PERIODIC RESPONSES OF NONLINEAR FEEDBACK SYSTEMS

by

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Memorandum No. ERL-M-258

19 December 1968

ELECTRONICS RESEARCH LABORATORY

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94720

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Abstract--For the scalar system of Fig. 1, with G linear and time invariant, and N_1 nonlinear and memoryless, an easily applied sufficient condition is derived which, for a periodic input, guarantees the existence of a periodic output of the same period. Error bounds for an approximate solution for the output are also derived.

I INTRODUCTION

Some ingenious techniques have been used to find approximate periodic responses for the nonlinear system of Fig. 1. Of these the best known among engineers is the describing function method [1] which has many desirable features for synthesis as well as analysis. On the negative side, this method, along with most other approximate methods, suffers from a lack of precision; in general, error bounds are not available and there is no assurance that the approximate results are even qualitatively correct.

Some attempts have been made to refine and justify some of these

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Research sponsored by the Joint Services Electronics Program Grant AFOSR-68-1488 and a NASA Traineeship.

approximate methods [2-5]. Of these the contributions of Sandberg [4] and Holtzman [5] are particularly important because in some cases they obtain error bounds for the approximate solutions, as well as a rigorous justification.

Both Sandberg and Holtzman consider versions of Fig. 1 and use contraction mapping fixed point theorems as their basic tool. Sandberg invokes a global contraction condition on the space of periodic real scalar functions of time which are square integrable over a period. His contraction condition (for a time-invariant system), involves only the slope of the nonlinear characteristic N_1 and the frequency response function of G, and may be tested in a simple direct way.

Holtzman uses a contraction condition which assures a contraction mapping on the space of continuous periodic vector functions of time in a neighborhood of the approximate solution; the norm on this space is the sup norm. The linear system may be, in the most general case, time-varying and finite dimensional while the nonlinearity need only be differentiable in a certain neighborhood of the origin. For this class of systems Holtzman does not derive an explicit contraction condition comparable with Sandberg's. To test the condition requires considerable preliminary analysis tailored to the specific problem at hand. Holtzman does suggest the possibility of simplifying the analysis in the case of scalar time-invariant linear systems.

The purpose of the present analysis is to adapt some of the best features of both analyses. In particular our system may be infinite dimensional, although for simplicity we assume scalar valued functions of time; otherwise our assumptions are like Holtzman's. In this case a contraction condition is derived which may be tested in a straightforward way. If further, like Sandberg, we restrict our linear system to be time-invariant, the contraction condition may be tested, without further recourse to functional analysis, in terms of the frequency response function of G and the slope of the nonlinearity.

II MATHEMATICAL BACKGROUND

We need the concepts of contraction mapping and Fréchet derivative and their application in a particular way to Banach spaces. For the convenience of the reader we include the familiar contraction mapping theorem in metric spaces.

A more complete discussion of the following material can be found in Kantorovich and Akilov [6].

Definition 1: Contraction Mapping

Given a complete metric space (X,d) and a closed set $W \subset X$, if the mapping $F:W \to W$ is such that $d(Fx,Fy) \leq rd(x,y)$ for all x,y in W and $r \in (0,1)$, then F is a contraction mapping on W.

Theorem 1. Contraction Theorem in a Metric Space

Given a complete metric space (X,d) and a closed set $W \subset X$, if $F:W \to W$ is a contraction mapping on W, then

a. There exists a unique x^* in W such that $x^* = Fx^*$.

b. For any x in W, $x^* = \lim_{n\to\infty} F^n x$, where $F^n = FF^{n-1}$ is the composition of F and F^{n-1}.

c. $d(x,x^*) \leq \left|\frac{1}{1-r}\right| d(Fx,x)$ for all x in W, where r is the contraction constant for F on W.

Proof of Theorem 1 is in Kantorovich and Akilov [6] on page 627. The element x^* is called a fixed point of F. There are several different fixed point theorems. The contraction theorem is the only one which contains an inherent bound on the distance from any particular point in the set to the fixed point. Notice that it also contains a constructive method for finding the fixed point.

Definition 2: Fréchet Derivative

Given the Banach spaces X and Y, and a mapping $F:X \to Y$, the Fréchet derivative of F at x, written F'_x, is defined such that

$$F'_x z = \lim_{h \to 0} \frac{F[x + hz] - Fx}{h} \quad , \quad \text{for all } z \in X ,$$

where the convergence is uniform in z for all z in X such that $\|z\| \leq 1$.

Notice that the Fréchet derivative $F'_x:X \to Y$ is a linear operator and its domain is all of X.

The application of Fréchet derivatives to contraction mappings is a consequence of the following lemma which is proved in Dieudonne [7] on page 155.

Lemma 1:

Given Banach spaces X and Y, a mapping $F:X \to Y$, and x_1 and $x_2 \in X$, define $S = \{x = \lambda x_1 + (1 - \lambda)x_2 | 0 \leq \lambda \leq 1\}$. If F'_x exists for all x in some open set containing S, then $\|Fx_1 - Fx_2\| \leq \|x_1 - x_2\| \sup_{x \in S} \|F'_x\|$.

Notice that the set S is simply a line segment connecting x_1 and x_2.

