STABILITY OF A NONLINEAR TIME-INVARIANT FEEDBACK SYSTEM UNDER ALMOST CONSTANT INPUTS

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STABILITY OF A NONLINEAR TIME-INVARIANT FEEDBACK SYSTEM UNDER ALMOST CONSTANT INPUTS

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Summary--We consider a multiple-input multiple-output feedback system consisting of a linear time-invariant subsystem and a memoryless time-invariant nonlinearity. The linear subsystem is represented by its impulse response which includes a unit step (i.e., an integrator). The nonlinearity is not required to be of the noninteracting type nor to be bounded away from zero by some sector condition. It is shown that for any "almost constant" input the error e_2 is bounded and goes to zero as $t \to \infty$.

I INTRODUCTION

The Popov criterion has been a significant departure in the stability theory of nonlinear feedback systems [1-4]. Many generalizations of the criterion have been published [5-13]. The present paper extends previous results in

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several directions. A multiple-input multiple-output feedback system is considered. The linear time-invariant subsystem is described by a convolution. The matrix impulse response contains a step at the origin, i.e., the subsystem contains an integrator. Furthermore the impulse response may contain an infinite number of jumps provided that their total variation be finite. Equivalently, the time derivative of the impulse response is required to belong to the convolution algebra $\mathcal{A}$ [14-15]. The nonlinear subsystem is required to be memoryless and time invariant. However the nonlinearities may be interacting. Also, even though there is an integrator the nonlinearities are not required to be bounded away from zero. The multiplier considered is of the form P + sQ where the constant matrices P and Q need not be diagonal. Finally, even though there are only very weak assumptions on the nonlinearities, the error will go to zero as long as the inputs are "almost constant" that is, the assumptions on the inputs require them to become asymptotically constant. The results presented below extend previously published ones in at least one of the following aspects: (1) the class of linear subsystems considered, (2) the class of nonlinearities considered, (3) the class of inputs considered and (4) the class of multipliers considered.

II NOTATIONS

In the following we shall encounter real numbers, vectors (in R^n) and elements of function spaces. The symbol $|\cdot|$ is used to denote the magnitude of a real number and the norm of a vector in R^n. We denote by R_+ the set of nonnegative real numbers. The developments that follow are valid independently of the choice of norm in R^n because all norms in R^n are equivalent. For function spaces we use the following norms: let $f: R_+ \longrightarrow R^n$, then by definition,

$$\|f\|_\infty \triangleq \sup_{t \geq 0} |f(t)|$$

and

$$\|f\|_1 \triangleq \int_0^\infty |f(t)|\,dt.$$

With these norms, the resulting normed spaces are denoted by L_n^∞ and L_n^1, respectively. When the symbols $|\cdot|$ and $\|\cdot\|$ are applied to a matrix or to a matrix-valued function, they denote the induced operator norms.

The subscript T, as in f_T, denotes the truncation of the function f at time T, namely,

$$f_T(t) = \begin{cases} f(t) & \text{for } 0 \leq t \leq T \\ 0 & \text{for } t > T \end{cases}$$

The scalar product of two functions of R_+ into R^n is denoted by $\langle \cdot , \cdot \rangle$; this scalar product can be truncated and we define

$$\langle x, y \rangle_T \triangleq \int_0^T x'(t)\, y(t)dt$$

III SYSTEM DESCRIPTION AND ITS ASSUMPTIONS

Consider the multiple-loop feedback system shown in Fig. 1. The variables u_1, u_2, e_1, e_2, y_1, y_2 denote functions defined on $[0,\infty)$ and having values in R^n. The block labeled G denotes a linear, time-invariant, nonanticipative subsystem whose input-output relation is given by a convolution:

$$y_1(t) = (G * e_1)(t) \qquad t \geq 0 \tag{1}$$

where G is an n x n matrix-valued function, identical to zero for $t < 0$. The block labeled ψ denotes a memoryless, time-invariant nonlinearity. The equations of the systems are

$$e_2 = u_2 + G * e_1 \tag{2}$$

$$e_1 = u_1 - \psi(e_2) \tag{3}$$

Upon imposing certain restrictions on G and ψ, we shall prove that, provided the inputs u_1 and u_2 belong to certain classes, the resulting functions e_1, y_1, e_2 and y_2 are in L_n^∞ ; furthermore $e_2(t)$, $y_2(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

