction. If $\{C(\hat{x},\delta\tilde{x},\Omega),\delta x^*\}$ is a $\delta\tilde{x}$-directed conical approximation, then there exists a nonzero vector $\psi = (\psi^0, \psi^1, \cdots, \psi^m)$ in E^{m+1} with $\psi^0 \leq 0$ such that:

$$\text{(i)} \quad \langle \psi^0 \nabla f(\hat{x}) + \sum_{i=1}^{m} \psi^i \nabla r^i(\hat{x}), \delta x\rangle \leq 0 \text{ for all } \delta x \text{ in } \overline{C(\hat{x},\delta\tilde{x},\Omega)}$$

$$\text{(ii)} \quad \psi^0(\langle \delta\tilde{x}, \frac{\partial^2 f(\hat{x})}{\partial x^2}\delta\tilde{x}\rangle + \langle \nabla f(\hat{x}), \delta x^*\rangle)$$

$$+ \sum_{i=1}^{m} \psi^i(\langle \delta\tilde{x}, \frac{\partial^2 r^i}{\partial x^2}(\hat{x})\delta\tilde{x}\rangle + \langle \nabla r^i(\hat{x}), \delta x^*\rangle) \leq 0$$

and $\psi^0 = 0$ if $\langle \nabla f(\hat{x}), \delta\tilde{x}\rangle < 0$.

Remark: Note that inequality (6) of theorem (5) may also be obtained from theorem (26). In fact, if $C(\hat{x},\Omega)$ is a conical approximation with map $\zeta(\delta x) = \epsilon\delta x + o(\epsilon\delta x)$, then, setting $\delta\tilde{x} = 0$, $C(\hat{x},0,\Omega) = C(\hat{x},\Omega)$, $\delta x^* = 0$, and $\zeta_0(\epsilon,\delta x) = \frac{\epsilon^2}{2}\delta x + o(\frac{\epsilon^2}{2}\delta x)$, we find that part (i) of theorem (26) yields the same result as theorem (5), but part (ii) carries no information. However, the inequality (26(i)) will often hold for cones

$C(\hat{x}, \delta\tilde{x}, \Omega)$ which are much larger than any conical approximation to Ω at $\hat{x}$.

It is appropriate at this stage to comment on the crucial differences between theorem (13) by Dubovickii and Miljutin [8,9] and theorem (26) of this section. Note that theorem (13) is essentially a disjointness condition in the domain space of the map $F(\cdot) = (f(\cdot), r^1(\cdot), \cdot \cdot \cdot, r^m(\cdot))$, (i.e., in E^n) while theorem (26) represents separation conditions in the range space of $F(\cdot)$ (i.e., in E^{m+1}), which requires simpler assumptions. Thus, to obtain from theorem (13) inequalities of the form (26(i)), (26(ii)) it is necessary to make fairly strong assumptions on each of the sets $C_0(\delta\tilde{x})$, $C_1(\delta\tilde{x})$ and $C_2(\delta\tilde{x})$, (see (10), (11), (12)). On the other hand, any time $C_1(\delta\tilde{x})$ is of the form $C_1(\delta\tilde{x}) = \delta x^* + C(\delta\tilde{x})$, where $C(\delta\tilde{x})$ is a convex cone, we find that $\{C(\delta\tilde{x}), \delta x^*\}$ is a $\delta\tilde{x}$-directed conical approximation to Ω at $\hat{x}$, and we obtain (26(i)), (26(ii)) immediately.

Before further discussion of theorem (26), we consider the zero gradient case. It is assumed that at most one gradient corresponding to an equality constraint is zero. We define the ray R in E^{m+1} by $R = \{y \mid y^0 \leq 0 \text{ and } y^i = 0 \text{ for } i = 1, \cdot \cdot \cdot, m\}$.

(27) <u>Theorem</u>: Suppose that $\hat{x}$ is an optimal solution to the Basic Problem (1) and that $C(\hat{x}, \Omega)$ is a conical approximation to Ω at $\hat{x}$ with

nonempty interior $\overset{\circ}{C}(\hat{x},\Omega)$.

(i) If $\nabla f(\hat{x}) = 0$, then the ray R has no points in the interior of the set

(28) $$L_0 = \{y \mid y^0 = \langle \delta x, \frac{\partial^2 f}{\partial x^2}(\hat{x})\delta x\rangle,\ y^i = \langle \nabla r^i(\hat{x}), \delta x\rangle,\ i = 1, \cdots, m,\ \delta x \in \overset{\circ}{C}(\hat{x},\Omega)\}.$$

(ii) If $\nabla r^1(\hat{x}) = 0$, then the ray R has no points in the interior of the set

(29) $$L_1 = \{y \mid y^0 = \langle \nabla f(\hat{x}), \delta x\rangle,\ y^1 = \langle \delta x, \frac{\partial^2 r^1}{\partial x^2}(\hat{x})\delta x\rangle,\ y^i = \langle \nabla r^i(\hat{x}), \delta x\rangle,\ i = 2, \cdots, m,\ \delta x \in \overset{\circ}{C}(\hat{x},\Omega)\}.$$

(iii) If $\nabla f(\hat{x}) = \nabla r^1(\hat{x}) = 0$, then the ray R has no points in the interior of the set

(30) $$L_2 = \{y \mid y^0 = \langle \delta x, \frac{\partial^2 f}{\partial x^2}(\hat{x})\delta x\rangle,\ y^1 = \langle \delta x, \frac{\partial^2 r^1}{\partial x^2}(\hat{x})\delta x\rangle,\ y^i = \langle \nabla r^i(\hat{x}), \delta x\rangle,\ i = 2, \cdots, m,\ \delta x \in \overset{\circ}{C}(\hat{x},\Omega)\}.$$

Remark: The above theorem remains true even when $\overset{\circ}{C}(\hat{x},\Omega)$ is replaced by the relative interior of $C(\hat{x},\Omega)$. Also, if only the case $\nabla f(\hat{x}) = 0$ is considered, it can be shown that the following is true.

(31) Theorem [12]: Suppose that $\hat{x}$ is an optimal solution to the Basic Problem (1) and that $C(\hat{x},\Omega)$ is a conical approximation to Ω at $\hat{x}$. If $\nabla f(\hat{x}) = 0$, then the ray R has no points in the interior of the set

(32) $L_0' = \{y | y^0 = \langle \delta x, \frac{\partial^2 f}{\partial x^2}(\hat{x})\delta x\rangle,\ y^i = \langle \nabla r^i(\hat{x}), \delta x\rangle$ for $i = 1, \cdots,$

$m,\ \delta x \in C(\hat{x},\Omega)\}$.

