EXISTENCE OF SADDLE POINTS IN DIFFERENTIAL GAMES

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1. Introduction. We consider games in which there are two players I and II whose respective states $x(t) \in R^n$, $y(t) \in R^m$ at time t obey the differential equations (1) and (2) respectively.

(1) $\dot{x}(t) = f(x(t), u(t), t)$

(2) $\dot{y}(t) = g(y(t), v(t), t)$

The control functions u and v are constrained by $u(t) \in U$ $v(t) \in V$ where $U \subset R^p$, $V \subset R^q$ are fixed compact subsets. The game starts at time $t = 0$ in some specified initial states $x(0) = x_0$, $y(0) = y_0$ and ends at a specified time T, at which instant I receives from II a certain amount -- the payoff. We consider two kinds of payoff. The payoff of the first kind is the value of a functional $\mu(x, y)$ where x and y are the trajectories of the two players. The payoff of the second kind is the smallest time t for which the triple $(x(t), y(t), t)$ belongs to a specified closed subset

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$F \subset R^n \times R^m \times R$ where it is assumed that $R^n \times R^m \times \{T\} \subset F$ and $T < \infty$. At each time t player I selects a control $u(t) \in U$ based upon his observations of the trajectory of II up to time t in such a way as to maximize the payoff; conversely at each time t player II selects a control $v(t) \in V$ based upon his observations of $x(\tau)$, $0 \leq \tau \leq t$, in such a way as to minimize the payoff. Games with payoff of the first kind have been called games of prescribed duration [1], while games with payoff of the second kind have been called pursuit-evasion games (player I is the evader, II is the pursuer). Now it is difficult to make precise the notion of a strategy for the players which takes into account the information available to them at each instant of time. In this paper we shall propose a precise definition of a strategy (which agrees with our intuition) and we justify it by demonstrating the existence of a saddle point. Our definition is an extension of that given in [2] in a direction suggested by Roxin [3].

Whereas the technique that we use to prove the saddle-point theorems (Theorems 7, 8, 9) is borrowed to a large extend from Fleming [4], the spirit of this paper is closer to the approach of Ryll-Nardzewski [5]. In the next section we state standard assumptions on the systems (1) and (2) which guarantee compactness of the space of trajectories of the two players. In Section 3 we define classes of strategies with differing information patterns and prove an important (although easy) result which allows us to compare these different classes of strategies. In Section 4 we use this result to give a very simple proof of Fleming's theorem for a payoff of the first kind, namely we show that the optimal payoff for the majorant and minorant games (see [4]) converge to the same limit V_F as the discrepancy in the information patterns vanishes. In Section 5 we propose our definition of the game and show existence of saddle-points for a payoff of the first kind (Theorem 7). The value of the game agrees with that of Fleming. As a corollary to this result in Section   we obtain existence of saddle-point for payoffs of the second kind. In Section 7 we give one example which seems to show that our definition cannot be made more attractive.

2. Conditions on the differential systems. We make the following assumptions on the differential systems (1). Corresponding assumptions are made (but not stated) regarding (2).

(i) For each fixed t, f is continuous in (x, u) for all $(x, u) \in R^n \times U$

(ii) There is a measurable function k, integrable on finite intervals, such that for every $u \in U$ and $x, \hat{x}$ in R^n,

$$|f(x, u, t) - f(\hat{x}, u, t)| \leq k(t)\,|x - \hat{x}|$$

(Here and throughout $|\ |$ denotes Euclidean norm in R^n or R^m)

(iii) There are positive numbers M and N, and a measurable function ℓ, integrable on finite intervals such that for every x in R^n, and u in U,

$$|f(x,u,t)| \leq \ell(t)(M+N)$$

and finally

(iv) Convexity condition: For every x in R^n, t in R, the set

$$f(x,U,t) = \{f(x,u,t) \mid u \in U\}$$

is convex.

A measurable function $u(v)$ is said to be an admissible control if $u(t) \in U(v(t) \in V)$ for all t. A solution x of (1) (y of (2)) is said to be an admissible trajectory if it arises from an admissible control.

<u>Definition:</u> Let $X_T(x_0)$ denote the set of all admissible trajectories x of (1) which are defined on $[0,T]$ and which start at x_0 at time 0 i.e., $x(0) = x_0$. Similarly we define $Y_T(y_0)$.

We consider $X_T(x_0)$ as a subset of the Banach spaces C_T^n -- the space of all continuous functions from $[0,T]$ into R^n under the max norm. Similarly $Y_T(y_0)$ is a subset of C_T^m. The next result is well-known (see for example [6] or [7]); the first part is a consequence of the assumption that the sets $f(x,U,t)$ and $g(y,V,t)$ are convex whereas the second part follows from the assumption that f,g are Lipschitz.

<u>Theorem 1.</u> (i) If $X_0 \subset R^n$ and $Y_0 \subset R^m$ are compact then

$$\bigcup_{x_0 \in X_0} X_T(x_0) \subset C_T^n \quad \text{and} \quad \bigcup_{y_0 \in Y_0} Y_T(y_0) \subset C_T^m$$

are compact.

(ii) $X_T(\cdot)$, $Y_T(\cdot)$ are continuous functions of their arguments. (Here continuity is with respect to the Hansdorff metric.)

Let X_0, Y_0 be compact sets and define $X_T = \bigcup_{x_0 \in X_0} X_T(x_0)$, $Y_T = \bigcup_{y_0 \in Y_0} Y_T(y_0)$. Let $u_0 \in U$ and $v_0 \in V$ be fixed. Let $\delta \geq 0$. Suppose that $x \in X_T$ is obtained from an admissible control u. Let $\Pi_\delta^X(x) \in X_T$ be the solution of (1) corresponding to the control u_δ where $u_\delta(t) = u_0$ $0 \leq t \leq \delta$ and $u_\delta(t) = u(t-\delta)$, $\delta < t \leq T$, and the initial condition $x(0)$ at 0. Similarly define the function $\Pi_\delta^Y : Y_T \to Y_T$. Note that if $x \in X_T(x_0)$ then $\Pi_\delta^X(x) \in X_T(x_0)$ and if $y \in Y_T(y_0)$ then $\Pi_\delta^Y(y) \in Y_T(y_0)$. The proof of the next result requires arguments which are standard in the theory of differential equations. Hence the proof is omitted.

