THE DISTRIBUTION OF INTERVALS BETWEEN ZEROS FOR A STATIONARY GAUSSIAN PROCESS

by

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1. Introduction. This note supplements an earlier paper [1], where the distribution for the interval between two successive zeros was found for a zero-mean Gaussian process with the following covariance function:

$$(1) \qquad \rho(\tau) = Ex(t)x(t+\tau) = \frac{3}{2} e^{-|\tau|/\sqrt{3}} \left(1 - \frac{1}{3} e^{-(2/\sqrt{3})|\tau|}\right) .$$

This earlier work was based on a time change for the process after which a formula of McKean [2] was used to derive the desired result. A second look at McKean's paper has revealed that the distribution of the interval between any two zeros (not only successive ones) can be found in a similar way. However, except for the case of two successive zeros, I have not been able to carry out a final integration to reduce the distribution to a closed-form expression.

2. An extension to McKean's formula. Let $\{W(t),\ t \geq 0\}$ be a standard Brownian motion (Gaussian, $EW(t)W(s) = \min(t, s)$). All stochastic processes considered in this paper are assumed to be separable. Define zero-crossing times $\sigma_n(\eta_0)$ as follows:

$$(2) \qquad \sigma_0 = 0$$

$$\sigma_{n+1}(\eta_0) = \min\left\{t > \sigma_n(\eta_0); \quad \eta_0 t + \int_0^t W(s)\, ds = 0\right\} , \quad \eta_0 \geq 0 .$$

The magnitudes of the slopes at the crossings are given by

$$(3) \qquad h_n(\eta_0) = (-1)^n[\eta_0 + W(\sigma_n)]$$

since slopes at successive crossings must have opposite signs.

Let $f(t, a)$ be the density for the distribution of $\sigma_1(1)$ and $h_1(1)$, i.e.,

$$\text{(4)} \qquad \text{Prob}\,\{\sigma_1(1) \in dt,\ h_1(1) \in da\} = f(t, a)\,dt\,da \quad .$$

McKean derived the formula

$$\text{(5)} \qquad \int_0^\infty f(t, a)e^{-\alpha t}\,dt = \int_0^\infty \frac{K_{i\gamma}(\sqrt{8\alpha})K_{i\gamma}(\sqrt{8\alpha}\ a)}{2\cosh(\pi\gamma/3)}\,dO$$

K_ν = modified Bessel function

$dO = 2\pi^{-2}\gamma \sinh \pi\gamma$

and upon inverting the Laplace transform, obtained

$$\text{(6)} \qquad f(t, a) = \frac{3a}{\pi\sqrt{2}\,t^2}\, e^{-2/t(1-a+a^2)} \int_0^{4a/t} \frac{e^{-3\theta/2}}{\sqrt{\pi\theta}}\,d\theta \quad .$$

For a standard Brownian motion $W(t)$, $CW(t/C^2)$ is again a standard Brownian motion. From this scaling property we can show that $(\sigma_n(\eta_0), h_n(\eta_0))$ have the same probability law as $(\eta_0{}^2\sigma_n(1), \eta_0 h_n(1))$. Therefore, if we denote the joint density function of $(\sigma_n(\eta_0), h_n(\eta_0))$ by $\pi_n(t, \eta|\eta_0)$, we find

$$\text{(7)} \qquad \begin{aligned} \pi_1(t, \eta|\eta_0)\,dt\,d\eta &= \text{Prob}\,(\sigma_1(\eta_0) \in dt,\ h_1(\eta_0)\ d\eta) \\ &= \frac{dt}{\eta_0{}^2}\frac{d\eta}{\eta_0} f\left(\frac{t}{\eta_0{}^2}, \frac{\eta}{\eta_0}\right) \end{aligned}$$

where f is given by (6).