It is clear from lemma 1 that if F maps X into X and $\|F'_x\| \leq r < 1$ for all $x \in X$, then F is a contraction on X. It is not necessary that $\|F'_x\| < 1$ for all $x \in X$, as the next theorem shows.

Theorem 2: Contraction in a Banach Space (Holtzman)

Given a Banach space $(X, \|\cdot\|)$ and a mapping $f: X \to X$, if

a. $F: X \to X$ has a Frechet derivative F'_x for all $x \in X$

b. there exists an $x_0 \in X$ and $f: R^+ \to R^+$, non-decreasing, such that $\|F'_x\| \leq f\left(\|x - x_0\|\right)$ for all $x \in X$

c. there exists an $r \in [0,1)$ such that $f\left(\frac{k}{1-r}\right) \leq r$, where $k \geq \|Fx_0 - x_0\|$

then there exists a unique $x^* \in \Omega$ such that $x^* = Fx^*$ where $\Omega = \left\{x \in X \mid \|x - x_0\| \leq \frac{k}{1-r}\right\}$.

The proof follows Holtzman and is included as a convenience to the reader.

Proof:

For all $x \in \Omega$,
$$\begin{aligned}\|F'_x\| &\leq f\left(\|x - x_0\|\right)\\ &\leq f\left(\frac{k}{1-r}\right)\\ &\leq r < 1\end{aligned}$$

For any $x_1, x_2 \in \Omega$,
$$\begin{aligned}\|Fx_1 - Fx_2\| &\leq \|x_1 - x_2\| \sup_{x\in\Omega} \|F'_x\|\\ &\leq r\,\|x_1 - x_2\|\end{aligned}$$

For any $x \in \Omega$,
$$\begin{aligned}\|Fx - x_0\| &\leq \|Fx - Fx_0 + Fx_0 - x_0\|\\ &\leq \|Fx - Fx_0\| + \|Fx_0 - x_0\|\\ &\leq r\,\|x - x_0\| + k\\ &\leq r\left(\frac{k}{1-r}\right) + k\\ &\leq \frac{k}{1-r} .\end{aligned}$$

Therefore $Fx \in \Omega$ and hence $F: \Omega \to \Omega$. Since $\|Fx_1 - Fx_2\| \leq r\,\|x_1 - x_2\|$ for all $x_1, x_2 \in \Omega$, F is a contraction on Ω, a closed set in a complete metric space. Hence by Theorem 1 the result is immediate.

This theorem is important since it gives a method for finding the set Ω on which F is a contraction. Notice that the distance from the given point x_0 to the fixed point is less than or equal to $\frac{k}{1-r}$.

Lemma 2:

Given Banach spaces X and Y, if $F:X \to Y$ is a bounded linear map, then $F'_x = F$ for all $x \in X$.

The proof follows from a straightforward application of the definition.

Lemma 3:

Given Banach spaces X, Y and Z, if $F:X \to Y$ is Fréchet differentiable at x_0 and $G:Y \to Z$ is Fréchet differentiable at $y_0 = Fx_0$, then GF is Fréchet differentiable at x_0 and $(GF)'_{x_0} = G'_{y_0} F'_{x_0}$.

III APPLICATION TO A NONLINEAR SYSTEM

The system (S) to be considered is shown in Fig. 1 subject to the following assumptions. Let C_{ω_0} be the set of all continuous periodic functions, $R \to R$, with period $T = 2\pi/\omega_0$. Assume the norm $\|x\| \triangleq \sup_{t\in R} |x(t)|$. With this norm, C_{ω_0} is a Banach space.

Assumptions

S1. The inputs u_1 and $u_2 \in C_{\omega_0}$.

S2. G is a bounded linear operator mapping C_{ω_0} into itself.

S3. N_1 is a map of C_{ω_0} into itself such that $y_1(t) = p(t)n(e_1(t))$, where $p(t) \in C_{\omega_0}$, and the function $n:R \to R$ is everywhere differentiable.

Although it seems plausible, it does not necessarily follow from the assumptions that the functions inside the loop, i.e., e_1, e_2, y_1, and y_2 belong to C_{ω_0}. The purpose of the following discussion is to find sufficient conditions assuring that this is the case.

The starting point for the analysis is Theorem 2, the contraction mapping theorem by Holtzman.

For the present problem we let $X = C_{\omega_0}$ and use the sup norm. We next identify the map F. Referring to Fig. 1 and Assumption S3, note that

$$e_2(t) = u_2(t) + p(t)n\Big(u_1(t) - y_2(t)\Big) \quad . \tag{1}$$

Considering y_2 as an input, and e_2 as an output, define an operator N such that $e_2 = Ny_2$. N is specified by (1) and from Assumptions S1 and S3 it may be seen that N maps C_{ω_0} into itself.

The composition of operators GN also maps C_{ω_0} into itself, by virtue of Assumption S2, and, referring again to Fig. 1, $y_2 = GNy_2$. By associating y_2 with x, and GN with F, it is apparent the theorem is relevant to the question of whether $y_2 \varepsilon C_{\omega_0}$; if so, it follows that e_1, y_1, and e_2 are also elements of C_{ω_0}.

While one may attempt to satisfy Holtzman's conditions for each particular system considered, it is the goal of the present analysis to find an equivalent condition which may be used without further recourse to functional analysis.