For ease of reference, we state the more explicit assumptions concerning the system of Fig. 1 as follows:

G1. The derivative (in the distribution sense) of G belongs to the convolution algebra $\mathcal{Q}$: By that we mean $\dot{G}$ is of the form

$$\dot{G}(t) = \begin{cases} R\delta(t) + \sum_{i=1}^{\infty} R_i \delta(t-t_i) + \dot{G}_a(t) \triangleq R\delta(t) + \dot{G}_\alpha(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{4}$$

where R and the R_i are constant n x n matrices such that $\sum_{i=0}^{\infty} |R_i| < \infty$, $0 < t_1 < t_2 < \ldots$, $\dot{G}_a$ is a function in L^1. Note that $\dot{G}_\alpha(t) \triangleq \sum_{i=1}^{\infty} R_i \delta(t-t_i) + \dot{G}_a(t)$ for $t > 0$, and zero elsewhere.

G2. The open-loop impulse response matrix of G is further assumed to be of the form

$$G(t) = \begin{cases} R1(t) + G_\alpha(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{5}$$

where 1(t) denotes the unit step. It is further assumed that $G_\alpha \in L^1$. The assumptions pertaining to the nonlinear subsystem are as follows:

N1. $\psi : R^n \longrightarrow R^n$ and is continuous.

N2. For some constant real matrix P

$$\psi(\xi)' P \xi > 0 \qquad \text{for all } \xi \in R^n \tag{6}$$

where ' denotes transposition and

$$\psi(\xi) = 0 \qquad \text{iff } \xi = 0$$

N3. There is a function $V \in C^1$ mapping R^n into R such that

$$V(\xi) \geq 0 \qquad \text{for all } \xi \in R^n$$

and for some constant real matrix Q

$$Q'\psi(\xi) = \nabla V(\xi) \qquad \text{for all } \xi \in R^n \tag{7}$$

Concerning the feedback system we make the following assumptions:

U. The inputs are subjected to the conditions

$$u_1,\ \dot{u}_1 \in L_n^1 \quad \text{and} \quad \dot{u}_2,\ \ddot{u}_2 \in L_n^1$$

(where the derivatives are calculated in the interval $(0,\infty)$) and, for any such inputs, e_1, e_2, y_1, and $y_2 \in L_{ne}^2$; roughly speaking, for such inputs there is no finite escape time.

IV COMMENTS

I. The class of open-loop impulse responses allowed by G1 and G2 include many special cases: a typical element of G has the behavior shown in Fig. 2. The class of allowed impulse responses include those of linear time-invariant differential systems whose transfer function goes to zero as $|s| \longrightarrow \infty$, of plants with several transportation lags, of systems with tapped delay lines, of distributed systems. Note also that G has a unit step and that, as a consequence of G1 and G2, the elements of G are in L^∞, also $G(t) \longrightarrow R$ as $t \longrightarrow \infty$: indeed, from G1 and G2 it follows that $|G_\alpha(t)| \longrightarrow 0$ as $t \longrightarrow \infty$.

II. As a consequence of the assumptions on the inputs, it is easily verified that $u_1 \in L_n^1 \cap L_n^\infty$, $\dot{u}_2 \in L_n^1 \cap L_n^\infty$ and $u_2 \in L_n^\infty$. We might describe u_2 as "an almost constant input" because $\dot{u}_2 \in L_n^1 \cap L_n^2$ and, consequently, $u_2(t) \longrightarrow u_{2\infty}$ (a constant vector in R^n) as $t \longrightarrow \infty$.

III. The absence of finite escape time can be guaranteed at the cost of an additional assumption on ψ. If for some constant k, $|\psi(\xi)| \leq k|\xi|, \forall \xi \in R^n$ then, together with $G \in L^\infty$, the Bellman-Gronwall lemma implies that e_2 is bounded by an increasing exponential; then exponential bounds on y_2, e_1, y_1 follow easily.

V MAIN RESULT

Theorem 1.

Let the system S of Fig. 1 satisfy the assumptions G1 - G2, N1 - N3, and U. If the constant matrices P and Q, defined in N2 and N3 are such that the matrix PR is real symmetric and positive definite and the Hermitian matrix

$$(P + j\omega Q)\hat{G}(j\omega) + \hat{G}'(-j\omega)(P' - j\omega Q') \tag{8}$$

is positive semi-definite for all $\omega \in R$, then e_1, e_2, y_1, y_2 and $\dot{e}_2$ are in L_n^∞. Furthermore $e_2(t), y_2(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

Proof.

