STABILITY OF LINEAR TIME-INVARIANT SYSTEMS

by

C. A. Desoer and M. Y. Wu

Department of Electrical Engineering
and Computer Sciences
Electronics Research Laboratory
University of California, Berkeley

Memorandum No. ERL-M 238

29 September 1967

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

Stability of Linear Time-invariant Systems

by

C. A. Desoer and M. Y. Wu

Department of Electrical Engineering
and Computer Sciences
Electronics Research Laboratory
University of California, Berkeley

ABSTRACT

The stability of a single-input, single-output, single-loop, linear, time-invariant system is related to the properties of its open-loop gain. The impulse response of the open-loop system may be of the form $g(t) = r + g_a(t) + \sum_{i=0}^{\infty} g_i\, \delta(t - t_i)$ where r is a nonnegative constant, g_a is integrable on $[0, \infty)$ and $\sum_{i=0}^{\infty} |g_i| < \infty$. If the Nyquist diagram of the open-loop gain does not go through nor encircle the critical point then the closed-loop system is input-output stable, in the several meanings explained in the paper.

The research reported herein was supported by the National Aeronautics and Space Administration under Grant NsG-354, suppl. 4.

(a) It is of the form

$$\varphi(t) = \begin{cases} \varphi_a(t) + \sum_{i=0}^{\infty} \varphi_i \, \delta(t - \tau_i) & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0 \end{cases} \tag{1}$$

where $\varphi_a(\cdot)$ is a locally integrable real-valued function and φ_i's are constant real numbers,

$$0 = \tau_0 < \tau_1 < \tau_2 < \dots \tag{2}$$

$$\sum_{i=0}^{\infty} |\varphi_i| \epsilon^{-\sigma\tau_i} < \infty \tag{3}$$

and

$$\int_0^{\infty} |\varphi_a(t)| \epsilon^{-\sigma t} \, dt < \infty . \tag{4}$$

When a real valued function such as φ_a is locally integrable and satisfies (4) we write $\varphi_a \epsilon L^1(\sigma)$. Similarly, $\varphi_a \epsilon L^{\infty}(\sigma)$ means that $\sup_{t \geq 0} |\varphi_a(t)| \epsilon^{-\sigma t} < \infty$.

(b) The elements of the algebra $\mathcal{Q}(\sigma)$ are added together and multiplied by scalars in the standard manner; thus, if φ, $\psi \epsilon \mathcal{Q}(\sigma)$ and λ is a scalar, then $\varphi + \psi \epsilon \mathcal{Q}(\sigma)$ and $\lambda \varphi \epsilon \mathcal{Q}(\sigma)$.

(c) The "multiplication" in $\mathcal{Q}(\sigma)$ is the convolution product. If φ, $\psi \epsilon \mathcal{Q}(\sigma)$, then the convolution of φ with ψ is also in $\mathcal{Q}(\sigma)$. In informal notation, the convolution of φ with ψ is

$$(\varphi * \psi)(t) = \begin{cases} \int_0^t \varphi(t-\tau)\,\psi(\tau)\,d\tau & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0 . \end{cases} \tag{5}$$

In performing this operation we recall that, for $\tau_1 \geq 0$ and $\tau_2 \geq 0$,

$$\delta(t-\tau_1) * \delta(t-\tau_2) = \delta[t-(\tau_1+\tau_2)]$$

(d) Any element φ in $\mathcal{Q}(\sigma)$ has a finite norm which is defined by

$$||\varphi||_\sigma = \int_0^\infty |\varphi_a(t)|\,\epsilon^{-\sigma t}\,dt + \sum_{i=0}^{\infty} |\varphi_i|\,\epsilon^{-\sigma\tau_i} < \infty \tag{6}$$

It is a standard exercise [7] to show that

$$||\varphi+\psi||_\sigma \leq ||\varphi||_\sigma + ||\psi||_\sigma \tag{7}$$

and

$$||\varphi * \psi||_\sigma \leq ||\varphi||_\sigma\,||\psi||_\sigma \tag{8}$$

C. With these preliminaries, we may state our assumptions on G: the input-output relation of the block G (shown in Fig. 1) is

$$\eta(t) = \begin{cases} (g * \xi)(t) & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0 \end{cases} \tag{9}$$

with

$$g(t) = \begin{cases} r + g_\ell(t) & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0 \end{cases} \tag{10}$$

where r is a non-negative constant, $g_\ell \in \mathcal{Q}(\sigma)$ and $g_a \in L^\infty(\sigma)$. In terms of the "error" e, the equation of the single-loop system is

$$e(t) = \begin{cases} u(t) - z(t) - k(e * g)(t) & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0 \end{cases} \tag{11}$$

The output g is given by $y = u - e$, hence

$$y(t) = \begin{cases} z(t) + k[(u - y) * g](t) & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0 \end{cases} \tag{12}$$

III. MAIN RESULTS

In this section we state our main results in the form of theorems and corollaries. The application of the theorems is illustrated in an example given in Section IV. All proofs are to be found in the appendix.

Theorem 1.