To apply Theorem 2 to the case at hand, the Fréchet derivative of GN is needed. By lemmas 2 and 3, $(GN)'_x = GN'_x$. Consider then the following lemma concerning N'_x, the Fréchet derivative of N evaluated at x.

Lemma 4:

Given the operator $N: e_2 = Ny_2$, as defined by Eq. (1) and subject to Assumption S3, then for all $x \subset C_{\omega_0}$

$$\|N'_x\| = \max_{t\in R} \left|p(t)n'\left(u_1(t) - x(t)\right)\right|$$

where $n'(\xi) = \dfrac{dn(\xi)}{d\xi}$.

Proof of Lemma 4:

By definition, the Fréchet derivative at x is the linear map with the property that for all z

$$\left(N'_x z\right)(t) = \lim_{h\to 0}\left(\frac{N[x + hz] - Nx}{h}\right)(t)$$

where the convergence is uniform in z for $\|z\| \leq 1$. Using the definition of N given in (1)

$$\begin{aligned}\left(N'_x z\right)(t) &= \lim_{h\to 0} \frac{1}{h}\left[u_2(t) + p(t)n\left(u_1(t) - x(t) - hz(t)\right) - u_2(t) - p(t)n\left(u_1(t) - x(t)\right)\right] \\ &= \lim_{h\to 0} p(t)\left[\frac{n\left(u_1(t) - x(t) - hz(t)\right) - n\left(u_1(t) - x(t)\right)}{hz(t)}\right] z(t) \\ &= p(t)\, n'\left(u_1(t) - x(t)\right) z(t) .\end{aligned}$$

By Assumption S3, n' exists and therefore the convergence is uniform in z for $\|z\| \leq 1$. Now by definition

$$\|N_x'\| = \sup_{\|z\|=1} \max_{t \in R} \left|\left(N_x' z\right)(t)\right|$$

$$= \sup_{\|z\|=1} \max_{t \in R} \left|p(t)n'\left(u_1(t) - x(t)\right)z(t)\right|$$

$$= \max_{t \in R} \left|p(t)n'\left(u_1(t) - x(t)\right)\right| .$$

This completes the proof of lemma 4.

We are now in a position to state and prove Theorem 3, which is the main result of the paper.

Theorem 3:

Let $x_0 \in C_{\omega_0}$ be an approximate solution for $x = GNx$. Assume there exist constants γ and k, $\gamma \geqq \max_{t \in R} |p(t)| \cdot \|G\|$, and $k \geqq |GNx_0 - x_0|$. If there exists an $r \in [0,1)$ such that

$$r \geqq \gamma \sup \left\{|n'(\rho)| : |\rho| \leq \|u_1\| + \|x_0\| + \frac{k}{1-r}\right\} \qquad \text{(I)}$$

then there exists a unique function $x^* \in \Omega$ such that $x^* = GNx^*$, where

$$\Omega = \left\{x \in C_{\omega_0} : \|x - x_0\| \leqq \frac{k}{1-r}\right\} .$$

In the theorem, $|G|$ refers to the operator norm of the linear operator G on the space C_{ω_0}.

Proof of Theorem 3:

By lemmas 2 and 3, the mapping GN, of C_{ω_0} into itself, has a Fréchet derivative equal to GN_x' for all $x \subset C_{\omega_0}$. Let

$$f(z) = \gamma \max \left\{ |n'(\rho)| : |\rho| \le \|u_1\| + \|x_0\| + z \right\}$$

then f maps R^+ into R^+ and is nondecreasing. Also,

$$\|GN_x'\| \le \|G\| \, \|N_x'\|$$

$$= \|G\| \max_{t \varepsilon R} \left| p(t) n'\left(u_1(t) - x(t)\right) \right| \qquad (2)$$

$$\le \|G\| \max_{t \varepsilon R} |p(t)| \max \left\{ |n'(\rho)| : |\rho| \le \|u_1\| + \|x\| \right\}$$

$$\le \gamma \max \left\{ |n'(\rho)| : |\rho| \le \|u_1\| + \|x_0\| + \|x - x_0\| \right\} \qquad (3)$$

$$= f\left(\|x - x_0\|\right)$$

Here, in line (2) use was made of lemma 4, and in line (3) the identity $x = x - x_0 + x_0$ and the triangle inequality was used.

Now, according to (I) of Theorem 3

$$r \ge f\left(\frac{k}{1-r}\right)$$

Then according to Theorem 2 there exists a unique $x^* \varepsilon \Omega$ such that $x^* = GNx^*$ and $\|x_0 - x^*\| \le \frac{k}{1-r}$. This completes the proof of Theorem 3.

The following points are to be noted regarding Theorem 3.

1. Since $x^* \varepsilon \Omega$, which is the set of all points within a $k/(1-r)$ radius of x_0, an error bound on the approximate solution is provided if the theorem is satisfied.

2. If $\gamma|n'(\rho)|$ is less than 1 for all ρ, then inequality (I) is globally satisfied and x_0 may be chosen arbitrarily with the assurance that the iterative procedure, $x_{n+1} = GNx_n$, converges to a fixed point x^*. There is therefore one and only one solution of (S) in C_{ω_0}.