Theorem 2. Let $$\mathcal{E}(\delta) = \mathrm{Sup}\{\, \|x - \Pi_\delta^X x\| \;\big|\; x \in X_T\} + \mathrm{Sup}\{\, \|y - \Pi_\delta^Y y\| \;\big|\; y \in Y_T\}$$

Then $\lim_{\delta \to 0} \mathcal{E}(\delta) = 0$. (Here and throughout $\| \;\|$ denotes norm in the Banach spaces C_T^n, C_T^m).

3. Strategies. Let x_0, y_0 be specified initial states. Throughout this paper the symbol δ (with or without subscripts) represents a number which is equal to $1/2^n$ for some integer $n \geq 0$. We now define three classes of strategies $A_\delta(x_0, y_0) = \{\alpha_\delta\}$, $A(x_0, y_0) = \{\alpha\}$, and $A^\delta(x_0, y_0) = \{\alpha_\delta\}$ for player I and three classes of strategies $B_\delta(x_0, y_0) = \{\beta_\delta\}$,

$B(x_0, y_0) = \{\beta\}$, and $B^{\delta}(x_0, y_0) = \{\beta^{\delta}\}$ for player II.

__Definition.__ (i) $A_{\delta}(x_0, y_0)$ is the set of all functions $\alpha_{\delta} : Y_T(y_0) \to X_T(x_0)$ such that if y, $\hat{y}$ are in $Y_T(y_0)$ with $y(\tau) = \hat{y}(\tau)$ for $0 \leq \tau \leq i\delta T$ then $\alpha_{\delta} y(\tau) = \alpha_{\delta} y(\tau)$ for $0 \leq \tau \leq (i+1)\delta T$; $i = 0, 1, \ldots, \frac{1}{\delta} - 1$.

(ii) $A^{\delta}(x_0, y_0)$ is the set of all functions $\alpha^{\delta} : Y_T(y_0) \to X_T(x_0)$ such that if y, $\hat{y}$ are in $Y_T(y_0)$ with $y(\tau) = \hat{y}(\tau)$ for $0 \leq \tau \leq i\delta T$ then $\alpha^{\delta} y(\tau) = \alpha^{\delta} y(\tau)$ for $0 \leq \tau \leq i\delta T$; $i = 0, 1, \ldots, \frac{1}{\delta}$.

(iii) $A(x_0, y_0)$ is the set of all functions $\alpha : Y_T(y_0) \to X_T(x_0)$ such that if $y, \hat{y}$ are in $Y_T(y_0)$ with $y(\tau) = \hat{y}(\tau)$ for $0 \leq \tau \leq t$ then $\alpha y(\tau) = \alpha \hat{y}(\tau)$ for $0 \leq \tau \leq t$; $0 \leq t \leq T$.

The sets of strategies $B_{\delta}(x_0, y_0)$, $B(x_0, y_0)$ and $B^{\delta}(x_0, y_0)$ are defined in the same way.

It is convenient to regard the strategies for I as subsets of $F(Y_T(y_0), X_T(x_0))$ -- the space of all functions from $Y_T(y_0)$ into $X_T(x_0)$ with the topology of pointwise convergence. Similarly we regard B_{δ}, B, B^{δ} as subsets of the topological space $F(X_T(x_0), Y_T(y_0))$. By the Tychonoff theorem $F(X_T(x_0), Y_T(y_0))$, $F(Y_T(y_0), X_T(x_0))$ are compact.

The first part of the next result is a direct consequence of the definition while the proof of the second part is a duplication of the arguments in Lemma 4.1 of [2].

__Theorem 3.__ If $\delta_1 \leq \delta_2$ then

(i) $A_{\delta_2} \subset A_{\delta_1} \subset A \subset A^{\delta_1} \subset A^{\delta_2}$

and

$$B_{\delta_2} \subset B_{\delta_1} \subset B \subset B^{\delta_1} \subset B^{\delta_2}.$$

(ii) The sets A_δ, A, A^δ are closed and hence compact subsets of $F(Y_T(y_0), X_T(x_0))$. Similarly the sets B_δ, B, B^δ are closed and hence compact subsets of $F(X_T(x_0), Y_T(y_0))$.

Recall the definition of the maps Π_δ^X, Π_δ^Y and the function $\mathcal{E}(\delta)$ in Theorem 2.

Theorem 4. (Approximation Theorem). (i) If $\alpha^\delta \in A^\delta$, $\beta^\delta \in B^\delta$ then $(\Pi_\delta^X \circ \alpha^\delta)$ and $(\alpha^\delta \circ \Pi_\delta^Y)$ belong to A_δ, $(\Pi_\delta^Y \circ \beta^\delta)$ and $(\beta^\delta \circ \Pi_\delta^X)$ belong to B_δ.

(ii) $||\alpha^\delta(x) - (\Pi_\delta^X \circ \alpha^\delta)(x)|| \leq \mathcal{E}(\delta)$, for $\alpha^\delta \in A^\delta$, $x \in X_T(x_0)$ and $||\beta^\delta(x) - (\Pi_\delta^Y \circ \beta^\delta)(y)|| \leq \mathcal{E}(\delta)$, for $\beta^\delta \in B^\delta$, $y \in Y_T(y_0)$.

Proof. (i) is a consequence of the definition while (ii) follows from Theorem 2.