Let the closed-loop system shown in Fig. 1 be described by (11) and (12). Assume that $g_\ell \in \mathcal{A}(0)$ and $g_a \in L^\infty(0)$. Let $k > 0$. Let $\hat{g}(s)$ be the Laplace transform of g. Under these conditions, if

$$\inf_{\mathrm{Re}\, s \geq 0} \left| 1 + k\hat{g}(s) \right| > 0 \tag{13}$$

then the closed-loop impulse response h (i.e., the response y for $z \equiv 0$ and $u(t) = \delta(t)$) is in $\mathcal{A}(0)$.

Corollary 1.

In Theorem 1, if, in addition, $g(t) = r + g_a(t)$ with $g_a \in L^1(0) \cap L^\infty(0)$ and $g_a(t) \to 0$ as $t \to \infty$, then the closed-loop impulse response h is also in $L^1(0) \cap L^\infty(0)$ and $h(t) \to 0$ as $t \to \infty$.

Theorem 2.

Let the closed-loop system shown in Fig. 1 be described by (11) and (12). Let $z \equiv 0$ and $k > 0$. Let $g_\ell \in \mathcal{A}(0)$ and $g_a \in L^\infty(0)$. Under these conditions

(a) If (13) holds and $u \in L^\infty(0)$, then $y \in L^\infty(0)$; i.e., if (13) holds, the closed-loop response with zero initial condition to a bounded input is bounded. Equivalently, the closed-loop transfer function is strictly stable (In the sense of [2], p. 414);

(b) If (13) holds and $u \in L^p(0)$, with $1 \leq p < \infty$, then $y \in L^p(0)$;

(c) If (13) holds and $u \in \mathcal{Q}(0)$, then $y \in \mathcal{Q}(0)$;

(d) If (13) holds and $u(t) = 1(t)$, then, provided $r > 0$, $y(t) = \mathcal{A}(t) \to 1$ as $t \to \infty$;

(e) If (13) holds and u is continuous on $[0, \infty)$ with $u(0) = 0$, then y is continuous;

(f) If (13) holds and $u \in L^\infty(0)$ and $u(t) \to 0$ as $t \to \infty$, then $y \in L^\infty(0)$ and $y(t) \to 0$ as $t \to \infty$;

(g) If (13) holds and $u \in L^\infty(0)$ and $u(t) \to u_\infty$ as $t \to \infty$, then $y \in L^\infty(0)$ and $y(t) \to u_\infty$ as $t \to \infty$.

Corollary 2.

Let the closed-loop system shown in Fig. 1 be described by (11) and (12). Let $u \equiv 0$ and $k > 0$. Let $g_\ell \in \mathcal{Q}(0)$ and $g_a \in L^\infty(0)$. Under these conditions,

(a) If (13) holds and $z \in L^\infty(0)$, then $y \in L^\infty(0)$;

(b) If (13) holds and $z \in L^p(0)$, with $1 \leq p < \infty$, then $y \in L^p(0)$;

(c) If (13) holds and $z \in \mathcal{Q}(0)$, then $y \in \mathcal{Q}(0)$;

(d) If (13) holds and $z \in L^\infty(0)$ and if furthermore $g(t) = r + g_a(t)$ with

$g_a \in L^1(0) \cap L^\infty(0)$, then $y(t) \to 0$ as $t \to \infty$.

Corollary 3.

Let the closed-loop system of Fig. 1 be described by (11) and (12). Let $g_\ell \in \mathcal{Q}(\sigma)$ and $g_a \in L^\infty(0)$. Let $k > 0$. Under these conditions,

(a) If (13) holds and $u, z \in L^\infty(0)$, then $y \in L^\infty(0)$;

(b) If (13) holds and $u, z \in L^p(0)$, with $1 \leq p < \infty$, then $y \in L^p(0)$;

(c) If (13) holds and $u, z \in \mathcal{Q}(0)$, then $y \in \mathcal{Q}(0)$.

Let us now make use of the exponential weighting factor $\epsilon^{-\sigma t}$. In doing so we have to set $r = 0$, thus, in the following $\hat{g}(s)$ does not have a pole at $s = 0$.

Theorem 3:

Let the closed-loop system of Fig. 1 be described by (11) and (12). Let $z \equiv 0$. Assume that $r = 0$ and $g_\ell \in \mathcal{Q}(\sigma)$ and $g_a \in L^\infty(\sigma)$, where σ is a constant parameter. Let $k > 0$ and let $\hat{g}(s)$ be the Laplace transform of g. Let the following condition

$$\inf_{\operatorname{Re} s \geq \sigma} |1 + k\hat{g}(s)| > 0$$

be satisfied. Under these conditions,

(a) If $u(t) = \delta(t)$, then $h \in \mathcal{Q}(\sigma)$;

(b) If $u \in L^{\infty}(\sigma)$, then $y \in L^{\infty}(\sigma)$;

(c) If $u \in L^{p}(\sigma)$, with $1 \leq p < \infty$, then $y \in L^{p}(\sigma)$;

(d) If $u \in \mathcal{Q}(\sigma)$, then $y \in \mathcal{Q}(\sigma)$

To visualize the meaning of these conclusions, suppose $\sigma = -1$, then $u \in L^{\infty}(-1)$ means that $u(t)\epsilon^{t}$ is bounded on $[0, \infty)$ and, provided the condition of the theorem is satisfied with $\sigma = -1$, $y(t)\epsilon^{t}$ will also be bounded on $[0, \infty)$.