It can be shown that theorems similar to (31) cannot be obtained for all situations covered by theorem (27), i.e., $\overset{\circ}{C}(\hat{x},\Omega)$ cannot be replaced by $C(\hat{x},\Omega)$. In fact, consideration of example (33), with $\Omega = \{(y,z) | (y - 1)^2 + (z - 1)^2 - 2 \le 0\}$, $\hat{x} = (0,0)$, and $C(\hat{x},\Omega) = \{(y,z) | y + z \ge 0\}$, will confirm this.

Theorems (26) and (27) represent two different approaches to second-order conditions. Theorem (26) is well structured and neatly supplements the first-order conditions in theorem (5). Theorem (27), on the other hand, is in rather awkward form, since the sets L_0, L_1, L_2 are in general neither convex nor even conical. However, in spite of this, theorem (27) answers some questions which theorem (26) does not, and in some cases leads to alternate expressions. We now demonstrate this.

Examination of theorem (26) indicates that the multiplier vector ψ depends on the critical direction $\widetilde{\delta x}$. However, it is clear from lemma (18) that we may be able to find a pair $\{C, \delta x^*\}$ which is a directed conical approximation for more than one critical direction $\widetilde{\delta x}$. The natural question to ask, then, is whether there is a multiplier vector ψ which will satisfy the conditions of theorem (26) for all these critical vectors $\widetilde{\delta x}$ and the given pair $\{C, \delta x^*\}$. Unfortunately, as will be seen from the following example, this is not always true.

(33) Example: Let $x = (y,z)$ and consider the Basic Problem (1) with

$f(x) = -(y - z)^2$, $r(x) = y^2 - z^2$ and $\Omega = \{x | y \geq 0,\ z \geq 0\}$. Clearly, the point $\hat{x} = (0,0)$ is an optimal solution since the only feasible points are defined by the intersection of the line $y - z = 0$ and the positive quadrant. Since $\nabla f(\hat{x}) = \nabla r(\hat{x}) = 0$, each $\delta\tilde{x}$ in Ω is a critical direction and we may take $C(\hat{x}, \delta\tilde{x}, \Omega) = \Omega$, $\delta x^* = (0,0)$. Now, since each gradient is zero, if there is a single multiplier vector ψ which satisfies theorem (26) for all $\delta\tilde{x}$ in Ω, it must satisfy $\psi^0\langle \delta\tilde{x}, \frac{\partial^2 f}{\partial x^2}(\hat{x})\delta\tilde{x}\rangle + \psi^1\langle \delta\tilde{x}, \frac{\partial^2 r}{\partial x^2}(\hat{x})\delta\tilde{x}\rangle \leq 0$ for all $\delta\tilde{x} \in \Omega$, with $\psi^0 \leq 0$. This is equivalent to requiring that the cone $V = \{v = (v^0, v^1) | v^0 = \langle \delta x, \frac{\partial^2 f}{\partial x^2}(\hat{x})\delta x\rangle,\ v^1 = \langle \delta x, \frac{\partial^2 r}{\partial x^2}(\hat{x})\delta x\rangle$, with $\delta x \in \Omega\}$ be separated from the ray R. It is trivial to verify that V is the set $\{(0,0)\} \cup \{v | v^0 < 0,\ v^0 + v^1 \geq 0\} \cup \{v | v^0 < 0,\ v^0 - v^1 \geq 0\}$, which cannot be separated from the ray R.

We see that in the above example the set Ω also serves as a simple conical approximation to Ω at $(0,0)$. Since Ω has an interior, and both $\nabla f(\hat{x})$ and $\nabla r(\hat{x}) = 0$, theorem (27) can be applied to answer the question as to when there is a single multiplier vector satisfying $\psi^0\langle \delta x, \frac{\partial^2 f}{\partial x^2}(\hat{x})\delta x\rangle + \psi^1\langle \delta x, \frac{\partial^2 r}{\partial x^2}(\hat{x})\delta x\rangle \leq 0$ for all $\delta x \in \Omega$. In particular, it yields the following modification of theorem (26).

(34) <u>Theorem</u>: Suppose that the Basic Problem (1) has only one equality constraint, i. e., $r : E^n \to E^1$ If $\hat{x}$ is an optimal solution to the Basic Problem (1), with $\nabla f(\hat{x}) = \nabla r(\hat{x}) = 0$, and if $C(\hat{x}, \Omega)$ is a conical approximation to Ω at $\hat{x}$ such that the set L_2 (30) is convex, then there exist scalars ψ^0, ψ^1 not both zero with $\psi^0 \leq 0$ such that:

$$\psi^0\langle \delta x, \frac{\partial^2 f}{\partial x^2}(\hat{x})\delta x\rangle + \psi^1\langle \delta x, \frac{\partial^2 r}{\partial x^2}(\hat{x})\delta x\rangle \leq 0 \text{ for all } \delta x \text{ in } \overline{C(\hat{x},\Omega)} .$$

Proof: Since there is only a single equality constraint and both $\nabla f(\hat{x})$ and $\nabla r(\hat{x}) = 0$, the set L_2 is conical, and by assumption also convex. Suppose L_2 is not separated from R. It follows that R has points in the interior of the set L_2, which contradicts theorem (27). Thus, R and L_2 must be separated, which proves the theorem.

When $r(\cdot) \equiv 0$, we have a somewhat simpler situation and theorem (31) leads to the following result.

(35) Theorem: Suppose that in the Basic Problem (1), $r(\cdot) \equiv 0$, and $\hat{x}$ is an optimal solution with $\nabla f(\hat{x}) = 0$. If $\{C_\alpha(\hat{x},\Omega) \mid \alpha \in A\}$, where A is an index set, is any collection of conical approximations to Ω at $\hat{x}$, then

$$\langle \delta x, \frac{\partial^2 f}{\partial x^2}(\hat{x})\delta x\rangle \geq 0 \text{ for all } \delta x \text{ in } \overline{\bigcup_{\alpha \in A} C(\hat{x},\Omega)} .$$

Proof: Let α be arbitrary in A, and let $C_\alpha(\hat{x},\Omega)$ be the corresponding conical approximation. Applying theorem (31), we see that the set L_0' is one dimensional and hence the statement of the theorem is that

$$\langle \delta x, \frac{\partial^2 f}{\partial x^2}(\hat{x})\delta x\rangle \geq 0 \text{ for all } \delta x \text{ in } C_\alpha(\hat{x},\Omega) .$$

Thus,

$$\langle \delta x, \frac{\partial^2 f}{\partial x^2}(\hat{x})\delta x\rangle \geq 0 \text{ for all } \delta x \text{ in } \bigcup_{\alpha \in A} C_\alpha(\hat{x},\Omega)$$

and by continuity this is also true for the closure, which completes the proof.