3. If (I) is not satisfied globally, it is important to pick a good first approximation x_0, both in order to satisfy the inequality (I) and to insure a small error bound. In this case although there is a unique fixed point in Ω there may be other solutions to (S) in $C_{\omega_0} \setminus \Omega$, the complement of Ω.

4. Once the approximate solution x_0 has been selected, GNx_0 can be calculated. Then k is simply a bound for $|(GNx_0)(t) - x_0(t)|$.

5. If $\gamma|n'(0)| \geq 1$ inequality (I) cannot possibly be satisfied.

6. There is a simple graphical method to determine if there is an r which satisfies inequality (I) and if so to find its best (minimum) value. Fig. 2 shows two curves plotted in an $r - \rho$ half-plane. For a given system, plot $\gamma|n'(\rho)|$ v.s. ρ and

$$\rho = \pm \left[\|u_1\| + \|x_0\| + \frac{k}{1-r} \right] \quad \text{v.s.} \quad r \geq 0$$

As shown in Fig. 2, draw the lowest horizontal segment between the walls of the "potential-well" such that the $\gamma|n'(\rho)|$ curve lies on or below the segment; this specifies the minimum r_1 which satisfies the inequality (I). r_2 also satisfies inequality (I), but will give a larger error bound than r_1. Neither r_3 or r_4 satisfy inequality (I).

7. This paper deals with the scalar case for ease of exposition. The arguments carry over to the case of vector valued functions with essentially

no change. If the norm

$$\|\underline{x}\| = \max_{i=1...n} \sup_{t} |x_i(t)|$$

is used, the graphical interpretation of the resulting inequality in Theorem 3 is only slightly changed.

IV TWO CLASSES OF LINEAR SYSTEMS

In Theorem 3, the only restrictions on G were that it be linear and map C_{ω_0} into itself. We now restrict our attention to linear time invariant systems. Such systems have an input-output relation of the form

$$x(t) = z(t) + \int_0^\infty g(t - \tau)e(\tau)d\tau , \quad \forall t \geq 0 \qquad (4)$$

where $e_{[0,\infty)}$ is the input, $x_{[0,\infty)}$ is the output, $z_{[0,\infty)}$ is the zero input response and $g_{(-\infty,\infty)}$ is the impulse response.

The two sided Laplace transform of g is

$$G(s) = \int_{-\infty}^\infty g(t)e^{-st}\, dt \qquad (5)$$

Notice that the symbol G is used for the mapping and G(s) for the Laplace transform of g.

Since we are interested only in inputs and outputs in C_{ω_0}, we shall henceforth consider all functions to be defined on $(-\infty,\infty)$.

We shall be interested in systems which have an initial state such that G maps C_{ω_0} into itself. We want G defined such that if e and x are in C_{ω_0} and have Fourier coefficients E_n and X_n respectively, then Ge = x implies $X_n = E_n G(jn\omega_0)$. Therefore we shall consider only systems which can be represented by the input-output relation

$$x(t) = \frac{1}{T}\int_0^T g^*(t - \tau)e(\tau)d\tau \ , \quad \forall t\varepsilon R \qquad (6)$$

where $g^*:R \to R$ satisfies the following assumptions:

G1. $g^* \varepsilon L^1_{[0,T]}$

G2. g^* has period T

G3. The nth Fourier coefficient of g^* is $G(jn\omega_0)$.

Under these conditions it is shown in Theorem A1 of Appendix A that

1. G maps C_{ω_0} into itself
2. If e and x are in C_{ω_0} and have Fourier coefficients E_n and X_n, then $X_n = E_n G(jn\omega_0)$
3. $\|G\| = \frac{1}{T}\int_0^T |g^*(t)|dt$

Convolutions of the form (6) are discussed in detail in Kaplan [8] for the case when both g^* and e are periodic and piecewise continuous.

We now consider two common cases in which such a periodic response function g^* can be defined.

Case 1

This is the class considered by Sandberg [4], except that he used the L^2 norm.

Assume

1. $g\varepsilon L^1_R$
2. x = Ge means

$$x(t) = \int_{-\infty}^{\infty} g(t - \tau)e(\tau)d\tau \ , \quad \forall t\varepsilon R \tag{7}$$

Comparison of (7) with (4) shows that we are considering the system to be in a state at t = 0 such that

$$z(t) = \int_{-\infty}^{0} g(t - \tau)e(\tau)d\tau \ , \quad \forall t \geq 0 \tag{8}$$

Under these conditions it is shown in lemma A1 of Appendix A that the convolution in (6) is equivalent to the convolution in (7) when

$$g^*(t) \triangleq T \sum_{n=-\infty}^{\infty} g(t + nT) \ , \quad \forall t\varepsilon R \tag{9}$$

Theorem A2 of Appendix A establishes that g* defined by (9) has the required properties G1, G2, and G3 and

$$\|G\| \leq \int_{-\infty}^{\infty} |g(t)|dt \tag{10}$$

Case 2

It is well known that even if the impulse response g does not belong to L^1_R periodic responses to periodic inputs are possible. For example consider the unstable linear system described by $\dot{y}(t) - y(t) = \cos \omega t$; the particular integral is defined and is of the form $A \cos(\omega t + \theta)$. From a mathematical point of view there is no difficulty in picking an initial state such that the transient is suppressed and hence, for this well chosen initial state, G maps a periodic input (i.e. cos ωt) into a periodic output.