4. Payoff of the first kind; Fleming's Theorem. Let $X_0 \subset R^n$, $Y_0 \subset R^m$ be fixed compact sets. Let $X_T = \bigcup_{x_0 \in X_0} X_T(x_0)$, $Y_T = \bigcup_{y_0 \in Y_0} Y_T(y_0)$. The payoff is a continuous real-valued function μ defined on the compact space $X_T \times Y_T$. Let $x_0 \in X_0$, $y_0 \in Y_0$ be specified initial states. Following Fleming [4], for each δ we define a majorant game $G^\delta(x_0, y_0)$ and a

minorant game $G_\delta(x_0, y_0)$ as follows: In the majorant game, player II picks a strategy $\beta_\delta \in B_\delta(x_0, y_0)$ and then player I picks a strategy $\alpha^\delta \in A^\delta(x_0, y_0)$. The outcome of these choices is a unique pair of trajectories $x \in X_T(x_0)$, $y \in Y_T(y_0)$ such that $\alpha^\delta(y) = x$ and $\beta_\delta(x) = y$. We shall denote these trajectories by $x = x(\alpha^\delta, \beta_\delta)$, $y = y(\alpha^\delta, \beta_\delta)$. The payoff is $\mu(x, y)$. In the minorant game, player I selects first a strategy $\alpha_\delta \in A_\delta(x_0, y_0)$ and then II picks a $\beta^\delta \in B^\delta(x_0, y_0)$. Again the outcome is a unique pair $x \in X_T(x_0)$, $y \in Y_T(y_0)$ such that $\alpha_\delta(y) = x$, $\beta^\delta(x) = y$. We shall denote these trajectories by $x = x(\alpha_\delta, \beta^\delta)$, $y = y(\alpha_\delta, \beta^\delta)$. The payoff is $\mu(x, y)$. Since I tries to maximize and II tries to minimize the payoff we define

$$V^\delta(x_0, y_0) = \operatorname*{Min}_{\beta_\delta \in B_\delta(x_0, y_0)} \; \operatorname*{Max}_{\alpha^\delta \in A^\delta(x_0, y_0)} \mu(x(\alpha^\delta, \beta_\delta), y(\alpha^\delta, \beta_\delta))$$

$$V_\delta(x_0, y_0) = \operatorname*{Max}_{\alpha_\delta \in A_\delta(x_0, y_0)} \; \operatorname*{Min}_{\beta^\delta \in B^\delta(x_0, y_0)} \mu(x(\alpha_\delta, \beta^\delta), y(\alpha_\delta, \beta^\delta))$$

From Theorem 3(i) it follows that

$$V_{\delta_2}(x_0, y_0) \le V_{\delta_1}(x_0, y_0) \le V^{\delta_1}(x_0, y_0) \le V^{\delta_2}(x_0, y_0)$$

whenever $\delta_1 \le \delta_2$. It follows that the two limits $\overline{V}(x_0, y_0) = \lim_{\delta \to 0} V^\delta(x_0, y_0)$ and $\underline{V}(x_0, y_0) = \lim_{\delta \to 0} V_\delta(x_0, y_0)$ exist. From the definition of the strategies it should be clear that an alternate definition of V^δ, V_δ is the following characterization which is closer to that of Fleming [4]

$$V^{\delta}(x_0, y_0) = \underset{y^1 \in Y_1(y_0)}{\text{Min}} \; \underset{x^1 \in X_1(x_0)}{\text{Max}} \; \underset{y^2 \in Y_2(y^1(\delta T))}{\text{Min}} \; \underset{x^2 \in X_2(x^1(\delta T))}{\text{Max}} \cdots$$

$$\underset{y^{1/\delta} \in Y_{1/\delta}(y^{1/\delta -1}((1-\delta)T))}{\text{Min}} \; \underset{x^{1/\delta} \in X_{1/\delta}(x^{1/\delta -1}((1-\delta)T))}{\text{Max}} \mu(x, y) \tag{3}$$

$$V_{\delta}(x_0, y_0) = \underset{x^1 \in X_1(x_0)}{\text{Max}} \; \underset{y^1 \in Y_1(y_0)}{\text{Min}} \cdots$$

$$\underset{x^{1/\delta} \in X_{1/\delta}(x^{1/\delta -1}((1-\delta)T))}{\text{Max}} \; \underset{y^{1/\delta} \in Y_{1/\delta}(y^{1/\delta -1}((1-\delta)T))}{\text{Min}} \mu(x, y) \tag{4}$$

where, $X_1(x_0)$ $(Y_1(y_0))$ is the set of all admissible trajectories $x^1(y^1)$ of (1) ((2)) defined on the interval $[0, \delta T]$ and starting at $x_0(y_0)$; and inductively if $x^i(y^i)$ has been chosen $X_{i+1}(x^i(i\delta T))$ $(Y_{i+1}(y^i(i\delta T)))$ is the set of all admissible trajectories x^{i+1} (y^{i+1}) defined on $[i\delta T, (i+1)\delta T]$ and starting at time $i\delta T$ in the state $x^i(i\delta T)$ $(y^i(i\delta T))$. The outcome (x, y) is defined by $x(t) = x^i(t)$ $(y(t) = y^i(t))$, $(i-1)\delta T \leq t \leq i\delta T$, $i = 1, 2, \ldots, \frac{1}{\delta}$. Since the various sets of trajectories X_i, Y_i are compact and vary continuously with initial conditions (by Theorem 1), and since μ is a continuous function it follows that V^{δ}, V_{δ} are well-defined and vary continuously with their arguments $(x_0, y_0) \in X_0 \times Y_0$.

The next lemma gives two other alternate expressions for V^{δ}, V_{δ}.

Lemma 1.