For the case $r(\cdot) \equiv 0$, if we assume that there is a family of conical approximations, $\{C_\alpha(\hat{x},\Omega) \mid \alpha \in A\}$, such that $\overline{\bigcup_{\alpha\in A} C(\hat{x},\Omega)} = STC(\hat{x},\Omega)$, † then we have obtained theorem (8) of Section II.

When $r(\cdot) \equiv 0$, we can also obtain a corollary of theorem (26), similar to theorem (35).

(36) <u>Theorem</u>: Suppose that in the Basic Problem (1), $r(\cdot) \equiv 0$. If $\hat{x}$ is optimal and A is a set such that for each $\alpha \in A$ there is a critical direction $\delta\tilde{x}_\alpha$ and corresponding directed conical approximation $\{C(\hat{x},\delta\tilde{x}_\alpha,\Omega),\delta x^*_\alpha\}$, then

(i) $\langle \nabla f(\hat{x}),\delta x\rangle \geq 0$ for all δx in $\overline{\bigcup_{\alpha\in A} C(\hat{x},\delta\tilde{x}_\alpha,\Omega)}$

and

(ii) $\langle \delta\tilde{x}_\alpha, \frac{\partial^2 f}{\partial x^2}(\hat{x})\delta\tilde{x}_\alpha\rangle + \langle\nabla f(\hat{x}),\delta x^*_\alpha\rangle \geq 0$ for all $\alpha \in A$.

<u>Proof</u>: Let $\alpha \in A$ be arbitrary and let $\{C(\hat{x},\delta\tilde{x},\Omega),\delta x^*_\alpha\}$ be the corresponding $\delta\tilde{x}_\alpha$-directed conical approximation. From theorem (26), since $\psi = (\psi^0)$ is nonzero, and satisfies $\psi^0 \leq 0$, we may take $\psi^0 = -1$. Thus, $\langle\nabla f(\hat{x}),\delta x\rangle \geq 0$ for all $\delta x \in C_\alpha(\hat{x},\delta\tilde{x}_\alpha,\Omega)$ and $\langle \delta\tilde{x}_\alpha, \frac{\partial^2 f}{\partial x^2}(\hat{x})\delta\tilde{x}_\alpha\rangle + \langle\nabla f(\hat{x}),\delta x^*_\alpha\rangle \geq 0$. Since α was arbitrary, the theorem is true.

Theorem (9) of Section II is obtained as a special case of (36). Thus, suppose $x: E^1 \to E^n$ is a twice differentiable function such that $x(\theta)$ is contained in Ω' for θ in $[0,\bar{\theta}]$, $\bar{\theta} > 0$, that $x(0)$ is optimal, and that $\frac{df}{d\theta}(x(0)) = 0$. Taking the critical direction $\delta\tilde{x} = \frac{dx(0)}{d\theta}$ and the corresponding directed conical approximation $\{C(\hat{x},\delta\tilde{x},\Omega'),\delta x^*\} = \{\{0\},\frac{d^2x(0)}{d\theta^2}\}$, we obtain from the second term of (36) that $\left\langle \frac{dx(0)}{d\theta}, \frac{\partial^2 f}{\partial\theta^2}(\hat{x})\frac{dx(0)}{d\theta}\right\rangle + \left\langle \nabla f(\hat{x}), \frac{d^2x(0)}{d\theta^2}\right\rangle \geqq 0$, which

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† The sequential tangent cone is always closed.

corresponds to the condition $\frac{d^2 f}{d\theta^2}(x(0)) \geqq 0$ of theorem (9).

Theorem (26) will now be applied to the Nonlinear Programming Problem (2) to obtain a generalization of Theorem (15). We note that as long as the total number of equality and inequality constraints is less than n, (where $x \in E^n$) and Ω has an interior, critical directions with nontrivial directed conical approximations will exist.

(37) <u>Theorem</u>: If $\hat{x}$ is an optimal solution to the Nonlinear Programming Problem (2) and $IC(\hat{x},\Omega)$ is not empty, then for each critical direction $\delta\tilde{x} \in \overline{IC(\hat{x},\Omega)}$ there exists a vector δx^* in E^n, multipliers $\psi^0, \psi^1, \ldots, \psi^m$ not all zero, with $\psi^0 \leqq 0$, and multipliers $u^i \leqq 0$ satisfying $u^i = 0$ if $i \notin \tilde{I}(\hat{x}, \delta\tilde{x})$, such that:

$$\text{(i)} \quad \left\langle \delta\tilde{x}, \frac{\partial^2 g^i}{\partial x^2}(\hat{x})\delta\tilde{x} \right\rangle + \langle \nabla g^i(\hat{x}), \delta x^* \rangle \leqq 0 \quad \text{for } i \in \tilde{I}(\hat{x}, \delta\tilde{x})$$

$$\text{(ii)} \quad \psi^0 \nabla f(\hat{x}) + \sum_{i=1}^{m} \psi^i \nabla r(\hat{x}) + \sum_{i=1}^{k} u^i \nabla g^i(\hat{x}) = 0$$

$$\text{(iii)} \quad \psi^0\left(\left\langle \delta\tilde{x}, \frac{\partial^2 f}{\partial x^2}(\hat{x})\delta\tilde{x}\right\rangle + \langle \nabla f(\hat{x}), \delta x^* \rangle\right)$$

$$+ \sum_{i=1}^{m} \psi^i\left(\left\langle \delta\tilde{x}, \frac{\partial^2 r^i}{\partial x^2}(\hat{x})\delta\tilde{x}\right\rangle + \langle \nabla r^i(\hat{x}), \delta x^* \rangle\right) \leqq 0$$

and $\psi^0 = 0$ if $\langle \nabla f(\hat{x}), \delta\tilde{x} \rangle < 0$.

<u>Proof</u>: Let $\delta\tilde{x} \in \overline{IC(\hat{x},\Omega)}$. By lemma (24), there is a δx^* which satisfies (i), and moreover $\{\delta x \mid \langle \nabla g^i(\hat{x}), \delta x \rangle < 0,\ i \in \tilde{I}(\hat{x}, \delta\tilde{x})\},\ \delta x^*\}$ is a $\delta\tilde{x}$-directed conical approximation. Thus, by theorem (26), there is a nonzero multiplier ψ, with $\psi^0 \leqq 0$, such that:

$\langle \psi^0 \nabla f(\hat{x}) + \sum_{i=1}^{m} \psi^i \nabla r^i(\hat{x}), \delta x\rangle \leq 0$ for all $\delta x \in \{\delta x \,|\langle \nabla g^i(\hat{x}), \delta x\rangle \leq 0$ for $i \in \tilde{I}(\hat{x}, \delta\tilde{x})\}$, and, in addition, this multiplier satisfies condition (iii). Applying Farka's lemma [13] to the inequality above, there are scalars $-u^i \geq 0$ for $i \in \tilde{I}(\hat{x}, \delta\tilde{x})$ such that

$$\psi^0 \nabla f(\hat{x}) + \sum_{i=1}^{m} \psi^i \nabla r^i(\hat{x}) = \sum_{i \notin \tilde{I}(\hat{x}, \delta\tilde{x})} -u^i \nabla g^i(\hat{x}) .$$

Defining $u^i = 0$ for $i \notin \tilde{I}(\hat{x}, \delta\tilde{x})$ completes the proof.

