Copyright © 1967, by the author(s).
All rights reserved.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission.

HOMOGENEOUS GAUSS-MARKOV RANDOM FIELDS

by

E. Wong

Memorandum No. ERL-M239

20 November 1967

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

1. INTRODUCTION

In this paper we consider Gaussian random fields which are: (a) homogeneous with respect to the motions of an n-dimensional space of constant curvature, and (b) Markovian in the sense of Lévy [1]. The principal result of this paper is the characterization of such random fields in terms of their covariance functions. We recall that in one dimension a similar question has the very simple answer that the covariance function of a stationary Gauss-Markov process must be an exponential. The answer in the n-dimensional case is nearly as simple, and will be given in this paper.

Let $(\Omega, \mathcal{Q}, \mathcal{P})$ be a fixed probability space, and let $\{x(\omega, z),\ \omega \in \Omega,\ z \in V_n\}$ be a family of Gaussian random variables with an n-dimensional parameter space V_n. We shall only consider three cases: (a) $V_n = R^n$, Euclidean space. (b) $V_n = S^n$, sphere. (c) $V_n = H^n$, hyperbolic space. Let $G(V_n)$ be the full group of motions in V_n which preserve distances. Suppose that for any finite set $A = \{z_i\} \subset V_n$,

$$\{x(\cdot, z_i),\ z_i \in A\} \text{ and } \{x(\cdot, g\, z_i),\ z_i \in A\}$$

have the same distribution whenever $g \in G(V_n)$. Then we say $\{x(\cdot, z), z \in V_n\}$ is a homogeneous random field.

Markovian property in higher dimensions was introduced by Lévy [1] in connection with Brownian motion. Let ∂D be a smooth closed

---

The research reported herein was supported by the U.S. Army Research Office -- Durham under Grant DA-ARO-D-31-124-G776 and Contract DAHCO4-67-C-0046.

The author is with the Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, California.

surface of dimension n - 1 in V_n, separating V_n into a bounded part D^-, and a possibly unbounded part D^+. A random field $\{x(z), z \in V_n\}$ is said to be Markovian of degree $\le p + 1$, if for any such ∂D every approximation $\hat{x}(z)$ to $x(z)$ in a neighborhood of ∂D which satisfies

$$|\hat{x}(z) - x(z)| = o\ (\delta^p) \quad \delta = \text{distance } (z, \partial D)$$

also has the property that given $\hat{x}(\cdot)$, $x(z)$ and $x(z')$ are independent whenever $z \in D^-$ and $z' \in D^+$.

A random field is Markovian of degree p, if it is Markovian of degree $\le$ p, but not $\le$ p - 1. In this paper we are primarily concerned with Markovian fields of degree 1. For this special case it is more convenient to define the Markovian property by: Given $\{x(z), z \in \partial D,\}$ $x(z), z \in D^-$ and $x(z)$, $z \in D^+$ are independent. If $x(z)$ has continuous sample functions, this definition clearly reduces to that of Lévy. This latter definition is more convenient when we have occasion later to consider the possibility of defining Markovian property for generalized random fields.

Since Gaussian distributions are uniquely determined by second order properties, whether a Gaussian random field is Markovian or not is completely determined by its covariance function. While it would be nice to give a necessary and sufficient condition on the covariance function for a Gaussian random field to be Markovian, we are able to do this only when the random field is homogeneous.