(i) $$V^{\delta}(x_0, y_0) = \underset{\alpha^{\delta} \in A^{\delta}(x_0, y_0)}{\mathrm{Max}} \quad \underset{\beta_{\delta} \in B_{\delta}(x_0, y_0)}{\mathrm{Min}} \quad \mu(x, y) \tag{5}$$

$$V_{\delta}(x_0, y_0) = \underset{\beta^{\delta} \in B^{\delta}(x_0, y_0)}{\mathrm{Min}} \quad \underset{\alpha_{\delta} \in A_{\delta}(x_0, y_0)}{\mathrm{Max}} \quad \mu(x, y) \tag{6}$$

(ii) $$V^{\delta}(x_0, y_0) = \underset{\beta_{\delta} \in B_{\delta}(x_0, y_0)}{\mathrm{Min}} \quad \underset{x \in X_T(x_0)}{\mathrm{Sup}} \quad \mu(x, \beta_{\delta}(x)) \tag{7}$$

$$V_{\delta}(x_0, y_0) = \underset{\alpha_{\delta} \in A_{\delta}(x_0, y_0)}{\mathrm{Max}} \quad \underset{y \in Y_T(y_0)}{\mathrm{Inf}} \quad \mu(\alpha_{\delta}(y), y) \tag{8}$$

Sketch of Proof: We shall prove (5) and (7). A proof of (5) can be obtained by noting that for any sets W, Z and any real-valued function γ on $W \times Z$, the following equality holds:

$$\underset{z \in Z}{\mathrm{Inf}} \ \underset{w \in W}{\mathrm{Sup}} \ \gamma(z, w) = \underset{s \in S}{\mathrm{Sup}} \ \underset{z \in Z}{\mathrm{Inf}} \ \gamma(s(z), z)$$

where S is the set of all functions s from Z into W. This equality together with the representation (3) of V^{δ} and the definitions of α^{δ}, β_{δ} can then be used to give (5).

Evidently $V^{\delta}(x_0, y_0)$ is at least as large as the right-hand side of (7). On the other hand if $\alpha^{\delta} \in A^{\delta}(x_0, y_0)$, $\beta_{\delta} \in B_{\delta}(x_0, y_0)$ and if $x = x(\alpha^{\delta}, \beta_{\delta})$, $y = y(\alpha^{\delta}, \beta_{\delta})$ is the outcome then

$$(x, y) = (x, \beta_{\delta}(x))$$

and so the right-hand-side of (7) is bigger than V^{δ}.

Following Fleming we propose the following definition:

Definition: The game has a value $V_F(x_0, y_0)$ provided that the two limits $\overline{V}(x_0, y_0) = \lim_{\delta \to 0} V^{\delta}(x_0, y_0)$ and $\underline{V}(x_0, y_0) = \lim_{\delta \to 0} V_{\delta}(x_0, y_0)$ are equal. In that case we define the (Fleming) value of the game: $V_F(x_0, y_0) = \overline{V}(x_0, y_0)$.

Lemma 2. Let $\eta > 0$. Then there is a δ^* such that for all $\delta < \delta^*$ and all $(x_0, y_0) \in X_0 \times Y_0$,

$$0 \leq V^{\delta}(x_0, y_0) - V_{\delta}(x_0, y_0) \leq \eta .$$

Proof. Since μ is continuous on the compact space $X_T \times Y_T$ there is $\mathcal{E}^* > 0$ such that

$$|\mu(x, y) - \mu(\hat{x}, \hat{y})| \leq \eta \tag{9}$$

whenever $||x - \hat{x}|| \leq \mathcal{E}^*$, $||y - \hat{y}|| \leq \mathcal{E}^*$; $x, \hat{x} \in X_T$; $y, \hat{y} \in Y_T$. Let $\delta^* > 0$ be such that for all $\delta < \delta^*$, $\mathcal{E}(\delta) < \mathcal{E}^*$ where $\mathcal{E}(\delta)$ is the function defined in Theorem 4 (ii). Now let $\delta < \delta^*$, $(x_0, y_0) \in X_0 \times Y_0$ be fixed. Let $\alpha^{\delta}_{opt} \in A^{\delta}(x_0, y_0)$ be such that

$$V^{\delta}(x_0, y_0) \leq \mu\,(x(\alpha^{\delta}_{opt}, \beta_{\delta}),\ y(\alpha^{\delta}_{opt}, \beta_{\delta})) \text{ for all } \beta_{\delta} \in B_{\delta}(x_0, y_0) \tag{10}$$

The existence of α^{δ}_{opt} follows from (5). Let $\underline{\alpha}_{\delta} = \Pi^X_{\delta} \circ \alpha^{\delta}_{opt}$. Then $\underline{\alpha}_{\delta} \in A_{\delta}(x_0, y_0)$ by Theorem 4 (i). Let $\beta^{\delta} \in B^{\delta}(x_0, y_0)$ be arbitrary and suppose that $x \in X_T(x_0)$, $y \in Y_T(y_0)$ are such that

$$\underline{\alpha}_\delta(y) = x\,, \qquad \beta^\delta(x) = y\,.$$

Let $\hat{x} = \alpha^\delta_{opt}(y)$, and let $\underline{\beta}_\delta = \beta^\delta \circ \Pi^X_\delta$. Then $x = \Pi^X_\delta(\hat{x})$ and $\underline{\beta}_\delta \in B_\delta$ and furthermore,

$$\alpha^\delta_{opt}(y) = \hat{x}, \quad \underline{\beta}_\delta(\hat{x}) = y\,.$$

It follows from (10) that

$$V^\delta(x_0, y_0) \leq \mu(\hat{x}, y)\,.$$

But $||x - \hat{x}|| = ||\Pi^X_\delta(\hat{x}) - \hat{x}|| \leq \mathcal{E}(\delta) \leq \mathcal{E}*$, so that by (9)

$$V^\delta(x_0, y_0) \leq \mu(x, y) + \eta$$

Since $\underline{\alpha}_\delta \in A_\delta$ and since $\beta^\delta \in B^\delta$ is arbitrary it follows that

$$\begin{aligned} V^\delta(x_0, y_0) &\leq \eta + \underset{\alpha_\delta \in A_\delta}{\text{Max}} \ \underset{\beta^\delta \in B^\delta}{\text{Min}} \ \mu\left(x(\alpha_\delta, \beta^\delta), y(\alpha_\delta, \beta^\delta)\right) \\ &= \eta + V_\delta(x_0, y_0)\,. \end{aligned}$$

The lemma is proved.