(38) Corollary 1. If in the statement of theorem (37), δx^* satisfies $\langle \delta\tilde{x}, \frac{\partial^2 g^i(\hat{x})}{\partial x^2} \delta\tilde{x}\rangle + \langle \nabla g^i(\hat{x}), \delta x^*\rangle = 0$ for $i \in \tilde{I}(\hat{x}, \delta\tilde{x})$, then condition (iii) may be replaced by:

$$\text{(iii')} \quad \langle \delta\tilde{x}, (\psi^0 \frac{\partial^2 f}{\partial x^2}(\hat{x}) + \sum_{i=1}^{m} \psi^i \frac{\partial^2 r^i}{\partial x^2}(\hat{x}) + \sum_{i=1}^{k} u^i \frac{\partial^2 g^i}{\partial x^2}(\hat{x}))\delta\tilde{x}\rangle \leq 0.$$

Proof: Let u^i be the scalars given in the statement of the theorem. Then $\sum_{i=1}^{k} u^i(<\delta\tilde{x}, \frac{\partial^2 g^i(\hat{x})}{\partial x^2} \delta\tilde{x}> + <\nabla g^i(\hat{x}), \delta x^*>) = 0$, and therefore this term may be added to (iii) without changing the sign of the inequality. However, from condition (ii) we have $\psi^0\langle \nabla f(\hat{x}), \delta x^*\rangle + \sum_{i=1}^{m} \psi^i\langle \nabla r^i(\hat{x}), \delta x^*\rangle + \sum_{i=1}^{k} u^i\langle \nabla g^i(\hat{x}), \delta x^*\rangle = 0$, which gives condition (iii') above. This completes the proof.

A sufficient condition which ensures that $IC(\hat{x}, \Omega)$ is not empty and that a δx^* satisfying the hypothesis of (38) exists is that the vectors $\nabla g^i(\hat{x})$, $i \in I(\hat{x})$, are linearly independent. However, while this assumption simplifies theorem (37), we are again faced with the question of determining when the multiplier vector ψ does not depend on the critical direction $\delta\tilde{x}$. The following, a generalization of (14), will be shown to be a sufficient condition for a single multiplier vector to satisfy theorem (37) for a class of critical directions.

(39) <u>Assumption</u>: Let $\delta\tilde{x}$ be a given critical direction in $\overline{IC(\hat{x},\Omega)}$ with index set $\tilde{I}(\hat{x},\delta\tilde{x})$, and let $K(\delta\tilde{x})$ be the set $\{\delta x \mid \delta x$ is a critical direction in $\overline{IC(\hat{x},\Omega)}$ and $\tilde{I}(\hat{x},\delta\tilde{x}) \subset \tilde{I}(\hat{x},\delta x)\}$. We assume that for every δx in $K(\delta\tilde{x})$ there is a function $x : E^1 \to E^n$ and a $\bar{\theta} > 0$ such that $x(0) = \hat{x}$, $\frac{dx(0)}{d\theta} = \delta x$, $x(\theta)$ is feasible for $\theta \in [0,\bar{\theta}]$, and , in addition, $g^i(x(\theta)) = 0$ for $i \in \tilde{I}(\hat{x},\tilde{\delta} x)$.

(40) <u>Corollary 2</u>. Suppose that $\hat{x}$ is an optimal solution to the Nonlinear Programming Problem (2), and that for the critical direction $\delta\tilde{x}$ in $\overline{IC(\hat{x},\Omega)}$, assumption (39) holds. Then there exists multipliers $\psi^0, \psi^1, \ldots, \psi^m$ not all zero, with $\psi^0 \leqq 0$, and multipliers $u^i \leqq 0$ satisfying $u^i = 0$ if $i \notin \tilde{I}(\hat{x},\delta \tilde{x})$, such that:

$$\text{(i)}\quad \psi^0 \nabla f(\hat{x}) + \sum_{i=1}^{m} \psi^i \nabla r^i(\hat{x}) + \sum_{i=1}^{k} u^i \nabla g^i(\hat{x}) = 0$$

$$\text{(ii)}\quad \langle \delta x, (\psi^0 \frac{\partial^2 f}{\partial x^2}(\hat{x}) + \sum_{i=1}^{m} \psi^i \frac{\partial^2 r^i}{\partial x^2}(\hat{x}) + \sum_{i=1}^{k} u^i \frac{\partial^2 g^i}{\partial x^2}(\hat{x}))\delta x \rangle \leqq 0$$

for all δx belonging to $K(\delta\tilde{x})$,

and $\psi^0 = 0$ if $< \nabla f(\hat{x}), \delta\tilde{x} > \; < \; 0$.

<u>Proof</u>: Assumption (39) is a sufficient condition for the existence of a δx^* satisfying the requirements of corollary (38) for the critical direction $\delta\tilde{x}$, thus (i) and (ii) above are satisfied for $\delta\tilde{x}$, and some multiplier (ψ,u). Now, let δx be an arbitrary vector in $K(\delta\tilde{x})$, with

corresponding function $x(\cdot)$, and let (ψ, u) be as given above for $\delta\tilde{x}$. Consider the function $\ell(\theta)$ defined by:

$$\ell(\theta) = \psi^0 f(x(\theta)) + \sum_{i=1}^{m} \psi^i r^i(x(\theta)) + \sum_{i=1}^{k} u^i g^i(x(\theta)) .$$

By assumption (39), for θ in $[0,\bar{\theta}]$ we have $\ell(\theta) = \psi^0 f(x(\theta))$. If $\psi^0 = 0$, then $\dfrac{d^2\ell(0)}{d\theta^2} = 0$; thus, from (37(ii)), we obtain

$$\frac{d^2\ell(0)}{d\theta^2} = \psi^0\langle \delta x, \frac{\partial^2 f}{\partial x^2}(\hat{x})\delta x\rangle + \sum_{i=1}^{m} \psi^i\langle \delta x, \frac{\partial^2 r^i}{\partial x^2}(\hat{x})\delta x\rangle + \sum_{i=1}^{k} u^i\langle \delta x, \frac{\partial^2 g^i}{\partial x^2}(\hat{x})\delta x\rangle = 0 ,$$

which satisfies (ii) above. If $\psi^0 = -1$, then we have $\ell(\theta) = -f(x(\theta))$ and, since by (37(ii)), $\dfrac{df(x(0))}{d\theta} = 0$, we require $\dfrac{d^2\ell(\theta)}{d\theta^2} = -\dfrac{d^2 f}{d\theta^2}(x(\theta)) \leq 0$. (cf. theorem (9).) Again, from (37(ii)), the inequality (ii) above is obtained. This completes the proof.