2. SECOND-ORDER PROPERTIES

There is no essential loss of generality in assuming that V_n has curvature 0, 1, and -1 corresponding to R^n, S^n, H^n respectively. With the assumption we can adopt a coordinate system $(\varphi_1, \ldots, \varphi_{n-1}, r)$, $\varphi = (\varphi_1, \ldots, \varphi_{n-1}) \in S^{n-1}$, $r \in [0, \infty)$ for R^n, H^n, and $r \in [0, \pi)$ for S^n. We express the Riemannian metric in the form of the differential arc length ds

$$(1) \qquad ds^2 = dr^2 + g^2(r) \sum_{i=1}^{n-1} \left(\prod_{k=i+1}^{n-1} \sin^2 \varphi_k \right) d\varphi_i^{\,2}$$

where $g(r) = r$, $\sin r$, $\sinh r$ for R^n, S^n and H^n respectively. The length of a sectionally smooth (piecewise differentiable in terms of coordinates) curve is found by integrating ds along the curve. The distance $d(z_1, z_2)$ between two points $z_1, z_2 \in V_n$ is the infimum of the lengths of all sectionally smooth curves connecting z_1 and z_2. It can be shown that for the three cases being considered, we have

$$(2) \quad d((\varphi, r), (\varphi', r')) = \begin{cases} \sqrt{r^2 + r'^2 - 2rr' \cos \theta(\varphi, \varphi')} \\ \cos^{-1}[\cos r \cos r' + \sin r \sin r' \cos \theta(\varphi, \varphi')] \\ \cosh^{-1}[\cosh r \cosh r' - \sinh r \sinh r' \cos \theta(\varphi, \varphi')] \end{cases}$$

for R^n, S^n and H^n respectively, where $\theta(\varphi, \varphi')$ is the spherical distance between φ and φ' on S^{n-1}.

Consider the full group G of one-to-one differentiable mappings of V_n onto itself which preserve distances. G acts transitively on V_n, i.e., taking any point into any other point. Hence, if we let K be the maximal subgroup leaving $(\cdot, 0)$ invariant, then V_n can be identified with the homogeneous coset space G/K. For a homogeneous Gaussian random field $\{x(\cdot, z), z \in V_n\}$, we have

$$E \quad x(\cdot, z) = E\, x(\cdot, gz)$$

for all $g \in G$. Hence, $E\, x(\cdot, z)$ = constant which we shall assume to be zero. Similarly, whenever $g \in G$

$$E\, x(\cdot, z) x(\cdot, z_o) = E\, x(\cdot, gz) x(\cdot, gz_o)$$

Since G acts transitively on V_n, there always exists g taking z_o into $(\cdot, 0)$ and z into $(0, d(z, z_0))$. Thus,

$$E\, x(\cdot, z)\, x(\cdot, z_o) = R(d(z, z_o)) \tag{3}$$

Analogous to Bochner's theorem in one dimension, the class of continuous covariance functions of the form of (3) can be put into a one-to-one correspondence with the class of all bounded non-decreasing functions defined on $[0, \infty)$ in the case of R^n and H^n, and the class of all non-negative functions defined on the integers in the case of S^n. This is done via a spectral representation for $R(\cdot)$. Now, let Δ denote the Laplace-Beltrami operator,

(4) $$\Delta(V_n) = \frac{1}{g^{n-1}(r)} \frac{\partial}{\partial r} \left[g^{n-1}(r) \frac{\partial}{\partial r} \right] + \frac{1}{g^2(r)} \Delta(S^{n-1})$$

where $\Delta(S^{n-1})$ can be recursively generated

(5) $$\Delta(S^n) = \frac{1}{\sin^{n-1}(\varphi_n)} \frac{\partial}{\partial \varphi_n} \left[\sin^{n-1}(\varphi_n) \frac{\partial}{\partial \varphi_n} \right] + \frac{1}{\sin^2 \varphi_n} \Delta(S^{n-1})$$

It is well known that Δ commutes with any g in G, and every differential operator commuting with G is a polynomial in Δ with constant coefficients. Let $L^2(S^{n-1})$ denote the set of all square-integrable functions on S^{n-1} (with respect to the uniform measure). Under the actions of the group $G(S^{n-1})$, $L^2(S^{n-1})$ breaks up into a direct sum of orthogonal invariant subspaces $H_0 \oplus H_1 \oplus H_3 \oplus \ldots$ with

$$1 = \dim(H_0) < \dim(H_1) \leq \ldots\ldots$$

Each H_m can be provided with an orthonormal basis $\{h_{m\ell}\}$, $\ell \leq \dim(H_m)$ so that