First let us make a simplifying observation. By the linearity of G the effect of u_1 on e_2 can be replaced by an equivalent input (at the second summation point) given by $G * u_1 \stackrel{\Delta}{=} v_2$. Since $G \in L^\infty$ and $\dot{G} \in \mathcal{A}$, and also u_1, $\dot{u}_1 \in L_n^1$ we have (taking derivatives in the distribution sense)

$$\dot{v}_2 = \dot{G} * u_1 \in L_n^1$$

$$\ddot{v}_2 = \dot{G} * \dot{u}_1 \in L_n^1$$

Thus the equivalent input satisfies the assumptions required of u_2. Therefore, we set $u_1 \equiv 0$ and replace u_2 by $u \triangleq u_2 + G * u_1$. We write

$$\phi(t) \triangleq \psi[e_2(t)] \quad \text{for } t \geq 0 \tag{9}$$

Consequently the system equation obtained from (2) and (3) is

$$e_2 = u - G * \phi \tag{10}$$

Differentiating both sides and using (5), we obtain

$$\dot{e}_2 = \dot{u} - [R + G_\alpha(0_+)]\phi - \dot{G}_\alpha * \phi \tag{11}$$

Consider the function w defined by

$$w(\phi,t) = P[(G * \phi)(t) - R\int_0^t \phi(t')dt'] + [QD(G * \phi)](t) \tag{12}$$

where D denotes the time-derivative operator. Let

$$\eta(t) \triangleq \int_0^t \phi(t')dt' \tag{13}$$

then with (10) and (11)

$$w = P(u - e_2 - R\eta) + Q(\dot{u} - \dot{e}_2) \tag{14}$$

Let us truncate these expressions to [0,T] and calculate using Parseval's theorem

$$<w,\phi>_T = \int_0^T w(t)'\phi(t)dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}_T^*(j\omega)\hat{w}_T(j\omega)d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}_T^*(j\omega)\left[(P + j\omega Q)\hat{G}(j\omega) - \frac{1}{j\omega} PR\right]\hat{\phi}_T(j\omega)d\omega \quad (15)$$

where we used the fact that over [0,T], $w(\phi,t)_T = w(\phi_T,t)$. Since the integrand in (15) depends only on the hermitian part of the bracketed matrix, and since $-PR/(j\omega)$ is a skew hermitian matrix, we obtain

$$<w,\phi>_T = \frac{1}{4\pi} \int_{-\infty}^{\infty} \hat{\phi}_T^*(j\omega)[(P+j\omega Q)\hat{G}(j\omega)+\hat{G}^*(j\omega)(P'-j\omega Q')]\hat{\phi}_T(j\omega)d\omega \quad (16)$$

Finally, using the assumption (8) of the theorem we conclude that

$$<w, \phi>_T \geq 0 \qquad \text{for all } T \geq 0 \quad (17)$$

Use now (14) into (17)

$$<Pu + Q\dot{u}, \phi>_T - <PR\eta, \phi>_T$$
$$- <Pe_2, \phi>_T - <Q\dot{e}_2, \phi>_T \geq 0 \quad \text{for all } T \geq 0 \quad (18)$$

Applying integration by parts to the first term of (18) and noting that $u \in L_n^{\infty}$ and $\dot{u} \in L_n^1 \cap L_n^{\infty}$ and $\ddot{u} \in L_n^1$, we obtain successively

$$
\begin{aligned}
<Pu + Q\dot{u}, \emptyset>_T &= <Pu + Q\dot{u}, \dot{\eta}>_T \\
&= \left(Pu(T) + Q\dot{u}(T)\right)'\eta(T) - <P\dot{u} + Q\ddot{u}, \eta>_T \\
&\leq \left| Pu(T) + Q\dot{u}(T)\right|\left|\eta(T)\right| + \left\|P\dot{u} + Q\ddot{u}\right\|_1 \left\|\eta_T\right\|_\infty \\
&\leq \left(|P||u(T)| + |Q||\dot{u}(T)| + |P|\,\|\dot{u}\|_1 + |Q|\,\|\ddot{u}\|_1\right)\|\eta_T\|_\infty \\
&\leq k\|\eta_T\|_\infty
\end{aligned} \tag{19}
$$

where $\|\eta_T\|_\infty \overset{\Delta}{=} \sup_{t\in[0,T]} |\eta(t)|$ and k is a constant depending on P, Q, u, $\dot{u}$, $\ddot{u}$ and is independent of T.