More generally, if the input is an element of C_{ω_0}, with Fourier components E_n the particular integral for the linear system is in the form

$$\sum_{n=-\infty}^{\infty} G(jn\omega_0)E_n \; e^{jn\omega_0 t}$$

and the condition

$$\sum_{n=-\infty}^{\infty} |G(jn\omega_0)| \; < \; \infty$$

is sufficient to insure its existence. Again, in the usual case, there is no difficulty in finding an initial condition which suppresses the transient.

Assuming

$$\sum_{n=-\infty}^{\infty} |G(jn\omega_0)| \; < \; \infty \tag{11}$$

and defining

$$g^*(t) \;\triangleq\; \sum_{n=-\infty}^{\infty} G(jn\omega_0) \; e^{jn\omega_0 t} \;, \quad \forall t \epsilon R \tag{12}$$

it is shown in Theorem A3 of Appendix A that g* has the required properties and

$$\|G\| \le \sum_{-\infty}^{\infty} |G(jn\omega_0)| \tag{13}$$

Kaplan [8] provides a table for the easy computation of g* when G(s) is rational. A few entries from the table are given in Appendix B. Due to a slight difference in definition, our table differs from Kaplan's by a factor T.

It is useful to know that when G satisfies the assumption of both cases 1 and 2, the two convolution operators are identical; i.e. for almost all $t \in R$

$$g^*(t) = \sum_{n=-\infty}^{\infty} g(t + nT) = \sum_{n=-\infty}^{\infty} G(jn\omega_0)\, e^{jn\omega_0 t} . \tag{14}$$

In this case g* may be computed from either expression.

V EXAMPLE

Consider the system in Fig. 3, where $u(t) = \cos 2t$, $y(t) = be^3(t)$, and G is the map satisfying

$$\ddot{x} + 5x = y \tag{15}$$

The corresponding impulse response g is sinusoidal so it is not an element of L^1_R. Therefore the conditions of Case 1 are not satisfied.

Associated with the differential equation (15) is the transfer function

$$G(s) = \frac{1}{s^2 + 5}$$

so (11) is satisfied. Therefore the periodic response function g^* can be defined by (12) in Case 2. To calculate g^* we use the table in Appendix B.

$$g^*(t) = \frac{\pi}{2\sqrt{5}} \frac{\sin\sqrt{5}\,t - \sin\sqrt{5}\,(t-\pi)}{1-\cos\sqrt{5}\,\pi}$$

$$= 2.04 \sin(\sqrt{5}\,t + 1.20) \qquad (16)$$

By direct calculation

$$\|G\| = \frac{1}{T}\int_0^T |g^*(t)|dt = 1.54 \qquad (17)$$

so let $\gamma = 1.54$ in Theorem 3.

We next find an approximate solution, x_o, for the system in Fig. 3. Try a solution in the form $x_o(t) = A \cos 2t$, and use the method of harmonic balance to find A.

$$\ddot{x}_o + 5x_o = -4A \cos 2t + 5A \cos 2t$$

$$= A \cos 2t$$

$$(Nx_o)(t) = be_o^3(t) = b[\cos 2t - A \cos 2t]^3$$

$$= b(1-A)^3 \cos^3 2t$$

$$= \frac{1}{4} b(1-A)^3(3 \cos 2t + \cos 6t) \qquad (18)$$

Balancing the fundamental requires

$$A = \frac{3}{4} b(1 - A)^3 \ . \tag{19}$$

Note that there is always a unique real solution for A. If b is positive there is a solution in (0,1). If b is negative, there is a solution in $(1,\infty)$.

We next calculate $k = \|GNx_o - x_o\|$. Nx_o is given in (18) and GNx_o may be most easily found by using the property regarding Fourier components under the map G. We find

$$GNx_o = \frac{1}{3} A \left(3 \cos 2t - \frac{1}{31} \cos 6t\right) \ , \tag{20}$$

hence

$$GNx_o - x_o = -\frac{1}{93} A \cos 6t \ , \tag{21}$$

and

$$k = \frac{|A|}{93} \ . \tag{22}$$

Next the question of the existence of an $r \in [0,1)$ is investigated by using the graphical test illustrated by Fig. 2.

Since $n'(\rho)$ in the present case is $3b\rho^2$, the graph of $\gamma|n'(\rho)|$ is a parabola and for b sufficiently small the required r may certainly be found.

As a specific numerical example let $b = 0.08$. Then $A = 0.0510$, $k = 5.43 \times 10^{-4}$, and $r = 0.352$ is approximately the smallest value of r satisfying inequality (I). By Theorem 3, there is a function $x^* \in C_{\omega_0}$ satisfying $x^* = GNx^*$ and

$$\|x^* - x_o\| \leq \frac{k}{1 - r} = 8.40 \times 10^{-4} \tag{23}$$

The maximum instantaneous error between the approximate and true solution is 8.40×10^{-4}, which is only 1.65% of the amplitude of x_o.