IV. EXAMPLE

The purpose of this example is to show that the condition (13) of Theorems 1 and 2 can be checked graphically à la Nyquist. For simplicity, suppose that the transfer function of the forward block G is given by

$$\hat{g}(s) = \frac{4}{s(s+1)(s+2)} + 0.03\,\epsilon^{-31.4 s}$$

Observe that $\hat{g}(s)$ is analytic in the closed right half plane except at $s = 0$ where it has a simple pole with residue 2. Condition (13) is equivalent to the requirement that the map of the closed right-half s-plane into the $\hat{g}$-plane is bounded away from the critical point $-1/k$. This can be checked in the usual manner by plotting the Nyquist diagram of $\hat{g}$. In the present instance, the plot has the usual shape expected from a position servo except that (i) for $\omega \to \infty$, the diagram is asymptotic to the circle centered at the origin with radius 0.03 (call it C(0, 0.03)), and (ii) the image of the arbitrarily large circle $s = R\epsilon^{j\theta}$ (R large and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$) is a curve inside the circle C(0, 0.03). On Fig. 2 we show only the interesting part of the diagram. From it we conclude that condition (13) is satisfied provided that either $k < 1.38$ or $1.50 < k < 1.58$. Note that since this system is "Nyquist stable" its L^2-stability cannot be predicted on the basis of passivity arguments [8].

V. CONCLUSIONS

Under very general assumptions pertaining to the open-loop system we have shown that if the Nyquist diagram of the open-loop gain satisfies the nonencirclement condition, then the input-output properties of the closed-loop system satisfy all the properties expected from a stable linear system: a bounded (continuous, tending to zero, in L^p $1 \leq p \leq \infty$, resp.) input produces a bounded (continuous, tending to zero, in L^p $1 \leq p \leq \infty$, resp.) output. The formulation and results of this paper include previously known results as special cases.

VI. APPENDIX

Basic lemma. Let $f: R_+ \to R$ and be of exponential order (i.e., for some finite constants f_M and λ, $|f(t)| \leq f_M \epsilon^{\lambda t}$ for all $t \geq 0$). Let

$$g(t) = r + g_\ell(t) \qquad t \geq 0 \tag{14}$$

where r is a non-negative constant and $g_\ell \in \mathcal{Q}(\sigma)$ for some $\sigma \geq 0$. Thus

$$g_\ell(t) = g_a(t) + \sum_{i=0}^{\infty} g_i \, \delta(t - t_i) \qquad t \geq 0 \tag{15}$$

where

$$0 = t_0 < t_1 < t_2 < \dots$$

$$\int_0^{\infty} |g_a(t)|\,\epsilon^{-\sigma t}dt < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} |g_i|\,\epsilon^{-\sigma t_i} < \infty .$$

In addition, let $g_a \in L^{\infty}(\sigma)$ (i.e., $g'_{aM} \triangleq \sup_{t \geq 0} |g_a(t)|\epsilon^{-\sigma t} < \infty$). Let $k > 0$ and let $e(\cdot)$ satisfy

$$e(t) = \begin{cases} f(t) - k(g * e)(t) & \text{for} \quad t \geq 0 \\ \\ 0 & \text{for} \quad t < 0 \end{cases} \tag{16}$$

Under these conditions, if $1 + kg_0 \neq 0$, then e is of exponential order and consequently its Laplace transform $\hat{e}(s)$ is well defined in some half plane of the form $\operatorname{Re} s > \sigma_0$.

Proof of the basic lemma: Multiply (16) by $\epsilon^{-\sigma t}$ and use $\epsilon^{-\sigma t} = \epsilon^{-\sigma(t-\tau)}\epsilon^{-\sigma\tau}$ to obtain

$$e'(t) = f'(t) - k(g' * e')(t) \qquad t \geq 0 \tag{17}$$

where $e'(t) = e(t)\epsilon^{-\sigma t}$ and similar formulas for f' and g'. Using (14) and (15), we rewrite (17) as

$$(1 + kg_0)\,e'(t) = f'(t) - kr\int_0^t \epsilon^{-\sigma(t-\tau)}e'(\tau)\,d\tau - k\int_0^t g'_a(t-\tau)\,e'(\tau)\,d\tau$$

$$- k\sum_{i \geq 1} g_i'\,e'(t - t_i) \qquad t \geq 0 \tag{18}$$

Since $\sigma \geq 0$ and $g'_{aM} < \infty$, we obtain

$$|1+k\,g_0|\;|e'(t)| \leq |f'(t)| + k(r + g'_{aM}) \int_0^t |e'(\tau)|\,d\tau$$

$$+ k \sum_{i \geq 1} |g'_i|\;|e'(t-t_i)| \qquad t \geq 0 \tag{19}$$

Define $e_M(t)$ as $\sup\limits_{0 \leq \tau \leq t} |e'(\tau)|$. If we replace e' by e_M in the <u>right hand side</u> of (19) and if we bound $|f'(t)|$ by $f_M \epsilon^{\lambda t}$ the inequality is strengthened, and we have

$$|1+k\,g_0|\;|e'(t)| \leq f_M \epsilon^{\lambda t} + k(r + g'_{aM}) \int_0^t e_M(\tau)\,d\tau$$