For a more general n, the fact that $\eta_0 t + \int_0^t W(s)\,ds$ and its derivative are jointly Markovian leads to the recursive relationship

$$\text{(8)} \qquad \pi_{n+1}(t, \eta|\eta_0) = \int_0^t \int_0^\infty \pi_1(t - s, \eta|\zeta)\pi_n(s, \zeta|\eta_0)\,ds\,d\zeta$$

Denoting

$$\hat{\pi}_n(\alpha, \eta|\eta_0) = \int_0^\infty e^{-\alpha t} \pi_n(t, \eta|\eta_0)\, dt \tag{9}$$

we can transform (8) into

$$\hat{\pi}_{n+1}(\alpha, \eta|\eta_0) = \int_0^\infty \hat{\pi}_1(\alpha, \eta|\zeta)\hat{\pi}_n(\alpha, \zeta|\eta_0)\, d\zeta \quad . \tag{10}$$

The function $\hat{\pi}_1(\alpha, \eta|\zeta)$ can be found from (7) and (5) as follows:

$$\begin{aligned}\hat{\pi}_1(\alpha, \eta|\eta_0) &= \int_0^\infty e^{-\alpha t} \pi_1(t, \eta|\eta_0)\, dt \\ &= \int_0^\infty \frac{1}{\eta_0^{\,3}} f\left(\frac{t}{\eta_0^{\,2}}, \frac{\eta}{\eta_0}\right) e^{-\alpha t}\, dt \\ &= \int_0^\infty \frac{1}{\eta_0} f\left(\tau, \frac{\eta}{\eta_0}\right) e^{-\alpha \eta_0^{\,2} \tau}\, d\tau \\ &= \frac{1}{\eta_0} \int_0^\infty \frac{K_{i\gamma}(\sqrt{8\alpha}\, \eta_0) K_{i\gamma}(\sqrt{8\alpha}\, \eta)}{2 \cosh (\pi\gamma/3)}\, d0 \quad .\end{aligned} \tag{11}$$

From the Lebedev transform pair [3, vol. 2, p. 173]

$$\begin{aligned} g(y) &= \int_0^\infty f(x) K_{ix}(y) 2\pi^{-2}\, x \sinh \pi x\, dx \\ f(x) &= \int_0^\infty g(y)\, K_{ix}(y) y^{-1}\, dy \quad , \end{aligned} \tag{12}$$

we conclude that

$$(13)\qquad 2\pi^{-1}\, x \sinh \pi x \int_0^\infty K_{ix}(y) K_{ix'}(y) y^{-1}\, dy \;=\; \delta(x - x')$$

Hence, by using (11) repeatedly in (10), we get

$$(14)\qquad \hat{\pi}_n(\alpha, \eta \mid \eta_0) = \frac{1}{\eta_0} \int_0^\infty \frac{K_{i\gamma}(\sqrt{8\alpha}\,\eta_0) K_{i\gamma}(\sqrt{8\alpha}\,\eta)}{[2 \cosh (\pi\gamma/3)]^n}\, d\Theta$$

$$d\Theta \equiv 2\pi^{-2}\gamma \sinh \pi\gamma$$

which is a surprisingly simple formula.

3. Computation of $P_m(t)$. Let $x(t)$ be a zero-mean Gaussian process with its covariance function given by (1). Then, it is easy to verify by direct computation that $x(t)$ can be represented in terms of a standard Brownian motion $W(t)$ as

$$(15)\qquad x(t) = \sqrt{3}\, e^{-\sqrt{3}t} \int_0^{\exp(2/\sqrt{3})t} W(s)\, ds \quad .$$

It follows that whenever $t \geq t_0$, we can write

$$(16)\qquad \begin{aligned} x(t) &= x(t_0) e^{-\sqrt{3}(t-t_0)} \left[1 + \tfrac{3}{2} g(t - t_0)\right] \\ &\quad + \sqrt{3} e^{-\sqrt{3}(t-t_0)} \cdot \left[\tfrac{1}{2}\dot{x}(t_0) g(t - t_0) + \int_0^{g(t-t_0)} W(s)\, ds\right] \end{aligned}$$

where $W(s)$ is again a standard Brownian motion (but not the same one as in (15)), and $g(t)$ is given by

$$(17)\qquad g(t) = \exp (2/\sqrt{3})t - 1 \quad .$$

Now, let τ_0 denote the first zero of $x(t)$ for $t \geq 0$, and let τ_ν denote the νth zero after τ_0. Because of the stationarity of $x(t)$, the distribution of $\tau_{m+1} - \tau_0$ is the same as the distribution of the interval between any pair of zeros with m zeros in between. We shall denote the distribution Prob $(\tau_{m+1} - \tau_0 \leq t)$ by $F_m(t)$, and denote the density $\dot{F}_m(t)$ by $P_m(t)$. The principal concern of this paper is the computation of $P_m(t)$.