__Theorem 5. (Fleming).__ Under the assumptions (of Section 2) on the differential equations (1) and (2),

$$\overline{V}(x_0, y_0) = \underline{V}(x_0, y_0) \tag{11}$$

Furthermore $V_F(\ ,\ )$ is continuous on $X_0 \times Y_0$.

Proof. The equality (11) is a corollary of the preceding lemma whilst the continuity of V_F follows from the fact that V^δ is continuous and the fact that V^δ converges uniformly to $\overline{V}$.

Remarks: The class of systems considered by Fleming is more general than the class treated here since his systems are of the form $x = f(x, u, v)$ i.e., both players control the same object. However the conditions under which he can prove the existence of V_F are more restrictive. Also the class of payoff functions is more restrictive. (This generalization is important in view of the manner in which we consider pursuit-evasion problems). Incidentally this theorem proves a conjecture of Fleming (p. 207, [8]), (at least for the class of systems considered here) namely the function $V(x, T)$ defined in [8] is the same as $V(x, T)$ defined in [4].

5. The Fair Game: Existence of Saddle-points for payoffs of the first kind.

In this section we propose a direct definition of a game. Our definition is in some sense a limit of the games G^δ, G_δ as δ goes to zero. However our formulation is much closer to that of Ryll-Nardzewski [5].

As before let x_0, y_0 be specified initial states. Player I choose a strategy $\alpha \in A(x_0, y_0)$, player II chooses a strategy $\beta \in B(x_0, y_0)$. It would be natural to define the outcome of such choice to be any pair $x \in X_T(x_0)$, $y \in Y_T(y_0)$ such that

$$\alpha(y) = x, \quad \beta(y) = x$$

Unfortunately, the above pair of equations may have either no solution or it may have more than one solution. The existence of a solution (but not uniqueness) can be guaranteed if α, β are required to be continuous functions; but then as we shall show in Section 7 we cannot guarantee existence of optimal strategies. We therefore propose the following definition:

Definition: Let $\alpha \in A(x_0, y_0)$ and $\beta \in B(x_0, y_0)$. A pair $x \in X_T(x_0)$, $y \in Y_T(y_0)$ is said to be an outcome of (α, β) if there is a sequence $x_n \in X_T(x_0)$, $y_n \in Y_T(y_0)$ $n = 1, 2, 3, \ldots$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \alpha(y_n) = x; \ \lim_{n\to\infty} y_n = \lim_{n\to\infty} \beta(x_n) = y.$$

(Evidently if α and β are continuous at y, x respectively then $\alpha(y) = x$, $\beta(x) = y$).

Let $0(\alpha, \beta) = \{(x, y) \mid (x, y) \text{ is an outcome of } (\alpha, \beta)\}$.

Theorem 6. For each $\alpha \in A$, $\beta \in B$, $0(\alpha, \beta)$ is a non-empty closed subset of $X_T(x_0) \times Y_T(y_0)$.

Proof. The closed-ness of $0(\alpha, \beta)$ follows from standard diagonal arguments. We now show that $0(\alpha, \beta)$ is non-empty. Let δ_k, $k = 1, 2, \ldots$ be a sequence decreasing to zero and let $\alpha_{\delta_k} = (\Pi^X_{\delta_k} \circ \alpha) \in A_{\delta_k}$. Let (x_k, y_k) be the pair such that

$$\alpha_{\delta_k}(y_k) = x_k, \quad \beta(x_k) = y_k .$$

Since $X_T(x_0)$, $Y_T(y_0)$ are compact we can assume (taking subsequences if necessary) that there is $x \in X_T(x_0)$, $y \in Y_T(y_0)$ such that

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} \alpha_{\delta_k}(y_k) = x; \quad \lim_{k\to\infty} y_k = \lim_{k\to\infty} \beta(x_k) = y .$$

But $||\alpha_{\delta_k}(y_k) - \alpha(y_k)|| = ||(\Pi^X_{\delta_k} \circ \alpha)(y_k) - \alpha(y_k)|| \leq \mathcal{E}(\delta_k)$

by Theorem 4 (ii). Since $\lim_{k\to\infty} \mathcal{E}(\delta_k) = 0$, the assertion follows.

<u>Definition:</u> For each $\beta \in B(x_0, y_0)$, let $\mu^+(\beta) = \sup_{\alpha \in A(x_0,y_0)} \max_{(x,y) \in 0(\alpha,\beta)} \mu(x,y)$

and for each $\alpha \in A(x_0, y_0)$ let $\mu_-(\alpha) = \inf_{\beta \in B(x_0,y_0)} \min_{(x,y) \in 0(\alpha,\beta)} \mu(x,y)$.

Now let $V^+(x_0, y_0) = \min_{\beta \in B(x_0,y_0)} \mu^+(\beta)$

$$V_-(x_0, y_0) = \max_{\alpha \in A(x_0,y_0)} \mu_-(\alpha)$$

In order to show that the Min and Max in the definition of V^+, V_- actually exist the following result will be helpful.