It may be noted that with r much smaller than 1 a grosser estimate for r would not appreciably increase the error bound. It may be simpler, then, to use

$$\gamma' = \sum_{n=-\infty}^{\infty} |G(jn\omega_0)|$$

as an upper bound for γ or even to find an upper bound for γ'. If this is done in the present case we find $\gamma' < 2.6$ (rather than $\gamma = 1.54$) and correspondingly $\|x_o - x^*\| \leq 18.1 \times 10^{-4}$.

Comment 3 following the statement of Theorem 3 applies to this example. Thus the theory does not exclude other possible solutions which satisfy $\|x - x_o\| > 8.40 \times 10^{-4}$.

APPENDIX A

The purpose of this appendix is to establish the consequences of the properties of the periodic response function g^* and to show that the functions defined in Cases 1 and 2 satisfy the assumptions G1, G2 and G3 on g^*.

First we establish the consequences of the properties of g^*.

<u>Theorem A1.</u>

Given $g^*:R \to R$ satisfying

G1. $g^* \in L^1_{[0,T]}$

G2. g^* has period $T = 2\pi/\omega_0$

G3. The nth Fourier coefficient of g^* is $G(jn\omega_0)$.

Define G such that $Ge = x$ implies

$$x(t) = \frac{1}{T}\int_0^T g^*(t-\tau)e(\tau)d\tau$$

then

1. G maps C_{ω_0} into itself

2. If e and x are in C_{ω_0} and have Fourier coefficients E_n and X_n, then $X_n = E_n G(jn\omega_0)$

3. $$||G|| = \frac{1}{T}\int_0^T |g^*(t)|dt$$

Proof:

1. First notice that for $e \in C_{\omega_0}$,

$$x(t) = \frac{1}{T}\int_0^T g^*(t-\tau)e(\tau)d\tau$$

$$= -\frac{1}{T}\int_t^{t-T} g^*(\xi)e(t-\xi)d\xi$$

$$= \frac{1}{T}\int_0^T g^*(\xi)e(t-\xi)d\xi \qquad (24)$$

Now we show that $e \varepsilon C_{\omega_0}$ implies x is continuous.

$$|x(t+\delta) - x(t)| = \left|\frac{1}{T}\int_0^T g^*(\tau)e(t+\delta-\tau)d\tau - \frac{1}{T}\int_0^T g^*(\tau)e(t-\tau)d\tau\right|$$

$$= \frac{1}{T}\left|\int_0^T g^*(\tau)\Big[e(t+\delta-\tau) - e(t-\tau)\Big]d\tau\right|$$

$$\leq \frac{1}{T}\int_0^T |g^*(\tau)|d\tau \cdot \max_{\tau\varepsilon[0,T]} |e(t+\delta-\tau) - e(t-\tau)| \qquad (25)$$

The right hand side of (25) goes to zero as δ goes to zero since e is continuous. Therefore x is continuous.

Now we show that $e \varepsilon C_{\omega_0}$ implies x is periodic with period T. By (24),

$$x(t+T) = \frac{1}{T}\int_0^T g^*(\tau)e(t+T-\tau)d\tau$$

$$= \frac{1}{T}\int_0^T g^*(\tau)e(t - \tau)d\tau$$

$$= x(t)$$

Therefore $x \in C_{\omega_0}$ and G maps C_{ω_0} into itself.

2. By definition,

$$X_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_0^T \left[\frac{1}{T}\int_0^T g^*(\tau)e(t - \tau)d\tau\right] e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T^2}\int_0^T \int_0^T g^*(\tau)e^{-jn\omega_0\tau}\, e(t - \tau)\, e^{-jn\omega_0(t-\tau)}\, d\tau dt$$

$$= \frac{1}{T^2}\int_0^T g^*(\tau)e^{-jn\omega_0\tau}\left[\int_0^T e(t - \tau)e^{-jn\omega_0(t-\tau)}\, dt\right]d\tau \qquad (26)$$

$$= \frac{1}{T}\int_0^T g^*(\tau)e^{-jn\omega_0\tau}\, d\tau \cdot E_n \qquad (27)$$

$$= G(jn\omega_0)E_n$$

The change of order of integration in (26) is justified by Fubini's theorem, since

$$\int_0^T \int_0^T \left| g^*(\tau)e^{-jn\omega_0\tau} e(t - \tau)e^{-jn\omega_0(t-\tau)} \right| d\tau dt$$

$$= \int_0^T \int_0^T |g^*(\tau)e(t - \tau)| d\tau dt$$

$$\leq \max_t |e(t)| \int_0^T \int_0^T |g^*(\tau)| d\tau dt$$

$$< \infty$$

3. By definition,

$$\|G\| = \sup_{\substack{\|e\|=1 \\ e\varepsilon C_{\omega_0}}} \sup_t \left| \frac{1}{T} \int_0^T g^*(\tau)e(t - \tau)d\tau \right|$$

$$\leq \sup_{\substack{\|e\|=1 \\ e\varepsilon C_{\omega_0}}} \sup_t \frac{1}{T} \int_0^T |g^*(\tau)| d\tau \cdot \|e\|$$

$$= \frac{1}{T} \int_0^T |g^*(\tau)| d\tau \tag{28}$$

To show that the reverse inequality also holds, pick any $t\varepsilon R$ and let $\left\{e_t^n\right\}_{n=1}^{\infty}$ be a sequence of functions in C_{ω_0} with $\|e_t^n\| = 1$ for each n and such that for almost all $\tau \varepsilon [0,T]$