(6) $$\int_{S^{n-1}} h_{m\ell}(\varphi)\, h_{pk}(\varphi)\, dO = \delta_{mp}\delta_{k\ell}$$

$$\left(\begin{array}{l} dO = \text{uniform distribution} \\ \int_{S^{n-1}} dO = 1 \end{array} \right)$$

The functions $h_{m\ell}$ are the spherical harmonics which satisfy

(7) $$\Delta_{(S^{n-1})} h_{m\ell} = - n(m+n-2)\, h_{m\ell} \,.$$

Let Λ_m be the set of λ values for which

(8) $$\Delta_m \psi_m(\lambda, r) = - \lambda\, \psi_m(\lambda, r)$$

$$\Delta_m = \frac{1}{g^{n-1}(r)} \frac{d}{dr} \left[g^{n-1}(r) \frac{d}{dr} \right] - \frac{m(m+n-2)}{g^2(r)}$$

has a bounded solution. The eigen function $\psi_m(\lambda, r)$ is unique up to normalization. Then, $h_{m\ell}(\varphi)\psi_m(\lambda, r)$ satisfy $\Delta\psi = - \lambda\, \psi$. Hence, so do finite or suitably convergent sums (over $m\ell$) of such products. Indeed, $\{h_{m\ell}(\varphi)\psi_m(\lambda, r),\ m \geq 0,\ \ell \leq \dim(H_m)\}$ spans the space of solutions of $\Delta\psi = - \lambda\, \psi$. Now, $h_o(\varphi) = 1$, therefore $\Delta\psi_o(\lambda, r) = - \lambda\, \psi_o(\lambda, r)$. Since Δ commutes with G, we also have for any fixed z'

(9) $$\Delta\, \psi_o(\lambda, d(z, z')) = - \lambda\, \psi_o(\lambda, d(z, z'))$$

Indeed, with $\psi_o(\lambda, 0) = 1$ and a suitable normalization for $\psi_m(\lambda, r)$ $m \geq 1$, we have

(10) $$\psi_o(\lambda, d(z, z')) = \sum_{m=0}^{\infty} \sum_{\ell \leq \dim(H_m)} h_{m\ell}(\varphi) h_{m\ell}(\varphi') \psi_m(\lambda, r) \psi_m(\lambda, r') \,.$$

It is obvious that $\psi_o(\lambda, d(z, z'))$ is non-negative definite. Hence, positive sums of the form $\sum_{\lambda_i \in \Lambda_o} F_i\, \psi_o(\lambda_i, d(z, z'))$ are non-negative definite, and so are limits of convergent sequences of such sums. Conversely, any continuous covariance function can be approximated by a sequence of such sums. Putting these ideas together yields the spectral representation theorem.

Theorem 1 ([2, 3])

$R(d(z, z'))$ is a continuous covariance function if and only if it can be represented in the form

$$(11) \qquad R(d(z, z')) = \int_{\Lambda_o} \psi_o(\lambda, d(z, z'))\, F(d\lambda)$$

where $F(\cdot)$ is a bounded non-decreasing function defined on Λ_o.

3. HOMOGENEOUS GAUSS-MARKOV FIELDS

Equations (10) and (11) show that a homogeneous Gaussian field $\{x(\cdot, \varphi, r),\ (\varphi, r) \in V_n\}$ has a representation

$$(12) \qquad x(\cdot, \varphi, r) = \sum_{m=0}^{\infty} \sum_{\ell \leq \dim(H_m)} h_{m\ell}(\varphi)\, x_{m\ell}(\cdot, r)$$

where $\{x_{m\ell}(\cdot, r)\}$ are independent Gaussian one-dimensional processes, and

$$(13) \qquad E\, x_{m\ell}(\cdot, r) x_{pk}(\cdot, r') = \delta_{mp} \delta_{\ell k} \int_{\Lambda_m} \psi_m(\lambda, r) \psi_m(\lambda, r') F(d\lambda)$$