<u>Lemma 3.</u> $\mu^+(\beta) = \sup_{x \in X_T(x_0)} \mu(x, \beta(x))$ (12)

and $\mu_-(\alpha) = \inf_{y \in Y_T(y_0)} \mu(\alpha(y), y)$

Proof. We prove the first equality. Clearly $\mu^+(\beta)$ is at least as big as the right-hand-side of (12). Now let $\alpha \in A$ and let x, x_n be in $X_T(x_0)$; y, y_n in $Y_T(y_0)$ for $n = 1, 2, \ldots$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \alpha(y_n) = x ; \quad \lim_{n\to\infty} y_n = \lim_{n\to\infty} \beta(x_n) = y .$$

Then,

$$\lim_{n\to\infty} (x_n, \beta(x_n)) = (x, y)$$

It follows that $\mu^+(\beta) \leq \operatorname{Sup}_{x \in X_T(x_0)} (x, \beta(x))$.

Lemma 4. $\mu^+(\beta)$ is a lower semicontinuous function of $\beta \in B(x_0, y_0)$ $\mu_-(\alpha)$ is an upper semicontinuous function of $\alpha \in A(x_0, y_0)$.

Proof: We shall only prove the first half of the assertion since the proof for the second half is analogous. Let z be a real number and let

$$B_z = \{\beta \mid \beta \in B(x_0, y_0),\ \mu^+(\beta) \leq z\}$$

We must show that B_z is closed. Let $\{\beta(k)\}$ be a net in B_z converging to β in B, i.e., for each $x \in X_T(x_0)$ $\lim_k \beta(k)x = \beta(x)$. Let $x \in X_T(x_0)$. Then by definition $\mu(x, \beta(k)x \leq z$ for all k. It follows from the continuity of μ that $\mu(x, \beta(x) \leq z$. Hence $\mu^+(\beta) \leq z$.

Corollary: There is a $\beta^* \in B(x_0, y_0)$, $\alpha^* \in A(x_0, y_0)$ such that

(i) $\mu^{+}(\beta^*) \leq \mu^{+}(\beta), \quad \beta \in B$

$\mu_{-}(\alpha^*) \geq \mu_{-}(\alpha), \quad \alpha \in A$

(ii) $\mu^{+}(\beta^*) = V^{+}(x_0, y_0) = V_F(x_0, y_0) = V_{-}(x_0, y_0) = \mu_{-}(\alpha^*)$ and

(iii) $\displaystyle \operatorname*{Min}_{(x,y) \in 0(\alpha^*, \beta^*)} \mu(x, y) = \operatorname*{Max}_{(x,y) \in 0(\alpha^*, \beta^*)} \mu(x, y)$

<u>Proof.</u> (i) follows from the preceding lemma and the fact that $B(x_0, y_0)$ and $A(x_0, y_0)$ are compact spaces. Again from the same lemma and the definition of V^+ we see that

$$\mu^{+}(\beta^*) = V^{+}(x_0, y_0) = \operatorname*{Min}_{\beta \in B(x_0, y_0)} \ \operatorname*{Sup}_{x \in X_T(x_0)} \mu(x, \beta(x))$$

$$\leq \operatorname*{Min}_{\beta_\delta \in B_\delta(x_0, y_0)} \ \operatorname*{Sup}_{x \in X_T(x_0)} \mu(x, \beta_\delta(x))$$

$$= V^{\delta}(x_0, y_0)$$

where the last equality is the same as Eq. (7). Similarly

$$\mu_{-}(\alpha^*) = V_{-}(x_0, y_0) \geq V_\delta(x_0, y_0)$$

so that (ii) follows from Theorem 5. To prove (iii) it is enough to note that by definition of μ_- and μ^+,

$$\mu_{-}(\alpha^*) \leq \operatorname*{Min}_{(x,y) \in 0(\alpha^*, \beta^*)} \mu(x, y) \leq \operatorname*{Max}_{(x,y) \in 0(\alpha^*, \beta^*)} \mu(x, y) \leq \mu^{+}(\beta^*)$$

and then (iii) follows (ii).

We can now define the fair game and prove the existence of a saddle point. The game G is defined as follows: Player I selects a strategy $\alpha \in A(x, y)$ whilst II independently selects a $\beta \in B(x, y)$. The payoff is given by $\mu(x, y)$ where (x, y) is an arbitrarily chosen pair from $0(\alpha, \beta)$. The saddle-point theorem shows that the value is independent of the arbitrary choice of the outcome.

Theorem 7. (Saddle-Point Theorem) There exists $\alpha^* \in A(x_0, y_0)$, $\beta^* \in B(x_0, y_0)$ such that for all $\alpha \in A(x_0, y_0)$ and all $\beta \in B(x_0, y_0)$,

$$\max_{(x,y) \in 0(\alpha, \beta^*)} \mu(x, y) \leq \max_{(x,y) \in 0(\alpha^*, \beta^*)} \mu(x, y)$$

$$= \min_{(x,y) \in 0(\alpha^*, \beta^*)} \mu(x, y) \leq \min_{(x,y) \in 0(\alpha^*, \beta)} \mu(x, y)$$

Furthermore $\mu(x, y) = V_F(x_0, y_0)$ for all $(x, y) \in 0(\alpha^*, \beta^*)$.

Proof: By the definition of μ^+, μ_- we see that

$$\max_{(x,y) \in 0(\alpha, \beta^*)} \mu(x, y) \leq \mu^+(\beta^*), \quad \mu_-(\alpha^*) \leq \min_{(x,y) \in 0(\alpha^*, \beta)} \mu(x, y)$$

The result now follows from the previous Corollary.

Definition. Given two players I and II with dynamics (1) and (2) respectively, and a continuous payoff μ of the first kind, the (Fleming) value of the game corresponding to initial conditions (x_0, y_0) will be denoted by

$$V_F(\mu ; x_0, y_0)$$

6. Payoff of the second kind: Pursuit-Evasion Games: In this section we consider payoffs of the second kind. Before we define the game we introduce a definition which will be helpful in relating these games to the games considered in the last section.