A sufficient condition for the cone $IC(\hat{x},\Omega)$ to be nonempty and for assumption (39) to hold is that the vectors $\nabla r^i(\hat{x})$, $i = 1, \cdots, m$, and $\nabla g^i(\hat{x})$, $i \in I(\hat{x})$, are linearly independent. From (37(ii)) and the fact that ψ is nonzero, this is also sufficient to guarantee that ψ^0 in (37(ii)) must be nonzero.

To illustrate how theorem (37) augments the first-order theory for the Nonlinear Programming Problem (2), suppose that $\bar{x}$ is a candidate optimal solution, $\delta\tilde{x}$ is a critical direction, and either $I(\bar{x}) = \emptyset$, or $\delta\tilde{x} \in IC(\bar{x},\Omega)$. Then, since $\tilde{I}(\bar{x}, \delta\tilde{x}) = \emptyset$, (37 (ii)) requires that $\psi^0\nabla f(\bar{x}) + \sum_{i=1}^{m} \psi^i\nabla r^i(\bar{x}) = 0$. If $\nabla f(\bar{x}), \nabla r^1(\bar{x}), \ldots, \nabla r^m(\bar{x})$ are linearly independent, we conclude, since $\psi \neq 0$, that $\bar{x}$ cannot be optimal. Similarly, if $\langle \nabla f(\bar{x}), \delta\tilde{x} \rangle < 0$ and $\nabla r^1(\bar{x}), \ldots, \nabla r^m(\bar{x})$ are linearly independent we conclude that $\bar{x}$ cannot be optimal. However, if (37(ii)) is satisfied, then (37(iii)) would have to be examined, and this is an easy task whenever the multipliers are unique.

As a further illustration of theorem (26), consider the quadratic programming problem, minimize $\frac{1}{2} \langle x, Qx \rangle + \langle d, x \rangle$ subject to $Ax = b$, where Q is an $n \times n$ symmetric matrix and A is an $m \times n$ matrix with $m < n$. Assume that $\hat{x}$ is optimal and that the rows, a_i $i = 1, \ldots, m$, of A are linearly independent. Then, choosing a vector $\delta\tilde{x}$ such that $A\delta\tilde{x} = 0$, we may set $\{C(\hat{x}, \delta\tilde{x}, E^n), \delta x^*\} = \{E^n, 0\}$. Thus, from (26(i)), $\psi^0(Q\hat{x} + d) + \sum_{i=1}^{m} \psi^i a_i = 0$, and clearly ψ^0 must be strictly less than 0. From (26(ii)) we obtain $\langle \delta\tilde{x}, Q\delta\tilde{x} \rangle \geqq 0$, and since $\langle \delta\tilde{x}, Q\hat{x} + d \rangle = 0$, then $\langle d, \delta\tilde{x} \rangle = 0$ if $Q\delta\tilde{x} = 0$. In other words, we have the necessary conditions: (i) $\langle \delta\tilde{x}, Q\delta\tilde{x} \rangle \geqq 0$ for every $\delta\tilde{x}$ such that $A\delta\tilde{x} = 0$, and (ii) $\langle d, \delta\tilde{x} \rangle = 0$ for every $\delta\tilde{x}$ such that $A\delta\tilde{x} = 0$ and $Q\delta\tilde{x} = 0$. Note that these conditions do not involve $\hat{x}$. In fact, it can be shown that the existence of one feasible solution, together with conditions (i) and (ii), are sufficient conditions for the existence of an optimal solution to the quadratic programming problem above.

<u>Remark:</u> In view of theorems (26) and (27), one may be inclined to think that more information about a candidate optimal solution $\hat{x}$ could be obtained, and verification of the necessary conditions simplified, by transcribing the original problem into an equivalent form with simple structure

or many critical directions. Thus, for any problem of the form (1), an equivalent problem with a single equality constraint, $\tilde{r} : E^n \to E'$, can always be defined by letting $\tilde{r}(x) = \sum_{i=1}^{m} (r^i(x))^2$. Since $\nabla\tilde{r}(\hat{x}) = 2 \sum_{i=1}^{m} r^i(\hat{x})\nabla r^i(\hat{x}) = 0$, we can now always apply either theorem (27) or theorem (26) with the set of critical directions being $\{\delta x | \langle \nabla f(\hat{x}), \delta x \rangle \leqq 0\}$. Unfortunately, theorems (26) and (27) can be satisfied trivially for this new problem and so it is seen that the transcription does not increase the amount of information available about the optimal solutions of the original problem. (Since $\langle \delta x, \frac{\partial^2 \tilde{r}}{\partial x^2} (\hat{x})\ \delta x \rangle = 2 \sum_{i=1}^{m} \langle \nabla r^i(\hat{x}), \delta x \rangle^2 \geq 0$ for all δx, in theorem (26) we may take $\psi^0 = 0, \psi^1 = -1$, while in theorem (27), again by the above inequality, the ray R will have no points in the interior of L_1 or L_2.)

Thus, we have shown that most second-order necessary conditions of optimality are special cases of theorems (26) and (27), which we shall now proceed to prove.

IV. DERIVATION OF THE MAJOR THEOREMS

In this section, proofs for theorems (26) and (27) are given. Both proofs rely on the Brouwer fixed point theorem [14,16]. We shall use here the more convenient notations $\frac{\partial f}{\partial x} (\hat{x}) (\delta x)$, $\frac{\partial^2 f}{\partial x^2} (\hat{x}) (\delta x)$, etc., rather than the gradient and Hessian notation.

Proof of Theorem (26)

Let us assume that $\hat{x}$ is an optimal solution to the Basic Problem (1) and that $\delta\tilde{x}$ is a critical direction with directed conical approximation $\{C(\hat{x}, \delta\tilde{x}, \Omega), \delta x^*\}$. We define the map $F : E^n \to E^{m+1}$ by $F(x) = (f(x), r^1(x), \ldots, r^m(x))$, and the vector $\delta\tilde{y}$ in E^{m+1} by $\delta\tilde{y} = \frac{\partial^2 F}{\partial x^2} (\hat{x})(\delta \tilde{x}) + \frac{\partial F}{\partial x} (\hat{x})(\delta x^*)$.