Since PR is real symmetric and positive definite

$$
\begin{aligned}
<PR\eta, \emptyset>_T &= <PR\eta, \dot{\eta}>_T \\
&= \frac{1}{2}\eta'(T)PR\eta(T) \\
&\geq \frac{1}{2}\lambda_m|\eta(T)|^2
\end{aligned} \tag{20}
$$

where λ_m is the least eigenvalue of PR, a positive number. Now for the last scalar product in (18), we obtain successively

$$
\begin{aligned}
<Q\dot{e}_2, \emptyset>_T &= <\dot{e}_2(\cdot), Q'\psi[e_2(\cdot)]>_T \\
&= \int_0^T \dot{e}_2(t)\nabla V[e_2(t)]dt \\
&= \int_{e_2(0)}^{e_2(T)} de_2'\, \nabla V[e_2] \\
&= V[e_2(T)] - V[e_2(0)]
\end{aligned} \tag{21}
$$

Using all the results above in (18), we conclude that, for all $T \geq 0$,

$$\frac{1}{2}\lambda_m |\eta(T)|^2 + \langle Pe_2(\cdot), \psi[e_2(\cdot)]\rangle_T + V[e_2(T)] \leq k\|\eta_T\|_\infty + V[e_2(0)] \tag{22}$$

Now, by N2, $\langle Pe_2(\cdot), \psi[e_2(\cdot)]\rangle_T \geq 0$ for all $T \geq 0$ and, by N3, $V[e_2(T)] \geq 0$ for all $T \geq 0$ hence (22) gives us

$$\frac{1}{2}\lambda_m |\eta(T)|^2 \leq k\|\eta_T\|_\infty + V[e_2(0)] \tag{23}$$

We note that (23) is true for all $T \geq 0$ and since the right-hand side is a monotonically increasing function of T for all $T \geq 0$ we may replace $|\eta(T)|$ by $\|\eta_T\|_\infty$ in the left-hand side of (23) and we have

$$\frac{1}{2}\lambda_m \|\eta_T\|_\infty^2 \leq k\|\eta_T\|_\infty + V[e_2(0)] \tag{24}$$

The inequality (24) implies that $\|\eta_T\|_\infty$ is bounded by a constant independent of T, i.e., $\eta \in L_n^\infty$.

We now truncate (10) and take the L_n^∞-norms of both sides; we obtain

$$\|e_{2T}\|_\infty \leq \|u_T\|_\infty + \|(\dot{G} * \eta)_T\|_\infty \tag{25}$$

where we have used the facts that $\emptyset = \dot{\eta}$ and $G * \dot{\eta} = \dot{G} * \eta$ (in the distribution sense). Since $\dot{G} \in \mathcal{A}$ and $\eta \in L_n^\infty$, we have $\dot{G} * \eta \in L_n^\infty$ [14-15]. This combines with $u \in L_n^\infty$ to give us $e_2 \in L_n^\infty$. With this result if we observe that

$y_2(t) = \psi[e_2(t)]$, and ψ is a continuous map of R^n into R^n, we conclude that $y_2 \in L_n^\infty$ because ψ maps compact sets into compact sets. Finally $y_1 \in L_n^\infty$ because $y_1 = e_2 - u_2$ both of which are in L_n^∞.

The last step is to show that $e_2(t)$ and $y_2(t) \longrightarrow 0$ as $t \longrightarrow \infty$. Use (11) and take L_n^∞-norms of both sides and obtain

$$\|\dot{e}_{2T}\|_\infty \leq \|\dot{u}\|_\infty + (|R| + |G_\alpha(0_+)|)\|y_2\|_\infty + \|\dot{G}_\alpha\|\|y_2\|_\infty \qquad (26)$$

Each term on the right-hand side is finite and is independent of T, hence $\dot{e}_2 \in L_n^\infty$. As a consequence the bounded function $e_2(\cdot)$ is uniformly continuous on $[0,\infty)$. Now going back to (22), we have

$$\langle Pe_2(\cdot), \psi[e_2(\cdot)]\rangle_T < \infty \quad \text{for all } T \in [0,\infty) \qquad (27)$$