We note that $F_m(t)$ can be written as

$$(18)\quad \begin{aligned} F_m(t) = 1 - \int_0^\infty \text{Prob}\ (\tau_0 \in dt_0) \int_{-\infty}^\infty \text{Prob}\ (\dot{x}(t_0) \in d\eta_0 | \tau_0 \in dt_0) \\ \cdot\ \text{Prob}\ (\tau_{m+1} > t + t_0 | \tau_0 = t_0, \dot{x}(t_0) = \eta_0)\ . \end{aligned}$$

A comparison of (16) and (2) shows that

$$(19)\quad \begin{aligned} &\text{Prob}\ (\tau_{m+1} > t + t_0 | \tau_0 = t_0, \dot{x}(t_0) = \eta_0) \\ &\qquad = \text{Prob}\left(\sigma_{m+1}\left(\frac{\eta_0}{2}\right) > g(t)\right)\ . \end{aligned}$$

Furthermore (see [4] for a clarification of "horizontal window"),

$$(20)\quad \begin{aligned} \text{Prob}\ (\dot{x}(t_0) \in d\eta_0 | \tau_0 \in dt_0) &= \text{Prob}\ (\dot{x}(t_0) \in d\eta_0 | x(t_0) = 0 \text{ in the horizontal window sense}) \\ &= \tfrac{1}{2}|\eta_0| e^{-\frac{1}{2}\eta_0^2}\ d\eta_0 \end{aligned}$$

Hence, (18) becomes (after integrating out t_0)

$$(21) \qquad F_m(t) = 1 - \frac{1}{2}\int_{-\infty}^{\infty} |\eta_0| e^{-\frac{1}{2}\eta_0^2} \text{ Prob}\left(\sigma_{m+1}\left(\frac{\eta_0}{2}\right) > g(t)\right) d\eta_0 \ .$$

Although Prob $(\sigma_n(\eta) > t)$ was computed only for $\eta \geqq 0$ in §2, it is obvious that we must have the symmetry Prob $(\sigma_n(\eta) > t)$ = Prob $(\sigma_n(-\eta) > t)$. Hence (21) becomes

$$F_m(t) = 1 - \int_0^{\infty} \eta_0 e^{-\frac{1}{2}\eta_0^2} \text{ Prob}\left(\sigma_{m+1}\left(\frac{\eta_0}{2}\right) > g(t)\right) d\eta_0$$

$$(22) \qquad = 1 - \int_0^{\infty} \eta_0 e^{-\frac{1}{2}\eta_0^2} \text{ Prob}\left(\frac{\eta_0^2}{4}\sigma_{m+1}(1) > g(t)\right) d\eta_0$$

$$= 1 - \int_0^{\infty} d\eta_0\, \eta_0 e^{-\frac{1}{2}\eta_0^2} \int_{4g(t)/\eta_0^2}^{\infty} ds \int_0^{\infty} d\eta\, \pi_{m+1}(s, \eta|1) \ .$$

The density $P_m(t)$ can be expressed as

$$(23) \qquad P_m(t) = 4\dot{g}(t)\int_0^{\infty}\int_0^{\infty} \frac{1}{\eta_0} e^{-\frac{1}{2}\eta_0^2} \pi_{m+1}(4g(t)/\eta_0^2, \eta|1)\, d\eta_0\, d\eta \ .$$

The Laplace transform (14) can now be inverted to yield

$$(24) \qquad \pi_{m+1}(t, \eta|1) = \frac{1}{2t} e^{-\frac{2}{t}(1+\eta^2)} \int_0^{\infty} \frac{K_{i\gamma}\left(\frac{4\eta}{t}\right)}{\left(2\cosh\frac{\pi\gamma}{3}\right)^{m+1}}\, d\Theta \ .$$