$$+ \sum_{i \geq 1} |g'_i|\;|e_M(t-t_i)| \qquad t \geq 0 \tag{20}$$

The right hand side of (20) is strictly monotonically increasing with t, and (20) holds for all $t \geq 0$, therefore we may replace $|e'(t)|$ in the <u>left hand side</u> by $e_M(t)$. Further since e_M is monotonically increasing $e_M(t-t_i) \leq e_M(t-t_1)$ for all $t \geq 0$ and all i, hence

$$|1 + k\,g_0|\; e_M(t) \leq f_M \epsilon^{\lambda t} + k(r + g'_{aM}) \int_0^t e_M(\tau)\,d\tau$$

$$+ k \sum_{i=1}^{\infty} |g'_i|\; e_M(t-t_1) \qquad t \geq 0 \tag{21}$$

Define the non-negative numbers α, β and γ as follows :

$$\alpha \triangleq \frac{f_M}{|1+kg_0|}, \quad \beta \triangleq \frac{k(r+g'_{aM})}{|1+kg_0|} \quad \text{and} \quad \gamma \triangleq \frac{k\sum_{i=1}^{\infty}|g'_i|}{|1+k\,g_0|} \tag{22}$$

then (21) becomes

$$e_M(t) \leq \alpha\epsilon^{\lambda t} + \beta\int_0^t e_M(\tau)\,d\tau + \gamma\, e_M(t-t_1) \qquad t \geq 0 \tag{23}$$

If we show that e_M is of exponential order, e', hence e, will also be of exponential order. Without loss of generality, we shall assume that $\lambda \geq 2\beta > 0$.

To start a proof by induction we consider the case where $0 \leq t < t_1$. Since $e_M(t-t_1) = 0$ for $t < t_1$, (23) becomes

$$e_M(t) \leq \alpha\epsilon^{\lambda t} + \beta\int_0^t e_M(\tau)\,d\tau \qquad 0 \leq t < t_1 \tag{24}$$

By the Bellman-Gronwall inequality, we obtain

$$e_M(t) \leq \alpha\epsilon^{\lambda t} + \alpha\epsilon^{\beta t}\,\frac{\beta}{\lambda-\beta}\,[\epsilon^{(\lambda-\beta)t} - 1] \qquad 0 \leq t < t_1 \tag{25}$$

Since $\lambda \geq 2\beta$ and since $\frac{\beta}{\lambda-\beta} \leq 1$, we obtain

$$e_M(t) \leq 2\,\alpha\epsilon^{\lambda t} \qquad \text{for} \quad 0 \leq t < t_1 \tag{26}$$

For the induction step we suppose that

$$e_M(t) \leq 2\alpha(1+2\gamma)^{n-1}\epsilon^{\lambda t} \qquad \text{for} \quad (n-1)\,t_1 \leq t < n\,t_1 \tag{27}$$

and we must show that

$$e_M(t) \leq 2\alpha(1+2\gamma)^{n}\epsilon^{\lambda t} \qquad \text{for} \quad n\,t_1 \leq t < (n+1)\,t_1 \tag{28}$$

Now, for $n\,t_1 \leq t < (n+1)\,t_1$, using (27) in (23) we obtain

$$\begin{aligned} e_M(t) &\leq \alpha\epsilon^{\lambda t} + 2\alpha\gamma(1+2\gamma)^{n-1}\epsilon^{\lambda t} + \beta\int_0^t e_M(\tau)\,d\tau \\ &\leq \alpha(1+2\gamma)^n\epsilon^{\lambda t} + \beta\int_0^t e_M(\tau)\,d\tau \\ &\leq \alpha(1+2\gamma)^n\epsilon^{\lambda t} + \alpha(1+2\gamma)^n\frac{\beta}{\lambda-\beta}\epsilon^{\beta t}[\epsilon^{(\lambda-\beta)t} - 1] \\ &\leq 2\alpha(1+2\gamma)^n\epsilon^{\lambda t} \qquad \text{for} \qquad nt_1 \leq t < (n+1)\,t_1 \end{aligned} \tag{29}$$

Hence the induction step is established; e_M is of exponential order. In fact

$$e_M(t) \leq 2\alpha\exp\left[\lambda + \frac{1}{t_1}\ln(1+2\gamma)\right]t \qquad \text{for} \quad t \geq 0 \tag{30}$$

Thus if f is of exponential order, e defined by (16) is of exponential order and $\hat{e}(s)$ is well defined for $\operatorname{Re} s > \lambda + \frac{1}{t_1} \ell n(1+2\gamma)$.