The convex cones K, R in E^{m+1} are defined by:

(41) $K = \{\delta y \mid \delta y = \frac{\partial F}{\partial x}(\hat{x})(\delta x),\ \delta x \in C(\hat{x}, \delta\tilde{x}, \Omega)\}$

(42) $R = \{\delta y \mid \delta y^0 \leq 0 \text{ and } \delta y^i = 0 \text{ for } i = 1, \ldots, m\}$

Let us first assume that $\frac{\partial f}{\partial x}(\hat{x})(\delta\tilde{x}) = 0$. For this case, theorem (26) claims that the convex set $\delta\tilde{y} + K = \{\delta y \mid \delta y = \delta\tilde{y} + \delta y',\ \delta y' \in K\}$, and the ray R, must be separated. We shall therefore assume the contrary, and obtain a contradiction on the optimality of $\hat{x}$.

Now, if $\delta\tilde{y} + K$ and R are not separated, there must exist vectors $\delta x_1, \ldots, \delta x_{m+1}$ in $C(\hat{x}, \delta\tilde{x}, \Omega)$, with a corresponding $\epsilon_0 > 0$ and map $\zeta_{\delta\tilde{x}}(\cdot, \cdot)$ defined as in (16), such that:

(43) $\zeta_{\delta\tilde{x}}(\epsilon, \delta x) \in \Omega - \hat{x}$ for all $\epsilon \in (0, \epsilon_0]$ and $\delta x \in co\{\delta x_1, \cdots, \delta x_{m+1}\}$.

(44) The set $\bar{\Sigma} = co\{\delta\bar{y}_1, \cdots, \delta\bar{y}_{m+1}\}$ with

$$\delta\bar{y}_i = \frac{\partial^2 r(\hat{x})}{\partial x^2}(\delta\tilde{x}) + \frac{\partial r}{\partial x}(\hat{x})(\delta x^*) + \frac{\partial r}{\partial x}(\hat{x})(\delta x_i),$$

for $i = 1, \cdots, m + 1$, is a simplex† in E^m, containing the origin in its interior.

(45) $f(\hat{x} + \delta x) - f(\hat{x}) < 0$ for all $\delta x \in \zeta_{\delta\tilde{x}}(\epsilon, co\{\delta x_1, \cdots, \delta x_{m+1}\})$ and $\epsilon \in (0, \epsilon_0]$.

†A simplex in E^m is a convex polyhedron with m + 1 vertices, which has an interior.

Now let us consider the case $\frac{\partial f}{\partial x}(\hat{x})(\delta\tilde{x}) < 0$. With $\psi^0 = 0$, the claim of theorem (26) is that the origin in E^m is not an interior point of the set $\frac{\partial^2 r}{\partial x^2}(\hat{x})(\delta\tilde{x}) + \frac{\partial r}{\partial x}(\hat{x})(\delta x^*) + \frac{\partial r}{\partial x}(\hat{x})(C(\hat{x},\delta\tilde{x},\Omega))$. Again, if we assume the contrary, then there must exist vectors $\delta x_1, \cdots, \delta x_{m+1}$ in $C(\hat{x},\delta\tilde{x},\Omega)$ satisfying (43) and (44) above. Since $\frac{\partial f}{\partial x}(\tilde{x})(\delta\tilde{x}) < 0$, (45) will also be satisfied for all $\epsilon \in (0,\epsilon_0']$, for some $\epsilon_0' > 0$, and, therefore, there is an $\epsilon_0 > 0$ for which (43), (44) and (45) will be satisfied. Thus, whether $\frac{\partial f}{\partial x}(\hat{x})(\delta\tilde{x}) = 0$ or $\frac{\partial f}{\partial x}(\hat{x})(\delta\tilde{x}) < 0$ holds, the contrary assumption leads to the above conditions, which we shall now utilize to complete the contradiction.

Now, let $\bar{Y}$ be a m x m matrix whose i-th column is $\delta\bar{y}_i - \delta\bar{y}_{m+1}$, $i = 1, 2, \cdots, m$, and let X be a n x m matrix whose i-th column is $\delta x_i - \delta x_{m+1}$, $i = 1, 2, \cdots, m$. Then $\bar{Y}$ is nonsingular since $\bar{\Sigma}$ is a simplex. Hence, for every $\delta\bar{y} \in \bar{\Sigma}$ and $\epsilon \in (0,\epsilon_0]$

$$X\bar{Y}^{-1}(\delta\bar{y} - \delta\bar{y}_{m+1}) + \delta x_{m+1} \in \mathrm{co}\{\delta x_1, \cdots, \delta x_{m+1}\},$$

and

$$\zeta_{\delta\tilde{x}}(\epsilon, X\bar{Y}^{-1}(\delta\bar{y} - \delta\bar{y}_{m+1}) + \delta x_{m+1}) \in \Omega - \hat{x}.$$

For $\epsilon \in (0,\epsilon_0]$, we now define the map $G_\epsilon : \bar{\Sigma} \to E^m$ by:

$$(46)\quad G_\epsilon(\delta\bar{y}) = \delta\bar{y} - \frac{2}{\epsilon^2} r(\hat{x} + \zeta_{\delta\tilde{x}}(\epsilon, X\bar{Y}^{-1}(\delta\bar{y} - \delta\bar{y}_{m+1}) + \delta x_{m+1})).$$

Then, recalling the form of $\zeta_{\delta\tilde{x}}(\cdot,\cdot)$ (see (16)), we obtain:

$$(47)\quad G_\epsilon(\delta\bar{y}) = \delta\bar{y} - \frac{2}{\epsilon^2}\{r(\hat{x}) + \frac{\epsilon^2}{2}\frac{\partial r}{\partial x}(\hat{x})X\bar{Y}^{-1}(\delta\bar{y} - \delta\bar{y}_{m+1}) + \frac{\epsilon^2}{2}\frac{\partial r}{\partial x}(\hat{x})(\delta x_{m+1})$$
$$+ \frac{\epsilon^2}{2}\frac{\partial r}{\partial x}(\hat{x})(\delta x^*) + \frac{\epsilon^2}{2}\frac{\partial^2 r}{\partial x^2}(\hat{x})(\delta x^*) + \frac{\epsilon^2}{2}\frac{\partial^2 r}{\partial x^2}(\hat{x})(\delta\tilde{x}) + o'(\epsilon^2,\delta\bar{y})\} = -\frac{o'(\epsilon^2,\delta\bar{y})}{\epsilon^2}$$

where $\|o'(\epsilon^2,\delta\bar{y})\|/\epsilon^2 \to 0$ as $\epsilon \to 0$, uniformly for $\overline{\delta y} \in \bar{\Sigma}$. Hence, there exists on $\epsilon_1 \in (0,\epsilon_0]$ such that $G_{\epsilon_1}(\cdot)$ maps $\bar{\Sigma}$ into $\bar{\Sigma}$, and therefore, by Brouwer's fixed-point theorem, there is a $\overline{\delta y}*$ in $\bar{\Sigma}$ such that $G_{\epsilon_1}(\delta\bar{y}*) = \delta\bar{y}*$. But from (46), we see that the point

$$x^* = \hat{x} + \zeta_{\delta\tilde{x}}(\epsilon_1, X\bar{Y}^{-1}(\delta\bar{y}* - \delta\bar{y}_{m+1}) + \delta x_{m+1})$$

satisfies $r(x^*) = 0$, and since from (43) and (45), $x^* \in \hat{\Omega}$ and $f(x^*) < f(\hat{x})$, we have a contradiction.