Using (24) in (23) yields

$$P_m(t) = \frac{1}{2}\dot{g}(t)\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{\eta_0}{g(t)} e^{-\frac{1}{2}\eta_0^2} e^{-\frac{1}{2}\frac{\eta_0^2}{g(t)}(1+\eta^2)}$$

$$(25) \qquad \cdot \frac{K_{i\gamma}\left(\frac{\eta\eta_0^2}{g(t)}\right)}{\left(2\cosh\frac{\pi\gamma}{3}\right)^{m+1}}\, d\eta_0\, d\eta\, d\Theta$$

$$= \frac{1}{2}\,\dot{g}(t) \int_0^\infty \int_0^\infty \int_0^\infty \eta_0 e^{-\frac{1}{2}\eta_0^2(1+\eta^2+g(t))}$$

$$\cdot \frac{K_{i\gamma}(\eta\eta_0^2)}{\left(2\cosh\frac{\pi\gamma}{3}\right)^{m+1}}\, d\eta_0\, d\eta\, d\Theta \;.$$

Now, if we use the formula

$$K_{i\gamma}(a) = \int_0^\infty e^{-a\cosh u} \cos\gamma u\, du \tag{26}$$

in (25), we get a fourfold integral with variables of integration, η_0, η, γ, u. Integrating first with respect to η_0, then u, we find

$$P_m(t) = \frac{1}{2}\,\dot{g}(t) \int_0^\infty d\Theta \int_0^\infty d\eta \frac{1}{\left(2\cosh\frac{\pi\gamma}{3}\right)^{m+1}}$$

$$\cdot \frac{\pi \sin\left(\gamma \cosh^{-1}\frac{1+\eta^2+g(t)}{2\eta}\right)}{(\sinh\pi\gamma)\sqrt{[1+\eta^2+g(t)]^2 - 4\eta^2}} \tag{27}$$

$$\left(d\Theta \equiv \frac{2}{\pi^2}\,\gamma \sinh \pi\gamma\right)$$

$$= \frac{1}{\pi}\,\dot{g}(t) \int_0^\infty d\gamma \int_{\cosh^{-1}\sqrt{1+g(t)}}^\infty dx \frac{\gamma \sin\gamma x}{\sqrt{\sinh^2 x - g(t)}\,(2\cosh \pi\gamma/3)^{m+1}} \;.$$

The expression (27) can be integrated once more with respect to γ to yield

$$(28)\qquad P_m(t) = \frac{1}{\pi}\,\dot{g}(t)\int_{\cosh^{-1}\sqrt{1+g(t)}}^{\infty} \frac{1}{\sqrt{\sinh^2 x - g(t)}}\,(-1)\,\frac{d}{dx} f_m(x)\,dx$$

where $f_m(x)$ is given by

$$(29)\qquad \begin{aligned} f_0(x) &= \frac{3}{4}\,\frac{1}{\cosh \frac{3}{2} x} \\ f_1(x) &= \frac{9}{8\pi}\,\frac{x}{\sinh \frac{3}{2} x} \\ f_m &= \frac{1}{m}\left[\left(\frac{3x}{2\pi}\right)^2 + \left(\frac{m-1}{2}\right)^2\right] f_{m-2}(x), \qquad m \geqq 2 . \end{aligned}$$

I have not been able to carry out the integration in (28), except for $m = 0$. In reference 1, $P_0(t)$ was obtained in terms of complete elliptic integrals. [1, (25)]