Lemma Let $x(\cdot, r, \varphi)$ be a homogeneous Gauss-Markov random field. Then $\{x_{m\ell}(\cdot, r)\}$ defined by

$$(14)\qquad x_{m\ell}(\cdot, r) = \int_{S^{n-1}} h_{m\ell}(\varphi)\, x(\cdot, \varphi, r)\, dO$$

is a set of independent Gauss-Markov processes in one dimension, and there exist functions $f_m(r)$, $g_m(r)$ such that

$$(15)\qquad E\, x_{m\ell}(r) x_{m\ell}(r') = \int_{\Lambda_m} \psi_m(\lambda, r)\psi_m(\lambda, r')F(d\lambda)$$

$$= f_m(\max(r, r'))g_m(\min(r, r'))$$

<u>Proof:</u> We need only to prove that $x_{m\ell}(r)$ are Markov, i.e., that whenever $r > r' > r_o$, $x_{m\ell}(r)$ and $x_{m\ell}(r_o)$ are independent given $x_{m\ell}(r')$. Since for different m and ℓ, $x_{m\ell}(r)$ are independent processes, we need only to prove that $x_{m\ell}(r)$ and $x_{m\ell}(r_o)$ are independent given $x_{pk}(r')$ for all p, k. But given $x_{pk}(r')$ for all p, k is the same as given $x(\varphi', r')$ for all $\varphi' \in S^{n-1}$. Thus, what needs to be proved is the independence of $x_{m\ell}(r)$ and $x_{m\ell}(r_o)$ given $\{x(\varphi', r'), \varphi' \in S^{n-1}\}$. But from the definition of a Markovian random field, whenever $r > r' > r_o$, $x(\varphi, r)$ and $x(\varphi_o, r_o)$ are independent given $\{x(\varphi', r'), \varphi' \in S^{n-1}\}$. The proof of the Markovian nature of $x_{m\ell}(r)$ is completed by noting (14). Finally, the form given by (15) is the required form for the covariance function of a one-dimensional Gauss-Markov process [4].

We are now in a position to state a necessary and sufficient condition on the covariance function for a homogeneous Gaussian random field to be Markovian.

Theorem 2 Let $\{x(\cdot, z), z \in V_n\}$ be a homogeneous Gaussian random field with a continuous covariance function. Then, for $x(\cdot, z)$ to be Markovian, it is necessary and sufficient that

(16) $$E\, x(\cdot, z)x(\cdot, z') = C\, \psi_o(\lambda_o, d(z, z')),\ C > 0,\ \lambda_o \in \Lambda_o$$

Proof: Necessity. From (15) we have

(17) $$\int_{\Lambda_o} \psi_o(\lambda, r)\psi_o(\lambda, r')F(d\lambda) = f_o(r)g_o(r'),\ r > r'$$

For a fixed $r > r'$ it is easy to show that $\int_{\Lambda_o} \lambda\, \psi_o(\lambda, r)\psi_o(\lambda, r')F(d\lambda)$ is a convergent integral. Whence

(18) $$g_o(r')\ \Delta_o f_o(r) = \int_{\Lambda_o} \lambda\, \psi_o(\lambda, r)\psi_o(\lambda, r')F(d\lambda) = f_o(r)\ \Delta'_o g_o(r')$$

whenever $r > r'$. This means that

(19) $$\frac{1}{g_o(r')}\ \Delta'_o\, g_o(r') = \frac{1}{f_o(r)}\ \Delta_o f_o(r) = \text{constant}$$

or

(20) $$\Delta_o f_o(r) = \text{constant}\ f_o(r), \qquad r > 0$$

The only bounded solution of (20) is proportional to $\psi_o(\lambda_o, r)$. Since

(21) $$E\, x(\cdot, z)x(\cdot, z') = \int_{\Lambda_o} \psi_o(\lambda, d(z, z'))F(d\lambda) = g_o(0)f_o(d(z, z'))$$

(16) follows and the proof for necessity is complete.