Let $F \subset R^n \times R^m \times [0, \infty)$ be a non-empty closed set. For each $T < \infty$ define the function $\mu_T : X_T(x_0) \times Y_T(y_0) \to R$ by

$$\mu_T(x, y) = \mathrm{Min}\{ |x(t) - x| + |y(t) - y| + |t - t| \,\Big|\, (x, y, t) \in F, t \in [0, T] \}$$

It is easy to show that μ_T is continuous. Evidently $\mu_T(x, y)$ is non-negative and

$$\mu_T(x, y) = 0 \quad \text{if and only if} \quad (x(t), y(t), t) \in F \quad \text{for some } t\,. \tag{13}$$

We now define the game: There is given a closed set $F \subset R^n \times R^m \times [0, \infty)$ and a $T_{max} < \infty$ such that $(x, y, T_{max}) \in F$ for all $(x, y) \in R^n \times R^m$. The game is played on the fixed time interval $[0, T_{max}]$. Player I (the evader) selects a strategy $\alpha \in A(x_0, y_0)$ whilst II (the pursuer) independently selects a strategy $\beta \in B(x_0, y_0)$. The payoff given by

$$t(x, y)$$

where $(x, y) \in 0(\alpha, \beta)$ is chosen arbitrarily and $t(x, y)$ is the smallest capture time i.e.,

$$t(x, y) = \min\{t \mid (t, x(t), y(t)) \in F\}$$

Player I tries to maximize the payoff while II tries to minimize it. As before we define

$$V_-(x_0, y_0) = \sup_{\alpha \in A(x_0, y_0)} \ \inf_{\beta \in B(x_0, y_0)} \ \inf_{(x, y) \in 0(\alpha, \beta)} t(x, y)$$

$$V^+(x_0, y_0) = \inf_{\beta \in B(x_0, y_0)} \ \sup_{\alpha \in A(x_0, y_0)} \ \sup_{(x, y) \in 0(\alpha, \beta)} t(x, y)$$

<u>Theorem 8.</u> $V_-(x_0, y_0) = V^+(x_0, y_0)$

<u>Proof.</u> Evidently $V_-(x_0, y_0) \leq V^+(x_0, y_0)$. Let $\epsilon > 0$. Then from the definition of V_-, for every strategy α there is a strategy β and a $(x, y) \in (\alpha, \beta)$ such that

$$t(x, y) \leq V_-(x_0, y_0) + \epsilon .$$

i.e., there is a $t \leq T_\epsilon = V_-(x_0, y_0) + \epsilon$ such that

$$(x(t), y(t), t) \in F . \tag{14}$$

Now define the continuous function μ_{T_ϵ} on the set $X_{T_\epsilon}(x_0) \times Y_{T_\epsilon}(y_0)$ as in the beginning of this section, and consider the game defined on the fixed time interval $[0, T_\epsilon]$ with the continuous payoff function μ_{T_ϵ}.

By Theorem 7 this game has a value $V_F(\mu_{T_\epsilon}; x_0, y_0)$. However because of (13), and the argument leading to (14) we conclude that

$$V_F(\mu_{T_\epsilon}; x_0, y_0) = 0.$$

Going back to Theorem 7, the saddle-point property implies the existence of a strategy $\beta(\epsilon)$ such that for every $\alpha \epsilon A(x_0, y_0)$ and every $(x, y) \epsilon 0(\alpha, \beta(\epsilon))$

$$\mu_{T_\epsilon}(x, y) = 0.$$

From (13) we can then conclude that for every $\alpha \epsilon A(x_0, y_0)$ and every $(x, y) \epsilon 0(\alpha, \beta(\epsilon))$,

$$t(x, y) \leq T_\epsilon = V_-(x_0, y_0) + \epsilon$$

It follows that

$$V^+(x_0, y_0) \leq V_-(x_0, y_0) + \epsilon$$

Since $\epsilon > 0$ is arbitrary the theorem is proved.

Definition: Let $T^* = V^+(x_0, y_0) = V_-(x_0, y_0)$.

Theorem 9: There exists a strategy $\beta^* \epsilon B(x_0, y_0)$ such that

$$\sup_{\alpha \in A(x_0, y_0)} \sup_{(x,y) \in 0(\alpha, \beta^*)} t(x,y) = T^* \leq \sup_{\alpha \in A(x_0, y_0)} \sup_{(x,y) \in 0(\alpha, \beta)} t(x,y)$$

for all $\beta \in B(x_0, y_0)$ i, e there exists an optimal pursuit strategy.

<u>Proof:</u> Consider the game defined on the fixed time interval $[0, T^*]$ with the continuous payoff function μ_{T^*}. Clearly $V_F(\mu_{T^*}; x_0, y_0) = 0$ and so there exists a strategy β^* such that for all $\alpha \in A(x_0, y_0)$ and all $(x, y) \in 0(\alpha, \beta^*)$, $\mu_{T^*}(x, y) = 0$; this implies that $t(x, y) \leq T^*$. Q. E. D.

Unfortunately, trivial examples show that in general there does not exist a strategy $\alpha^* \in A(x_0, y_0)$ such that

$$T^* = \inf_{\beta \in B(x_0, y_0)} \inf_{(x,y) \in 0(\alpha^*, \beta)} t(x,y) \tag{15}$$

We can therefore only assert the following theorem.

<u>Theorem 10.</u> If there is a strategy $\alpha^* \in A(x_0, y_0)$ which is optimal for player I (i.e., satisfies (15)) then the pair (α^*, β^*) from a saddle point i.e., for all $\alpha \in A(x_0, y_0)$, $\beta \in B(x_0, y_0)$,

$$\sup_{(x,y) \in 0(\alpha, \beta^*)} t(x,y) \leq \sup_{(x,y) \in 0(\alpha^*, \beta^*)} t(x,y) = T^* = \inf_{(x,y) \in 0(\alpha^*, \beta^*)} t(x,y)$$

$$\leq \inf_{(x,y) \in 0(\alpha^*, \beta)} t(x,y)$$

Various conditions can be placed on the set of trajectories and the endzone F which guarantee existence of an optimal evasion strategy α^*.