4. Computation of $P_m(0)$. From (17), we have $g(0) = 0$ and $\dot{g}(0) = 2/\sqrt{3}$. Therefore, for $t = 0$, (27) becomes

$$(30)\qquad \begin{aligned} P_m(0) &= \frac{2}{\sqrt{3}\pi}\int_0^{\infty} d\gamma \int_0^{\infty} dx\,\frac{\gamma \sin \gamma x}{\sinh x(2\cosh \pi\gamma/3)^{m+1}} \\ &= \frac{1}{\sqrt{3}}\int_0^{\infty}\left(\frac{e^{\pi\gamma}-1}{e^{\pi\gamma}+1}\right)\frac{\gamma\,d\gamma}{(2\cosh \pi\gamma/3)^{m+1}} \\ &= \frac{1}{\sqrt{3}}\left(\frac{3}{\pi}\right)^2\int_1^{\infty}\left(\frac{x^3-1}{x^3+1}\right)\frac{\ln x}{x\left(x+\frac{1}{x}\right)^{m+1}}\,dx \\ &= \frac{1}{2\sqrt{3}}\left(\frac{3}{\pi}\right)^2\int_0^{\infty}\left(\frac{x^3-1}{x^3+1}\right)\frac{x^m \ln x}{(x^2+1)^{m+1}}\,dx \end{aligned}$$

which can be evaluated by contour integration to yield

$$\text{(31)}\qquad P_m(0) = \frac{1}{4\sqrt{3}}\left(\frac{3}{\pi}\right)^2(2\pi i) \sum \text{Residue} \left\{\frac{(z^3 - 1)z^m \ln z(1 - \ln z/2\pi i)}{(z^3 + 1)(z^2 + 1)^{m+1}}\right\}$$

where the summation is taken over the residues at the five poles $z = e^{\frac{\pi i}{2}}$, $e^{\frac{3\pi}{2}i}$, $e^{\pi i}$, $e^{\frac{\pi i}{3}}$, $e^{\frac{5\pi i}{3}}$. The expression (31) can be further elaborated to give

$$\text{(32)}\qquad P_m(0) = \frac{9}{2\sqrt{3}} \left\{\frac{(-1)^m}{3 \cdot 2^{m+1}} - \frac{10}{27} + \frac{i}{\pi m!} \frac{d^m}{dz^m}\left[\frac{z^m}{(z + i)^{m+1}} H(z)\right]\Bigg|_{z=e^{\pi i/2}} + \frac{1}{\pi}\frac{i}{m!}\frac{d^m}{dz^m}\left[\frac{z^m}{(z - i)^{m+1}} H(z)\right]\Bigg|_{z=e^{3\pi i/2}}\right\}$$

with

$$\text{(33)}\qquad H(z) = \frac{(z^3 - 1)}{(z^3 + 1)} \ln z\left[1 - \ln z/2\pi i\right] .$$

I have not been able to reduce (32) further.

For a stationary zero-mean Gaussian process with covariance function of the form

$$\text{(34)}\qquad \rho(\tau) = 1 - \frac{\tau^2}{2} + \alpha|\tau|^3 + o(|\tau|^3) ,$$

it is not hard to show that $P_m(0)$ is proportional to α. Longuet-Higgins [5] has obtained bounds for $\frac{1}{\alpha} P_m(0)$ for m up to 7. For m = 0, 1, 2, these bounds read

$$1.1556 < \frac{1}{\alpha} P_0(0) < 1.158$$

(35)
$$0.1971 < \frac{1}{\alpha} P_1(0) < 0.198$$

$$0.0491 < \frac{1}{\alpha} P_2(0) < 0.0556$$

Now, the covariance function (1) under consideration in this paper has the form of (34) with $\alpha = \frac{2}{3\sqrt{3}}$. Hence, the true values for $\frac{1}{\alpha} P_m(0)$ can be evaluated and compared against the bounds in (35). For m = 0, 1, 2, (32) can be evaluated to yield

$$P_0(0) = \frac{2}{3\sqrt{3}}\left(\frac{37}{32}\right) = \frac{2}{3\sqrt{3}}\left(1.15625\right)$$

(36)
$$P_1(0) = \frac{2}{3\sqrt{3}}\left(\frac{41}{64} - \frac{108}{64\pi}\right) \cong \frac{2}{3\sqrt{3}}\left(0.1972\right)$$

$$P_2(0) = \frac{2}{3\sqrt{3}}\left(\frac{121}{128} - \frac{81}{32\pi} - \frac{27}{32\pi^2}\right) \cong \frac{2}{3\sqrt{3}}\left(0.0541\right)$$

which are in agreement with (35).

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FOOTNOTES

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