$$e_t^n\ (t-\tau) \xrightarrow[n]{} \operatorname{sgn} g^*(\tau) \tag{29}$$

Then $|e_t^n\ (t-\tau)\, g^*(\tau)| \le |g^*(\tau)|$ for each n, $g^* \varepsilon L^1_{[o,T]}$, and $e_t^n\ (t-\tau) g^*(\tau) \xrightarrow[n]{} |g^*(\tau)|$ for almost all $t\ \varepsilon\ [o,T]$, so by the Lebesque dominated convergence theorem,

$$\int_0^T e_t^n\ (t-\tau) g^*(\tau) d\tau \xrightarrow[n]{} \int_0^T |g^*(\tau)| d\tau \tag{30}$$

Now by definition,

$$\|G\| = \sup_{\substack{\|e\|=1 \\ e\varepsilon C_{\omega_0}}} \sup_t \left| \frac{1}{T} \int_0^T g^*(\tau) e(t-\tau) d\tau \right|$$

For any given $t\ \varepsilon\ R$, $e_t^n\ \varepsilon\ C_{\omega_0}$ and $\|e_t^n\| = 1$, so

$$\|G\| \ge \left| \frac{1}{T} \int_0^T g^*(\tau)\ e_t^n\ (t-\tau)\ d\tau \right| \tag{31}$$

Comparing (31) and (30),

$$\|G\| \ \ge\ \frac{1}{T} \int_0^T |g^*(\tau)| d\tau \tag{32}$$

Comparing (28) and (32)

$$\|G\| = \frac{1}{T} \int_0^T |g^*(t)| dt$$

This completes the proof of Theorem A1.

Before proving Theorem A2, a lemma is required.

Lemma A1:

If $v:R \to C$ is a bounded periodic function with period T and $g:R \to R$ is in L^1_R, then

1. $$\int_{-\infty}^{\infty} |g(\tau)| d\tau = \int_0^T \sum_{n=-\infty}^{\infty} |g(\tau + nT)| d\tau$$

2. $$\int_{-\infty}^{\infty} g(\tau)v(t - \tau)d\tau = \int_0^T \left[\sum_{n=-\infty}^{\infty} g(\tau + nT)\right] v(t - \tau)d\tau$$

Proof:

1. First notice that

$$\int_{-\infty}^{\infty} g(\tau)v(t - \tau)d\tau = \sum_{n=-\infty}^{\infty} \int_{nT}^{nT+T} g(\tau)v(t - \tau)d\tau$$

$$= \sum_{n=-\infty}^{\infty} \int_0^T g(\tau + nT)v(t - \tau - nT)d\tau$$

$$= \sum_{n=-\infty}^{\infty} \int_0^T g(\tau + nT)v(t - \tau)d\tau \qquad (33)$$

The fact that v has period T was used in (33). Proceeding similarly,

$$\int_{-\infty}^{\infty} |g(\tau)v(t - \tau)| d\tau = \sum_{n=-\infty}^{\infty} \int_0^T |g(\tau + nT)v(t - \tau)| d\tau \qquad (34)$$

Now for any given $t \in R$, define $f_N:[0,T] \to R$ such that

$$f_N(\tau) = \sum_{n=-N}^{N} |g(\tau + nT)v(t - \tau)| \tag{35}$$

Then $\{f_N\}_{N=1}^{\infty}$ is a monotonically increasing sequence of functions, and $f_N \varepsilon L^1_{[0,T]}$ for each N.

Also

$$\begin{aligned}
\int_0^T f_N(\tau)d\tau &= \int_0^T \sum_{n=-N}^{N} |g(\tau + nT)v(t - \tau)|d\tau \\
&= \sum_{n=-N}^{N} \int_0^T |g(\tau + nT)v(t - \tau)|d\tau \\
&\leq \sum_{n=-\infty}^{\infty} \int_0^T |g(\tau + nT)v(t - \tau)|d\tau \\
&= \int_{-\infty}^{\infty} |g(\tau)v(t - \tau)|d\tau \\
&< \infty
\end{aligned} \tag{36}$$

(34) was used to establish (36). By the monotone convergence theorem for functions in $L_{[0,T]}$ (see Williamson [9], page 62), $f \triangleq \lim_{N\to\infty} f_N$ is in $L^1_{[0,T]}$ and

$$\lim_{N\to\infty} \int_0^T f_N(\tau)d\tau = \int_0^T f(\tau)d\tau$$

i.e.

$$\sum_{n=-\infty}^{\infty} \int_0^T |g(\tau + nT)v(t - \tau)|d\tau = \int_0^T \sum_{n=-\infty}^{\infty} |g(\tau + nT)v(t - \tau)|d\tau$$

Choosing $v(\tau) = 1$ for all $\tau \in R$ establishes Part 1.