Proof of Theorem 1:

It is not obvious a priori whether h has a Laplace transform. To start with suppose that $u(t) = 1(t)$, then, calling the solution $\mathscr{s}$, we obtain from (12)

$$\mathscr{s}(t) = \begin{cases} k(g * 1)(t) - k(g * \mathscr{s})(t) & \text{for} \quad t \geq 0 \\ \\ 0 & \text{for} \quad t < 0 \end{cases} \tag{31}$$

Since $g \in \mathcal{A}(0)$, $g * 1$ is bounded on $[0, \infty)$ by the function $\|g_\ell\|_0 + rt$. (In particular, $\hat{g}(s)$ is analytic in the open right half s-plane.) Hence by the basic lemma, $\mathscr{s}$ is of exponential order and so is its integral

$$\rho(t) \triangleq \int_0^t \mathscr{s}(\tau)\, d\tau .$$

Thus, for some finite σ_0, $\epsilon^{-\sigma_0 t} \rho(t)$ is a bounded continuous function on $[0, \infty)$. Hence ρ and all its derivatives (in the distribution sense) have a Laplace transform defined for $\operatorname{Re} s > \sigma_0$ ([4] Chap. 8 and Thm. 6, p. 239; [5] Sec. 8.3). Using the rules of Laplace transforms as they apply to distributions [4, 5, 6] we obtain from (31)

$$\hat{\mathscr{s}}(s) = \frac{k\hat{g}(s)}{1 + k\hat{g}(s)} \cdot \frac{1}{s} \qquad \operatorname{Re} s > \sigma_0 \tag{32}$$

The derivative of $\mathcal{A}$ (denoted by $\mathcal{A}'$) is h: indeed $\mathcal{A}$ is the step response of a linear time-invariant system, hence $\mathcal{A}'$ is equal to the impulse response. More formally, differentiate (31): (a) use the fact that for any two distributions in $\mathcal{D}'_+$ §, say S and T,

$$(S * T)' = S' * T = S * T'.$$

(b) Observe that

$$\frac{d}{dt} 1(t) = \delta(t)$$

(c) Recall that for any distribution [4, pp. 170-174].

$$S * \delta = S$$

hence in an informal notation

$$\mathcal{A}'(t) = \begin{cases} k g(t) - k(g * \mathcal{A}')(t) & \text{for} \quad t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{33}$$

This is precisely the equation satisfied by h, as can be seen by putting $u(t) = \delta(t)$ in (12). We conclude that $\mathcal{A}' = h$: this is a consequence of uniqueness which follows from that there are no divisors of zero in $\mathcal{D}'_+$ [4, p. 173]. Consequently $\hat{h}(s) = s\hat{\mathcal{A}}(s)$ and from (32) we get

---

§ $\mathcal{D}'_+$ is the space of distributions whose support are contained in $0 \leq t < \infty$ [4, p. 172; 5, p. 168].

$$\hat{h}(s) = \frac{k\hat{g}(s)}{1 + k\hat{g}(s)} \qquad \text{Re } s > \sigma_0 \tag{34}$$

To proceed we have to distinguish two cases:

Case 1: $r = 0$. In this case $\hat{g}(s) = \hat{g}_\ell(s)$, where by assumption $g_\ell \in \mathcal{A}(0)$. The assumption (13) implies that

$$\inf_{\text{Re } s \geq 0} |1 + k\hat{g}_\ell(s)| > 0 \tag{35}$$

It follows from results of Hille and Phillips [7, p. 150]† that $\mathcal{L}^{-1}\{1/[1+k\hat{g}_\ell(s)]\}$ is a well defined element of the algebra $\mathcal{A}(0)$. In the present case

$$\hat{h}(s) = k\hat{g}_\ell(s) \cdot \frac{1}{1 + k\hat{g}_\ell(s)} \tag{36}$$

thus $h(\cdot)$ is the convolution of $kg_\ell(\cdot)$ with $\mathcal{L}^{-1}\{1/[1+k\hat{g}_\ell(s)]\}$. Consequently $h \in \mathcal{A}(0)$.

Case 2: $r > 0$. Then $\hat{g}(s) = \frac{r}{s} + \hat{g}_\ell(s)$, and

$$\hat{h}(s) = \frac{k[\frac{r}{s} + \hat{g}_\ell(s)]}{1 + k[\frac{r}{s} + \hat{g}_\ell(s)]} \tag{37}$$

---

† In the notation of Hille and Phillips, our algebra $\mathcal{A}(0)$ is denoted by $L(1(\cdot)) + A(1(\cdot))$.

Multiplying numerator and denominator by $s/(s+kr)$ we obtain

$$\hat{h}(s) = \frac{\frac{kr}{s+kr} + \frac{ks}{s+kr} \cdot \hat{g}_\ell(s)}{1 + \frac{ks}{s+kr} \cdot \hat{g}_\ell(s)} \tag{38}$$

Call $\hat{n}(s)$ and $\hat{d}(s)$ the numerator and the denominator of this expression. Clearly, $\mathcal{L}^{-1}[\hat{n}(s)] \in \mathcal{Q}(0)$. Now,

$$\hat{d}(s) = \frac{s}{s+kr}[1 + k\hat{g}(s)] \tag{39}$$

noting that $\hat{g}(s)$ has a pole at $s = 0$ (with residue $r > 0$) we easily show that

$$\inf_{\mathrm{Re}\, s \geq 0} |1 + k\hat{g}(s)| > 0 \quad \text{implies that} \quad \inf_{\mathrm{Re}\, s \geq 0} |\hat{d}(s)| > 0$$

Hence from the results of Hille and Phillips, $\mathcal{L}^{-1}[1/\hat{d}(s)] \in \mathcal{Q}(0)$. And, as before, we conclude from (38) that $h \in \mathcal{Q}(0)$.

The proof of Corollary 1 is given in [1].