Sufficiency is rather trivial, because for a Gaussian random field with covariance function of the form (16) is degenerate in the following sense: Given smooth closed surface ∂D,

$$(22) \qquad x(\cdot, z) = E \left\{ x(\cdot, z) \,\middle|\, x(\cdot, z'), z' \in \partial D \right\}$$

with probability 1 for all $z \in V_n$. Hence, $x(\cdot, z)$ is Markovian, but only in a trivial sense.

4. GENERALIZED MARKOVIAN FIELDS

In this section we shall show that it is possible to define the Markovian property for certain generalized random fields, and give a necessary and sufficient condition for a homogeneous Gaussian generalized random field on R^n to be Markovian. This generalizes theorem 2 for R^n. Non-degenerate examples of homogeneous generalized Gauss-Markov fields do exist, and represent natural generalizations of the Ornstein-Uhlenbeck process in one dimension.

Let $\mathcal{S}(R^n)$ denote the Schwartz space of real-valued C^∞ functions of rapid descent. That is, $\mathcal{S}$ contains all real-valued functions f on R^n for which there exist finite constants C_{mk} such that

$$(23) \qquad \sup_{z \in R^n} |z|^m |D^k f(z)| \le C_{mk}$$

$$z = (z_1, z_2, \ldots, z_n), \quad k = (k_1, \ldots, k_n), \quad D^k = \frac{\partial^{k_1 + \ldots + k_n}}{\partial z_1^{k_1} \ldots \partial z_n^{k_1}}$$

Convergence in $\mathcal{S}$ of a sequence $\{f_\nu\}$ means

(a) $f_\nu \in \mathcal{S}$ for each ν

(b) $\sup_{z \in R^n} |z|^m |D^k f_\nu(z)| \le C_{mk}$ independent of ν

(c) For each k $\{D^k f_\nu\}$ converges uniformly on every compact set in R^n.

Let $\mathcal{H}$ be a Hilbert space of real Gaussian random variables with zero mean. We shall define a real zero-mean Gaussian generalized random field X to be a continuous linear map of $\mathcal{S}$ into $\mathcal{H}$. An isometry $g: R^n \to R^n$ induces a map $T_g: \mathcal{S} \to \mathcal{S}$ by

$$(T_g f)(z) = f(g^{-1} z)$$

The generalized random field X is said to be homogeneous if for all $g \in G(R^n)$ and $f_1, f_2 \in \mathcal{S}$

$$\begin{aligned} E\, X(T_g f_1)\, X(T_g f_2) &= E\, X(f_1) X(f_2) \\ &= B(f_1, f_2) \end{aligned} \tag{24}$$

We shall call B the covariance bilinear functional of X. A bilinear functional B on $\mathcal{S} \times \mathcal{S}$ is the covariance functional of a homogeneous Gaussian generalized random field X, if and only if

$$B(f_1, f_2) = \int_0^\infty \sum_{m,\ell} \hat{f}^{(1)}_{m\ell}(\lambda)\, \hat{f}^{(2)}_{m\ell}(\lambda)\, F(d\lambda) \tag{25}$$

where F is a non-decreasing function of slow growth on $[0, \infty)$, and

$$(26) \qquad \hat{f}_{m\ell}(\lambda) = \int_{S^{n-1}} dO \int_0^\infty dr\, f(\varphi, r)\, h_{m\ell}(\varphi) r^{n-1}\, \psi_m(\lambda, r)$$

$$m = 0, 1, \ldots$$

$$\ell \leq \dim H_m$$

The monotone function F will be called the spectral distribution of X [5]. In terms of $\hat{f}_{m\ell}(\lambda)$ we can write

$$(27) \qquad X(f) = \sum_{m,\ell} \int_0^\infty \hat{f}_{m\ell}(\lambda)\, \hat{x}_{m\ell}(d\lambda)$$

where $\left\{\hat{x}_{m\ell}\right\}$ is a family of independent $\mathcal{H}$-valued Borel measures on $[0, \infty)$ with

$$(28) \qquad E\, \hat{x}_{m\ell}(\Lambda)\, \hat{x}_{pq}(\Lambda') = \delta_{mp} \delta_{\ell q} F(\Lambda \cap \Lambda')$$