In other words, for some finite M, independent of T, we obtain

$$\int_0^T \psi[e_2(t)]' P\, e_2(t)\, dt \leq M \qquad \text{for all } T \geq 0 \qquad (28)$$

By N2, the integrand in (28) is positive whenever $e_2(t) \neq 0$. Since $e_2(\cdot)$ is uniformly continuous on $[0, \infty)$ and takes values in the closed ball $B(0, \|e_2\|_\infty)$ -- which is a compact set in R^n -- and since the function $\psi(\cdot)'P\,\cdot$ is a continuous mapping of R^n into R^n, the mapping $t \longmapsto \psi[e_2(t)]'P\, e_2(t)$ is uniformly continuous on $[0, \infty)$.

Hence (28) implies that $\psi[e_2(t)]' P\, e_2(t) \longrightarrow 0$ as $t \longrightarrow \infty$. In view of the continuity of ψ, the boundedness of e_2 and N2, this implies that $e_2(t) \longrightarrow 0$ as $t \longrightarrow \infty$ consequently so does $y_2(t) = \psi[e_2(t)]$. This concludes the proof of the theorem.

VI CONCLUSION

Under very general assumptions pertaining to the linear subsystem, the nonlinearity and the multiplier, we have established sufficient conditions of the stability of a multiple-input multiple-output time-invariant feedback system under almost constant inputs. The results extended previously published ones in several directions: the linear time-invariant subsystem is decribed by a convolution. The matrix impulse response contains a step at the origin, i.e., the subsystem contains an integrator. Furthermore, the impulse response may contain an infinite number of jumps provided that their total variation be finite. The memoryless time-invariant nonlinearities may be interacting and although there is an integrator the nonlinearities are not required to be bounded away from zero. In contrast to most previous work on generalizations of Popov criterion to the multiple feedback systems, the multiplier considered need not be a scalar function or a diagonal matrix. Finally, even though the inputs are present, the error will go to zero provided that the inputs are "almost constant".

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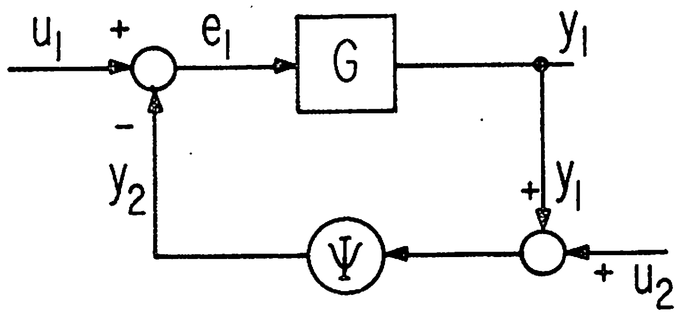

Fig. 1. Multiple-input multiple-output system under consideration.

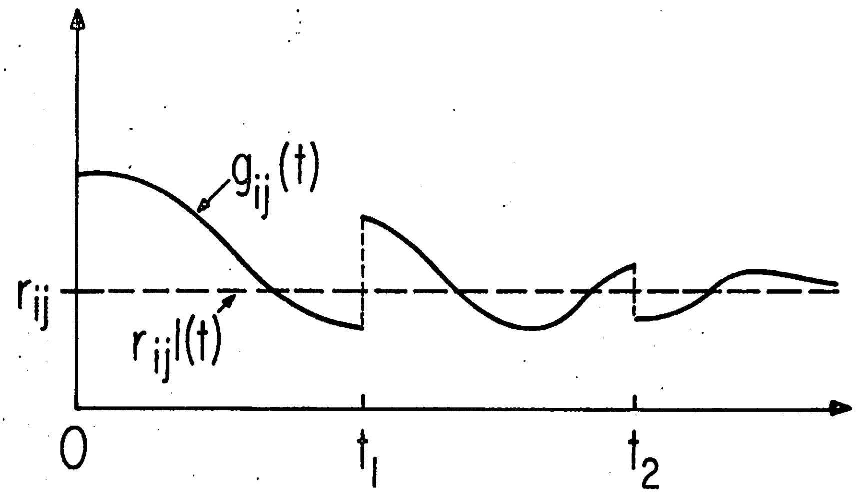

Fig. 2. Typical behavior of the ij-element of the impulse response matrix G(t).