Thus, if $\frac{\partial f}{\partial x}(\hat{x})(\delta\tilde{x}) = 0$, $\delta\tilde{y} + K$ and R (with K and R defined as in (41) and (42)), must be separated, while if $\frac{\partial f}{\partial x}(\hat{x})(\delta\tilde{x}) < 0$, the set

$$\frac{\partial^2 r}{\partial x^2}(\hat{x})(\delta\tilde{x}) + \frac{\partial r}{\partial x}(\hat{x})(\delta x*) + \frac{\partial r}{\partial x}(\hat{x})(C(\hat{x},\delta\tilde{x},\Omega))$$

is contained in a half-space in E^m, and hence the statements of theorem (26) follow.

<u>Proof of Theorem (27)</u>

We shall require the following lemma.

(48) <u>Lemma</u>: Let A_0 and A_1 be $n \times n$ symmetric matrices and let a_0, $a_1, \cdots, a_m$ be vectors in E^n. Suppose that for $i = 0, 1, 2$ the functions $H_i : E^n \to E^{m+1}$ are defined by:

(i) $H_0^0(x) = \langle x, A_0 x\rangle$, and $H_0^j(x) = \langle a_j, x\rangle$ for $j = 1, \cdots, m$

(ii) $H_1^0(x) = \langle a_0, x\rangle$, $H_1^1(x) = \langle x, A_1 x\rangle$, and $H_1^j(x) = \langle a_j, x\rangle$ for $j = 2, \cdots, m$

(iii) $H_2^{0}(x) = \langle x, A_0 x\rangle$, $H_2^{1}(x) = \langle x, A_1 x\rangle$, and $H_2^{j}(x) = \langle a_j, x\rangle$ for $j = 2, \cdots, m$.

Let C be a convex cone in E^n.

If for any $i \in \{0,1,2\}$ the point $y_0 = (-1, 0, \cdots, 0)$ is an interior point of the set $H_i(C)$, then there exists an $\tilde{x}$ in C such that $H_i(\tilde{x}) = y_0$, and the Jacobian matrix $\frac{\partial H_i}{\partial x}(\tilde{x})$ has rank m + 1.

Proof: We shall prove the lemma only for $H_2(\cdot)$; the proofs for the other two cases are similar.

Thus, assume that y_0 is an interior point of $H_2(C)$. From this, it follows that:

(i) there is an $\bar{x} \in C$ such that $H_2(\bar{x}) = y_0$, and

(ii) $A_0\bar{x}, a_2, \cdots, a_m$ are linearly independent.

If all the vectors $A_0\bar{x}, A_1\bar{x}, a_2, \cdots, a_m$ are linearly independent, we are finished. If they are not, then we shall construct a vector $\tilde{x}$ satisfying (47). Thus, if $A_0\bar{x}, A_1\bar{x}, a_2, \cdots, a_m$ are linearly dependent, then since $\langle \bar{x}, A_0\bar{x}\rangle \neq 0$, we must have

$$A_1\bar{x} = \sum_{i=2}^{m} \beta^i a_i . \tag{49}$$

Now, we may choose x_1, x_2 in C such that $H_2(x_1) = (-1, -\gamma, 0, 0, \cdots, 0)$ and $H_2(x_2) = (-1, \gamma, 0, 0, \cdots, 0)$ for some $\gamma > 0$. For $u \geq 0$ and $\lambda \in [0,1]$, let $x(\lambda,u) = \bar{x} + u(\lambda x_1 + (1-\lambda)x_2)$. Observe that $x(\lambda,u) \in C$, and that $\langle a_i, x(\lambda,u)\rangle = 0$ for $i = 2, \cdots, m$. Using the symmetry of A_1, and

(49), we find that $\langle x(\lambda,u),A_1x(\lambda_1 u) = u^2\langle \lambda x_1 + (1-\lambda)x_2, A_1(\lambda x_1 + (1-\lambda)x_2)\rangle$, hence there is a $\lambda^* \in (0,1)$ such that $\langle x(\lambda^*,u),A_1x(\lambda^*,u)\rangle = 0$ for all $u \geq 0$. Now, let $u^* > 0$ be chosen so that $\langle x(\lambda^*,u^*), A_0x(\lambda^*,u^*)\rangle < -\frac{1}{2}$ and let $\tilde{x} = \alpha x(\lambda^*,u^*)$, where

$$\alpha = 1/[|\langle x(\lambda^*,u^*),A_0x(\lambda^*,u^*)\rangle|]^{1/2} > 0 .$$

Then $H_2(\tilde{x}) = (-1,0,0 \cdots)$ and $A_0\tilde{x}$, $A_1\tilde{x}$, $a_2, \cdots, a_m$ are linearly independent, since otherwise $A_1\bar{x} + u^*\lambda^*A_1x_1 + u^*(1-\lambda^*)A_1x_2 = \sum_{i=2}^{m} \tilde{\beta}^i a_i$, for some $\tilde{\beta}^i$, $i = 2, \cdots, m$, i.e., from (49), $\lambda^*A_1x_1 + (1-\lambda^*)A_1x_2 = \sum_{i=2}^{m} \bar{\beta}a_i$ for some $\bar{\beta}^i$, $i = 2, \cdots, m$. But this implies $\langle x_1,A_1x_2\rangle < 0$ and $\langle x_2,A_1x_1\rangle > 0$, which is impossible and hence the lemma is proved.