We note that the sequence $\left\{X(f_\nu)\right\}$ converges whenever $\left\{\hat{f}_\nu\right\}$ converges in $L^2(dFdO)$ norm. Here, $\hat{f}$ is given by

$$(29) \qquad \hat{f}(\varphi, \lambda) = \sum_{m,\ell} \hat{f}_{m\ell}(\lambda)\, h_{m\ell}(\varphi)$$

Therefore, if we define $\hat{X}$ by $\hat{X}(\hat{f}) = X(f)$, then $\hat{X}$ can be extended to a continuous linear map of $L^2(dFdO)$ into $\mathcal{H}$. In particular, if F is bounded, the corresponding ordinary random field can be recovered by setting

$$(30) \qquad x(\varphi_o, r_o) = \hat{X}(\xi_{\varphi_o, r_o})$$

$$\xi_{\varphi_o, r_o}(\varphi, \lambda) = \sum_{m,\ell} h_{m\ell}(\varphi_o) h_{m\ell}(\varphi) \psi_m(\lambda, r_o)$$

Let ∂D be a smooth n-1 closed surface in R^n and let $d\sigma$ be the differential surface area. For $f \in L^2(\partial D, d\sigma)$ define

$$\tilde{f}_{m\ell}(\lambda) = \int_{\partial D} f(t)\, \psi_m(\lambda, r(t))\, h_{m\ell}(\varphi(t))d\sigma \tag{31}$$

$$t = (t_1, \ldots, t_{n-1}) \quad \text{coordinates for } \partial D,$$

and let

$$\tilde{f}(\varphi,\lambda) = \sum_{m,\ell} h_{m\ell}(\varphi)\, \tilde{f}_{m\ell}(\lambda) \tag{32}$$

Suppose X is such that $\tilde{f} \in L^2(dFdO)$ whenever $f \in L^2(\partial D, d\sigma)$, then we can define

$$\begin{aligned} X_{\partial D}(f) &= \hat{X}(\tilde{f}) \\ &= \sum_{m,\ell} \int_0^\infty \tilde{f}_{m\ell}(\lambda)\, \hat{x}_{m\ell}(d\lambda) \end{aligned} \tag{33}$$

Clearly, $\left\{X_{\partial D}(f),\ f \in L^2(\partial D, d\sigma)\right\}$ serves to represent the surface data on ∂D. Once surface data is defined, Markovian property can again be defined.

Let X be a homogeneous Gaussian generalized random field with spectral distribution F. Suppose that whenever ∂D is a smooth closed n-1 surface in R^n and $f \in L^2(\partial D, d\sigma)$ then $\tilde{f} \in L^2(dFdO)$. Let $\mathcal{H}(\partial D) \subset \mathcal{H}$ denotes the closed linear manifold generated by $\left\{X_{\partial D}(f), f \in L^2(\partial D, d\sigma)\right\}$. We say X is Markovian if for any increasing sequence of nested surfaces

∂D_1, ∂D, ∂D_2,

$$\mathcal{H}(\partial D_2) - P_{\partial D}\mathcal{H}(\partial D_2) \text{ is orthogonal to } \mathcal{H}(\partial D_1)$$

where $P_{\partial D}\mathcal{H}(\partial D_2)$ denotes the image of $\mathcal{H}(\partial D_2)$ under the projection $P_{\partial D}$ on $\mathcal{H}(\partial D)$. In other words, X is Markovian, if given the surface data on ∂D inside and outside are independent since with Gaussian law orthogonality and independence are equivalent. The following result generalizes theorem 2 for R^n:

Theorem 3: Let $\{X(f),\ f \in \mathcal{S}\}$ be a homogeneous Gaussian generalized random filed on R^n with spectral distribution F. A necessary and sufficient condition for X to be Markovian is that

(34) $$\int_0^\infty \psi_o(\lambda, r)\, F(d\lambda) = R(r), \quad r > 0$$

defines a twice-differentiable function on $(0, \infty)$ which satisfies

(35) $$\frac{1}{r^{n-1}} \frac{d}{dr}\left[r^{n-1} \frac{dR(r)}{dr} \right] = \alpha R(r)$$

where α is a constant.

Remark: We note that R(r) need not be bounded, but when it is, the result reduces to that of theorem 2.

Proof: Necessity. Let ∂D be an n-1 sphere with radius r. Since the spherical functions $h_{mn} \in L^2(\partial D, d\sigma)$, we can define

(36) $$x_{m\ell}(r) = X_{\partial D}(h_{m\ell}), \quad m \geq 0, \quad \ell \leq \dim H_m$$

By an argument completely analogous to that of the lemma preceding theorem 2, we can show that $\left\{x_{m\ell}(r),\ 0 \le r < \infty\right\}$ is a family of independent one-dimensional Gauss-Markov processes. Hence, we must have

$$(37) \qquad E\, X_{o1}(r)\, X_{o1}(r_o) = f_o{}'(\max(r, r_o))\, g_o{}'(\min(r, r_o))$$

From (33) and (31), it follows that

$$(38) \qquad X_{o1}(r) = X_{\partial D}(h_{o1})$$

$$= r^{n-1} \int_0^\infty \psi_o(\lambda, r)\, x_{o1}(d\lambda)$$

Therefore, from (28) and (37)

$$(39) \qquad E\, X_{o1}(r) X_{o1}(r_o) = (r, r_o)^{n-1} \int_0^\infty \psi_o(\lambda, r)\psi_o(\lambda, r_o) F(d\lambda)$$

$$= f_o{}'(\max(r, r_o)) g_o{}'(\min(r, r_o))$$

or

$$\int_0^\infty \psi_o(\lambda, r)\psi_o(\lambda, r_o) F(d\lambda) = f_o(\max(r, r_o)) g_o(\min(r, r_o))$$

which is identical to (17). Hence, (19) holds once more and

$$(40) \qquad \Delta_o f_o(r) = \frac{1}{r^{n-1}} \frac{d}{dr} \left[r^{n-1} \frac{df_o(r)}{dr} \right]$$

$$= \text{constant } f_o(r)$$

Because $\psi_o(\lambda, 0) = 1$, $f_o(r) = R(r)$ and (35) follows.

Sufficiency. Assume $R(r) = \int_0^\infty \psi_o(\lambda, r) F(d\lambda)$ satisfies (35). Then,

(41) $$\Delta_z R(|z-z_o|) = \alpha R(|z-z_o|), \quad z \neq z_o$$

For any smooth closed n-1 surface ∂D separating z and z_o, (41) can be treated as an exterior Dirichlet problem with boundary data on ∂D. Let G(z, z') be the Green's function for this Dirichlet problem, then

(42) $$R(|z-z_o|) = \int_{\partial D} H(z,z')\, R(|z'-z_o|)\, d\sigma$$

$$z \in D^+, \quad z_o \in D^- \cup \partial D$$

where $H(z,z') = \partial_n{}' G(z,z')$ is the outward normal derivative of $G(z,z')$ with respect to z' on ∂D. Let $\{\partial D_1, \partial D, \partial D_2\}$ be an increasing family of nested surfaces. Then