Proof of Theorem 2:

(a) Since $z \equiv 0$ and $u \in L^{\infty}(0)$, by the basic lemma, y is of exponential order. Hence its Laplace transform is obtained from (12) and

$$\hat{y}(s) = \frac{k\hat{g}(s)}{1 + k\hat{g}(s)}\hat{u}(s) = \hat{h}(s)\hat{u}(s) \tag{40}$$

Therefore the output y is the convolution of the closed loop impulse response h with the input u, i.e., using informal notation

$$y(t) = \begin{cases} \int_0^t h(\tau)\, u(t-\tau)\, d\tau & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{41}$$

Since $u \in L^{\infty}(0)$, we set $u_M \triangleq \sup_{t \geq 0} |u(t)| < \infty$, and since $h \in \mathcal{Q}(0)$ with

$$h(t) = h_a(t) + \sum_{i=0}^{\infty} h_i\, \delta(t - t_i') \tag{42}$$

where $\int_0^{\infty} |h_a(t)| dt < \infty$ and $\sum_{i=0}^{\infty} |h_i| < \infty$, we conclude that

$$|y(t)| \leq u_M \left[\int_0^{\infty} |h_a(t)|\, dt + \sum_{i=0}^{\infty} |h_i| \right] < \infty ,$$

hence $y \in L^{\infty}(0)$.

(b) Let $L_p(y)$ denote the L^p-norm of y. Then from (41) and (42) we have

$$L_p(y) \leq L_p \left[|h_a * u| + \sum_{i=0}^{\infty} |h_i|\, |u(t - t_i')| \right]$$

(cont'd.)

$$\leq L_p(|h_a| * |u|) + \left(\sum_{i=0}^{\infty} |h_i| \right) L_p(u)$$

$$\leq L_1(h_a) \cdot L_p(u) + \left(\sum_{i=0}^{\infty} |h_i| \right) L_p(u)$$

$$\leq \left[L_1(h_a) + \sum_{i=0}^{\infty} |h_i| \right] L_p(u)$$

$$= ||h||_0 L_p(u) < \infty$$

because $h \in \mathcal{Q}(0)$ and $u \in L^p(0)$. Hence $y \in L^p(0)$.

(c) Since $z \equiv 0$ and $u \in \mathcal{Q}(0)$, by an argument similar to that used in the proof of Theorem 1, y has a Laplace transform in the distribution sense. $\hat{y}(s)$ is obtained from (12) and is given in (40). Since $h \in \mathcal{Q}(0)$ and $u \in \mathcal{Q}(0)$, we conclude from the closure property of the algebra $\mathcal{Q}(0)$ that $y \in \mathcal{Q}(0)$.

(d) Note that

$$\mathcal{s}(t) = \int_0^t h(\tau)\, d\tau \tag{43}$$

Since $h \in \mathcal{Q}(0)$ and is defined by (42). Given arbitrary $\epsilon > 0$ there exists a T depending on ϵ such that

$$\int_T^\infty |h_a(t)|\,dt < \epsilon/2 \quad \text{and} \quad \sum_{t_i \in (T,\infty)} |h_i| < \epsilon/2 .$$

Consequently,

$$\int_t^\infty |h(\tau)|\,d\tau < \epsilon \qquad \forall\ t > T$$

Thus $\lim_{t\to\infty} \int_0^t h(\tau)\,d\tau$ exists and the number $\mathcal{S}_\infty \triangleq \lim_{t\to\infty} \mathcal{S}(t)$ is well defined. Next we calculate $\mathcal{S}_\infty$ by the final-value theorem of Laplace transform and (43), (32), (38): we obtain $\mathcal{S}_\infty = \lim_{t\to\infty} \mathcal{S}(t) = \lim_{s\to 0} \frac{s}{s}\hat{h}(s) = 1$, provided that $r > 0$.

(e) Since u is continuous on $[0,\infty)$, to prove y is continuous on $[0,\infty)$ it suffices to show that e is continuous on $[0,\infty)$. Using (14) and (15), we rewrite (11) for $z \equiv 0$ as

$$e(t) = \begin{cases} u(t) - kr\displaystyle\int_0^t e(\tau)dt - k\int_0^t g_a(t-\tau)\,e(\tau)\,d\tau - k\sum_{i=0}^{\infty} g_i\, e(t-t_i) & \text{for } t \geq 0 \\ 0 & t < 0 \end{cases} \tag{44}$$

Since u is continuous on $[0,\infty)$ with $u(0) = 0$, clearly $e(0) = 0$ and $e(\cdot)$ is continuous on $[0, t_1]$. Hence for $i = 1, 2, \ldots$, $e(t-t_i)$ is continuous on $[0, 2t_1]$, $e(\cdot)$, the solution of (44) is also continuous on $[0, 2t_1]$. By iterating this argument we conclude that $e(\cdot)$ is continuous on $[0, \infty)$.