Let us now consider theorem (27) and assume that $\hat{x}$ is an optimal solution, and that $\frac{\partial f}{\partial x}(\hat{x})(\cdot) \equiv 0$, $\frac{\partial r^1(\hat{x})}{\partial x}(\cdot) \not\equiv 0$; or that $\frac{\partial f}{\partial x}(\hat{x})(\cdot) \not\equiv 0$, $\frac{\partial r^1}{\partial x}(\hat{x})(\cdot) \equiv \hat{0}$; or that $\frac{\partial f(\hat{x})}{\partial x}(\cdot)$ and $\frac{\partial r^1}{\partial x}(\hat{x})(\cdot) \equiv 0$. Since we need not distinguish in our proof between the above cases, it is convenient to define the indicator functions σ^i, $i = 0,1,2,\ldots,m$, as follows:

$$\sigma^0 = \begin{cases} 1 \text{ if } \frac{\partial f}{\partial x}(\hat{x})(\cdot) \not\equiv 0 \\ 0 \text{ otherwise} \end{cases}$$

(50) $$\sigma^1 = \begin{cases} 1 \text{ if } \frac{\partial r^1}{\partial x}(\hat{x})(\cdot) \not\equiv 0 \\ 0 \text{ otherwise} \end{cases}$$

$$\sigma^i = 1 \text{ for } i = 2, \cdots, m .$$

We define the map $H : E^n \to E^{m+1}$ by:

(51) $$H^0(\delta x) = \sigma^0 \frac{\partial f}{\partial x}(\hat{x})(\delta x) + \frac{1}{2}(1-\sigma^0)\frac{\partial^2 f}{\partial x^2}(\hat{x})(\delta x)$$

and

$$H^i(\delta x) = \sigma^i \frac{\partial r^i}{\partial x}(\hat{x})(\delta x) + \frac{1}{2}(1-\sigma^i)\frac{\partial^2 r^i}{\partial x^2}(\hat{x})(\delta x)^{\wedge}$$

for $i = 1, \cdots m$. The claim of theorem (27) is that the ray $R = \{y \in E^{m+1} | y^0 \leq 0,\ y^1 = \cdots = y^m = 0\}$ has no points in the interior of the set $L = \{y | y = H(\delta x),\ \delta x \in \overset{\circ}{C}(\hat{x},\Omega)\}$.

Let us assume the theorem is false. Then it follows from lemma (48) that there is a vector $\widetilde{\delta x}$ in $\overset{\circ}{C}(\hat{x},\Omega)$ such that $H(\widetilde{\delta x}) = (-1,0,0, \cdots,0)$ and that the Jacobian $\frac{\partial H}{\partial x}(\widetilde{\delta x})$ has rank $m + 1$. We may assume without loss of generality that the first $m + 1$ columns of $\frac{\partial H}{\partial x}(\widetilde{\delta x})$ are linearly independent. Hence, letting $\widetilde{\delta x} = (\widetilde{\delta x}', \widetilde{\delta x}'')$ where $\widetilde{\delta x}' = (\widetilde{\delta x}^1, \cdots, \widetilde{\delta x}^{m+1}), \widetilde{\delta x}'' = (\widetilde{\delta x}^{m+2}, \cdots, \widetilde{\delta x}^n)$, it follows from the implicit function theorem [15] that there are closed neighborhoods U, V of the origin in E^{m+1} such that $H(\cdot)$ is a 1-1 function from $\{\widetilde{\delta x}' + U\} \times \{\widetilde{\delta x}''\}$ onto $\delta y_0 + V$. We shall denote the continuous inverse of this function by $H^-(\cdot)$.

Since we may assume that U is sufficiently small, there is a linearly independent set of vectors $\delta x_1, \cdots, \delta x_{m+1}$ in $C(\hat{x},\Omega)$, with corresponding map $\zeta(\cdot)$ defined as in (3), such that $\{\widetilde{\delta x}' + U\} \times \{\widetilde{\delta x}''\} \subset co\{0, \delta x_1, \cdots, \delta x_{m+1}\}$. We now define, for $\alpha \in (0,1]$, the uniformly continuous map $G_\alpha(\cdot) : V \to E^{m+1}$ by:

$$(52)\quad G_\alpha(\delta y) = \delta y - D_\alpha^{-1}(F(\hat{x} + \zeta(\alpha H^-(\delta y_0 + \delta y))) - F(\hat{x})) + \delta y_0 ,$$

where $F(x) \triangleq (f(x), r^1(x), \cdots, r^m(x))$, $\delta y_0 = (-1, 0, 0, \cdots, 0)$, and $D_\alpha = [d(\alpha)_{ij}]$ is an $m + 1 \times m + 1$ nonsingular diagonal matrix such that for $i, j = 0, 1, 2, \ldots, m$

$$d(\alpha)_{ij} = \begin{cases} 0 & i \neq j \\ \alpha\sigma^i + (1 - \sigma^i)\frac{\alpha^2}{2} & i = j \end{cases}$$

with the σ^i defined as in (50).

Expanding (52), we obtain:

$$(53)\quad G_\alpha(\delta y) = \delta y - D_\alpha^{-1}\left\{\alpha \frac{\partial F}{\partial x}(\hat{x})(H^-(\delta y_0 + \delta y)) + \frac{\alpha^2}{2}\frac{\partial^2 F}{\partial x^2}(\hat{x})(H^-(\delta y_0 + \delta y)) + o(\alpha, \delta y)\right\} + \delta y_0 \ .$$

This may be rearranged to yield:

$$G_\alpha(\delta y) = \delta y_0 + \delta y - H(H^-(\delta y_0 + \delta y)) - D_\alpha^{-1}\tilde{o}(\alpha, \delta y) = -D_\alpha^{-1}\tilde{o}(\alpha, \delta y) \ ,$$

where, for $i = 0, \cdot\cdot\cdot, m$, $|\tilde{o}^i(\alpha, \delta y)|/(\alpha\sigma^i + (1 - \sigma^i)\frac{\alpha^2}{2}) \to 0$ as $\alpha \to 0$, uniformly for $\delta y \in V$. Thus, we may choose $\alpha^* \in (0,1]$ such that $\hat{x} + \zeta(\alpha^* H^-(\delta y_o + \delta y)) \in \Omega$ for all $\delta y \in V$, and $G_{\alpha^*}(\delta y) \in V$ for all $\delta y \in V$. From Brouwer's fixed point theorem, there is a $\delta y^* \in V$ such that $G_{\alpha^*}(\delta y^*) = \delta y^*$, and hence from (52), the point

$$x^* = \hat{x} + \zeta(\alpha^* H^-(\delta y_0 + \delta y^*))$$

satisfies $r(x^*) = 0$, and $f(x^*) < f(\hat{x})$. Since x^* is also in Ω, we have a contradiction of the optimality of $\hat{x}$, which completes the proof of the theorem.

Conclusion

We have constructed in this paper a theory of second-order conditions of optimality, which is consistent with the modern approach to first-order necessary conditions. Also, we have shown that this theory not only results in a number of new conditions of optimality, but also yields most, if not all, the previously known second order conditions. The application of our results to specific nonlinear programming or optimal control problems is reasonably straightforward and, consequently, was not emphasized in our treatment.

In conclusion, we should like to point out that a number of the results in this paper extend trivially to optimization problems in linear topological spaces. These extensions are obtained by stipulating the existence of suitable linear and bilinear functionals to replace the gradients and Hessians used in this paper.

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