One such condition is the following:

(C) As the initial states and time (x_0, y_0, t_0) approaches F the value $T^*(x_0, y_0, t_0)$ approaches 0.

In this case we can show that the function

$$T(\alpha) = \operatorname*{Inf}_{\beta \in B(x_0, y_0)} \quad \operatorname*{Inf}_{(x, y) \in 0(\alpha, \beta)} t(x, y)$$

is an upper semicontinuous function of $\alpha \in A(x_0, y_0)$ and hence there exists α^* such that $T(\alpha^*) \geq T(\alpha)$ for all α. Evidently then $T(\alpha^*) = T^*$ and α^* satisfies (15). We now sketch a proof to show that Condition (C) above implies the upper-semicontinuity of $T(\alpha)$.

<u>Definition.</u> Let $\alpha \in A(x_0, y_0)$. We say that a pair $(x, y) \in X_T(x_0) \times Y_T(y_0)$ is a possible outcome if there is a sequence y_n, $n = 1, 2, \ldots$ in $Y_T(y_0)$ converging to y such that $\alpha(y_n)$, $n = 1, 2, \ldots$ converges to x. Let $P0(\alpha)$ be the set of all possible outcomes.

It is easy to check that

$$T(\alpha) = \operatorname*{Inf}_{(x, y) \in P0(\alpha)} t(x, y)$$

Now let z be any real number and let

$$A_z = \{\alpha \,|\, \alpha \in A(x_0, y_0), \; T(\alpha) \geq z\}$$

We must show that A_z is a closed set. Let $\{\alpha(k)\}$ be a net in A_z

converging to α and let $(x,y) \in P0(\alpha)$ i.e., let $\{y_n\} \subset Y_T(y_0)$ be a sequence such that y_n converges to y and $\alpha(y_n)$ converges to x. Suppose that $t(x,y) = z - \epsilon$ for some $\epsilon > 0$. This means that

$$(x(z-\epsilon),\ y(z-\epsilon),\ z-\epsilon) \in F$$

Since $\lim_{n\to\infty} ||y_n - y|| = 0$ and $\lim_{n\to\infty} ||\alpha(y_n) - x|| = 0$, given $\eta > 0$ there is $N(\eta) < \infty$ sufficiently large such that†

$$\rho\{(\alpha(y_n)(z-\epsilon),\ y_n(z-\epsilon),\ z-\epsilon),\ F\} < \eta$$

whenever $n > N(\eta)$. Now $\lim_k \alpha(k)(y_n) = \alpha(y_n)$. Hence for k sufficiently large,

$$\rho\{(\alpha(k)(y_n)(z-\epsilon),\ y_n(z-\epsilon),\ z-\epsilon),\ F\} < 2\eta$$

But then by condition (C) $\quad T(\alpha(k)) \leqq z - \epsilon + \gamma(\eta)$ $\quad$ where $\lim_{\eta\to 0} \gamma(\eta) = 0$.

It follows that for all sufficiently large k, $T(\alpha(k)) < z$ which is a contradiction. Hence A_z is closed and so $T(\alpha)$ is upper semicontinuous. We can summarize our results as a theorem.

<ins>Theorem 11.</ins> Suppose that (1) and (2) satisfy the assumptions of Section 2 and also suppose that condition (C) holds. Then there exists $\alpha^* \in A(x_0, y_0)$, $\beta^* \in B(x_0, y_0)$ such that for all $\alpha \in A(x_0, y_0)$, $\beta \in B(x_0, y_0)$

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† $\rho\{(x,y,t),\ F\} = \min\{\ |x - \hat{x}| + |y - \hat{y}| + |t - \hat{t}|\ \big|\ (\hat{x}, \hat{y}, \hat{t}) \in F\}$

$$\sup_{(x,y)\in 0(\alpha,\beta^*)} t(x,y) \leq \sup_{(x,y)\in 0(\alpha^*,\beta^*)} t(x,y) = T^*$$

$$= \inf_{(x,y)\in 0(\alpha^*,\beta^*)} t(x,y) \leq \inf_{(x,y)\in 0(\alpha^*,\beta)} t(x,y).$$

7. An example. System Equations

$$\dot{x} = u, \quad |u| \leq 1$$

$$\dot{y} = v, \quad |v| \leq 1$$

$x(0) = y(0) = 0$, final time $T=1$. x, y, u, v, are real numbers; x is the state of player I, y is the state of player II. The payoff μ is just a function of the final states $x(1)$, $y(1)$ and is given by:

$$\mu(x,y) = \begin{cases} |x(1)| & \text{for} \quad x(1)\,y(1) \geq 0 \\ (1 - |y(1)|)\,|x(1)| & \text{for} \quad x(1)\,y(1) \leq 0. \end{cases}$$

Consider the strategy β^* for II givein by $\beta(x) = -x$ for all $x \in X_1$. Then

$$\mu(x, \beta^*(x)) \leq 1/4$$

Let $\alpha^*: Y_1 \to X_1$ be the strategy given by

$$(\alpha^* y)(t) = y(t) \quad \text{for} \quad t \leq 1/2$$

$$(\alpha^* y)(t) = \begin{cases} y(1/2) + t & \text{for} \quad t > 1/2 \quad \text{if} \quad y(1/2) \geq 0 \\ y(1/2) - t & \text{for} \quad t > 1/2 \quad \text{if} \quad y(1/2) < 0 \end{cases}$$

Then for all $y \in T_1$,

$$\mu(\alpha^*(y), y) \geq 1/4 .$$

Evidently (α^*, β^*) are optimal. Furthermore α^* is not continuous, although it can be approximated by continuous strategies; moreover every continuous strategy is inferior to α^*.

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