(f) Since by assumption $u \in L^{\infty}(0)$, let $u_M \triangleq \sup_{t \geq 0} |u(t)| < \infty$. Also by assumption $u(t) \to 0$ as $t \to \infty$, therefore for any $\epsilon > 0$, there exists $T_u(\epsilon)$ such that for $t > T_u(\epsilon)$, $|u(t)| < \epsilon$. Since $h \in \mathcal{Q}(0)$, for any $\epsilon > 0$, there exists $T_h(\epsilon)$ such that for $t > T_h(\epsilon)$

$$\int_t^{\infty} |h_a(\tau)| d\tau + \sum_{t_i \in [t, \infty)} |h_i| < \epsilon .$$

Let $t > T_h(\epsilon) + T_u(\epsilon)$, then

$$|y(t)| = \left| \int_0^t h(t-\tau) u(\tau) d\tau \right|$$

$$\leq \int_0^t |h(t-\tau)| |u(\tau)| d\tau$$

$$= \int_0^{t-T_h} |h(t-\tau)| |u(\tau)| d\tau + \int_{t-T_h}^{t} |h(t-\tau)| |u(\tau)| d\tau$$

In the first integral the argument of h varies from T_h to t, therefore, since $u \in L^{\infty}$, the first integral satisfies

$$\int_0^{t-T_h} |h(t-\tau)| |u(\tau)| d\tau \leq \epsilon \, u_M$$

In the second integral the argument of u varies from $t - T_h > T_u$ to t, hence over this interval $|u(t)| < \epsilon$ and the second integral satisfies

$$\int_{t-T_h}^{t} |h(t-\tau)|\,|u(\tau)|\,d\tau \leq \epsilon\,||h||_0$$

Thus we have shown that $t > T_h(\epsilon) + T_u(\epsilon)$ implies

$$|y(t)| \leq \epsilon\,(u_M + ||h||_0)$$

Therefore we have proved that $y(t) \to 0$ as $t \to \infty$.

(g) Set

$$u(t) = [u(t) - u_\infty] + u_\infty 1(t)$$

The first term is bounded and goes to zero, and the second is a multiple of the step. By linearity together with (d) and (f), conclusion (g) follows.

Proof of Corollary 2:

Since z in the present case satisfies the same assumption as u in Theorem 2. Hence, as before, y has a Laplace transform and $\hat{y}(s)$ is obtained from (12)

$$\hat{y}(s) = \frac{1}{1 + k\hat{g}(s)} \cdot \hat{z}(s) = \hat{z}(s) - \hat{h}(s)\hat{z}(s) \qquad (45)$$

Note that the second term on the right hand side of (45) has exactly the same properties as those of (40). Therefore (a) (b) and (c) of this corollary are the direct consequences of Theorem 2 and (45). The proof of (d) is given in [2].

Proof of Corollary 3:

Since the system is linear, corollary 3 is a direct consequence Theorem 2, corollary 2 and superposition.

Proof of Theorem 3.

Since $z \equiv 0$, if we multiply $\epsilon^{-\sigma t}$ to both sides of (12) we obtain

$$y'(t) = \begin{cases} k[g' * (u' - y')](t) & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0 \end{cases} \tag{46}$$

Where $y'(t) = \epsilon^{-\sigma t} y(t)$ and same formulas for u', g'.

By the same argument as in the proof of Theorem 1, we obtain the Laplace transform of y',

$$\hat{y}'(s) = \frac{k\hat{g}'(s)}{1 + k\hat{g}'(s)} \hat{u}'(s) \tag{47}$$

Observe that $\hat{g}'(s) = \hat{g}(s+\sigma)$. Hence the assumption $\inf\limits_{\mathrm{Re}\, s \geq \sigma} |1 + k\hat{g}(s)| > 0$ implies that $\inf\limits_{\mathrm{Re}\, s \geq 0} |1 + k\hat{g}'(s)| > 0$. By definition, $u \in \mathcal{Q}(\sigma)$ and

$u \in L^{\infty}(\sigma)$ etc. imply $u' \in \mathcal{Q}(0)$ and $u' \in L^{\infty}(0)$ etc. and similarly for g and y. Therefore the results of this theorem follow immediately from Theorem 1, Theorem 2 and the definition.

REFERENCES

[1] Desoer, C. A., "A General Formulation of the Nyquist Criterion," IEEE Trans. on Circuit Theory, vol. CT-12, No. 2, June 1965, pp. 230-234.

[2] Zadeh, L. A. and C. A. Desoer, "Linear System Theory," 1963; McGraw-Hill Book Co., New York.

[3] Nyquist, H., "Regeneration Theory," Bell System Tech. J., vol. 2, Jan. 1932, pp. 126-147.

[4] Schwartz, L., "Théorie des Distributions (Revised Edition)," Hermann Co., Paris, 1966.

[5] Zemanian, A. H., "Distribution Theory and Transform Analysis," McGraw Hill Book Co., New York, 1965.

[6] Lavoine, J., "Calcul Symbolique," CNRS, Paris, 1959.

[7] Hille, E. and R. S. Phillips, "Functional Analysis and Semi-Groups," (Revised Edition) AMS, Providence, R. I., 1957 [See especially p. 150. Note that our Algebra $\mathcal{A}(\sigma)$ is in their notation $L(\epsilon^{-\sigma t}) + A(\epsilon^{-\sigma t})$]

[8] Zames, G. "On the Input-Output Stability of Time-Varying Nonlinear Feedback Systems," Part I, IEEE Trans. on Automatic Control, vol. AC-11, No. 2, April 1966, pp. 228-238. (Theorem 3, p. 235, in particular).

LIST OF FOOTNOTES

† For a first reading the reader may assume $\sigma = 0$ in the following.