(43) $$f_D(z') = \int_{\partial D_2} H(z,z') f(z)\, d\sigma$$

maps $L^2(\partial D_2, d\sigma)$ into $L^2(\partial D, d\sigma)$, so that $X_{\partial D}(f_D)$ is well-defined whenever $f \in L^2(\partial D_2, d\sigma)$. Now, $X_{\partial D}(f_D)$ is the projection of $X_{\partial D_2}(f)$ on $\mathcal{H}(\partial D)$ because $X_{\partial D}(f_D) \in \mathcal{H}(\partial D)$ and

(44) $$E\left[X_{\partial D_2}(f) - X_{\partial D}(f_D)\right] X_{\partial D}(g)$$

$$= \int_{\partial D_2} \int_{\partial D} R(|z-z'|) f(z) g(z')\, d\sigma\, d\sigma'$$

$$- \int_{\partial D} \int_{\partial D} R(|z-z'|) f_D(z) g(z')\, d\sigma\, d\sigma'$$

$$= \int_{\partial D_2} d\sigma \int_{\partial D} d\sigma' f(z) g(z') \left[R(|z-z'|) - \int_{\partial D} H(z,z'') R(|z''-z'|)\, d\sigma''\right]$$

$$= 0$$

Similarly, we can show

$$(45) \qquad E\left[X_{\partial D_2}(f) - X_{\partial D}(f_D)\right] X_{\partial D_1}(g) = 0$$

$$g \in L^2(\partial D_1, d\sigma)$$

Therefore, $\left[X_{\partial D_2}(f) - P_{\partial D} X_{\partial D_2}(f)\right]$ is orthogonal to $X_{\partial D_1}(g)$ for every $g \in L^2(\partial D_1, d\sigma)$. This proves that X is Markovian. This proof for sufficiency parallels closely the arguments of McKean [6].

Equation (35) can be readily solved. Corresponding to a non-negative F measure in (34), there are only two possible forms for R. These are

$$(a) \qquad R(r) = A \frac{J_{n/2-1}(\nu_o r)}{(\nu_o r)^{n/2-1}}$$

$$(b) \qquad R(r) = A \frac{K_{n/2-1}(\nu_o r)}{(\nu_o r)^{n/2-1}}$$

Case (a) corresponds to an $F(\lambda)$ which has a single jump at $\lambda = \nu_o^2$, and was already covered by theorem 2. Case (b) corresponds to an unbounded F

$$(46) \qquad F(d\nu^2) = \frac{A}{\nu_o^n} \frac{\nu^{n-1} d\nu}{1 + \left(\frac{\nu}{\nu_o}\right)^2}$$

It is interesting to note that n = 1, which has been excluded from our discussion so far, corresponds to a bounded spectral distribution. One readily recognizes that in that case

$$R(\tau) = \frac{A}{\nu_o} \int_0^\infty \cos \nu\tau \; \frac{1}{1 + \left(\frac{\nu}{\nu_o}\right)^2} \, d\nu$$

$$= \frac{\pi}{2} A \, e^{-\nu_o |\tau|}$$

which is the well-known covariance function for the Ornstein-Uhlenbeck process.

Theorem 3 can be readily generalized to include S^n. It is probably also true for H^n, although we have no proof of that.

REFERENCES

1. P. Lévy, "A Special Problem of Brownian Motion, and a General Theory of Gaussian Random Function," Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley, Vol. 2, pp. 133-175, 1956.

2. A. M. Yaglom, "Second-Order Homogeneous Random Fields," Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley, Vol. 2, pp. 593-620, 1961.

3. M. A. Naimark, Normed Rings, P. Noordhoff N. V., Groningen, (See in particular Art. 31), 1959.

4. J. L. Doob, Stochastic Processes, John Wiley and Sons, Inc., New York, p. 234, 1953.

5. Kiyoshi Ito, "Isotropic Random Currents," Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley, Vol. 2, pp. 125-132, 1956.

6. H. P. McKean, Jr., "Brownian Motion with a Several Dimensional Time," Theory of Probability and its Applications, Vol. 8, pp. 335-365, 1963.