§ $\mathcal{D}'_+$ is the space of distributions whose support are contained in $0 \leq t < \infty$ [4, p. 172; 5, p. 168].

‡ In the notation of Hille and Phillips [7], our algebra $\mathcal{Q}(0)$ is denoted by $L(1(\cdot)) + A(1(\cdot))$.

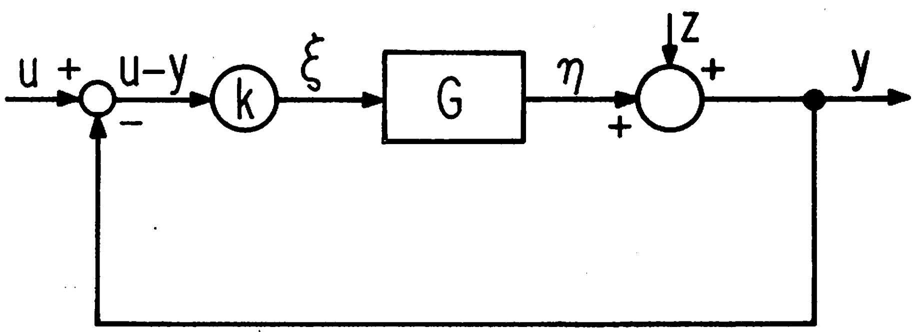

Fig. 1. System under consideration.

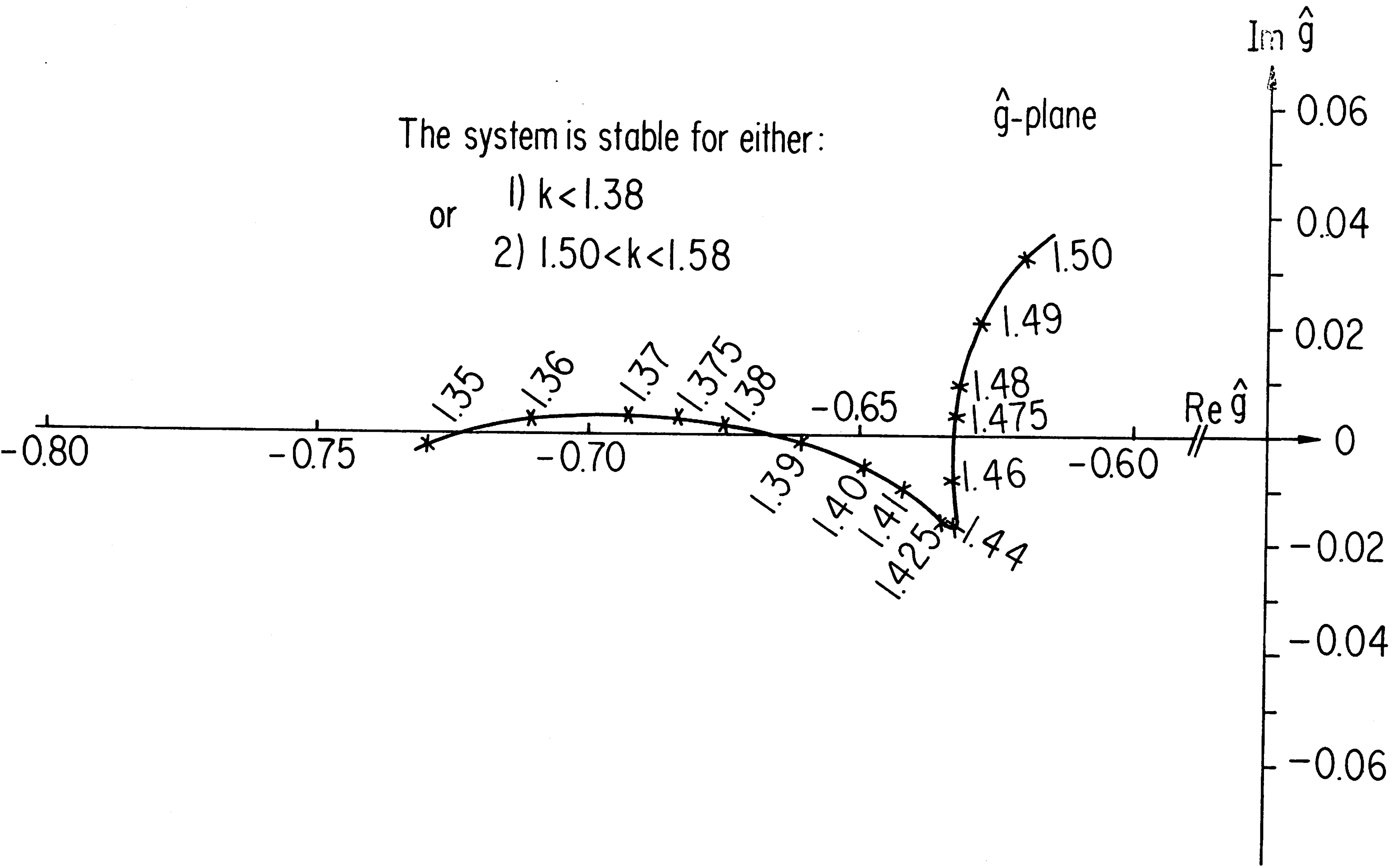

Fig. 2. Important part of the Nyquist diagram of the system considered in the example.