2. For f defined above to be in $L^1_{[0,T]}$,

$$\sum_{n=-\infty}^{\infty} |g(\tau + nT)v(t - \tau)|$$

must be finite almost everywhere. Now for the same t used in the definition of f_N in (35), define $h_N:[0,t] \to C$ such that

$$h_N(\tau) = \sum_{n=-N}^{N} g(\tau + nT)v(t - \tau) \tag{37}$$

Now $h_N \in L^1_{[0,T]}$, $|h_N(\tau)| \leq f(\tau)$ for each N and $f \in L^1_{[0,T]}$, so by the Lebesque dominated convergence theorem,

$$\lim_{N\to\infty} \int_0^T h_N(\tau)d\tau = \int_0^T \lim_{N\to\infty} h_N(\tau)d\tau$$

i.e.

$$\sum_{n=-\infty}^{\infty} \int_0^T g(\tau + nT)v(t - \tau)d\tau = \int_0^T \sum_{n=-\infty}^{\infty} g(\tau + nT)v(t - \tau)d\tau \tag{38}$$

Finally, using (33) in (38)

$$\int_{-\infty}^{\infty} g(\tau)v(t - \tau)d\tau = \int_0^T \sum_{n=-\infty}^{\infty} g(\tau + nT)v(t - \tau)d\tau$$

This completes the proof of lemma A1.

We can now prove that g^* defined in (9) in Case 1 has the required properties.

<u>Theorem A2</u>:

If $g \in L^1_R$ and g^* is defined by (9), then

G1. $g^* \in L^1_{[0,T]}$

G2. g^* has period T

G3. The nth Fourier coefficient of g^* is $G(jn\omega_0)$

and in addition

4. $||G|| \leq \int_{-\infty}^{\infty} |g(t)|dt$

Proof:

G1. By definition,

$$g^*(t) = T \sum_{n=-\infty}^{\infty} g(t + nT)$$

so,

$$\int_0^T |g^*(t)|dt = \int_0^T \left| T \sum_{n=-\infty}^{\infty} g(t + nT) \right| dt$$

$$\leq T \int_0^T \sum_{n=-\infty}^{\infty} |g(t + nT)| dt$$

$$= T \int_{-\infty}^{\infty} |g(t)| dt \qquad (39)$$

$$< \infty$$

Lemma A1 was used in (39) to justify the interchanging of summation and integration.

G2. $$g^*(t + T) = T \sum_{n=-\infty}^{\infty} g(t + T + nT)$$

$$= T \sum_{n=-\infty}^{\infty} g(t + nT)$$

$$= g^*(t)$$

G3. By definition the nth Fourier coefficient of g^*

$$= \frac{1}{T} \int_0^T g^*(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T} \int_0^T T \sum_{n=-\infty}^{\infty} g(t + nT) e^{-jn\omega_0 t} dt$$

$$= \int_{-\infty}^{\infty} g(t)e^{-jn\omega_0 t}\, dt \qquad (40)$$

$$= G(jn\omega_0) \qquad (41)$$

Lemma A1 was used in (40) with $v(t) = e^{-jn\omega_0 t}$ and (5) was used in (41).

4. By Theorem A1,

$$\|G\| = \frac{1}{T}\int_0^T |g^*(t)|dt$$

$$= \frac{1}{T}\int_0^T \left|T \sum_{n=-\infty}^{\infty} g(t + nT)\right|dt \qquad (42)$$

$$\leq \int_0^T \sum_{n=-\infty}^{\infty} |g(t + nT)|dt$$

$$= \int_{-\infty}^{\infty} |g(t)|dt \qquad (43)$$

The definition (9) of g* was used in (42) and lemma A1 was used in (43). This completes Theorem A2.

Now we prove that g* in Case 2 as defined in (12) has the required properties.

Theorem A3:

If g* is defined by (12), and (11) is satisfied, then

G1. $g^* \in L^1_{[0,T]}$

G2. g^* has period T

G3. The nth Fourier coefficient of g^* is $G(jn\omega_0)$

and in addition

4. $$\|G\| \leq \sum_{n=-\infty}^{\infty} |G(jn\omega_0)|$$

Proof:

Let $f_N:[0,T] \to R$ be defined by

$$f_N(t) = \sum_{n=-N}^{N} G(jn\omega_0)e^{jn\omega_0 t}$$

Then $f_N \in C_{\omega_0}$ for each N. By (11),

$$\sum_{n=-\infty}^{\infty} |G(jn\omega_0)|$$

is finite so $\{f_N\}_{N=1}^{\infty}$ is a Cauchy sequence in the Banach space C_{ω_0}. Therefore g^* is in C_{ω_0}, so g^* has period T and is in $L^1_{[0,T]}$. This proves both G1 and G2.

G3. By definition

$$g^*(t) = \sum_{n=-\infty}^{\infty} G(jn\omega_0)e^{jn\omega_0 t}$$

and g^* is in $L^1_{[0,T]}$, so its nth Fourier coefficient exists and is clearly $G(jn\omega_0)$.

4. By Theorem A1, and the definition of g* in (12)

$$\|G\| = \frac{1}{T}\int_0^T |g^*(t)|dt$$

$$= \frac{1}{T}\int_0^T \left|\sum_{n=-\infty}^{\infty} G(jn\omega_0)e^{jn\omega_0 t}\right|dt \tag{44}$$

$$\leq \frac{1}{T}\int_0^T \sum_{n=-\infty}^{\infty} |G(jn\omega_0)|dt$$

$$= \sum_{n=-\infty}^{\infty} |G(jn\omega_0)|$$

This completes the proof of Theorem A3.

Finally we establish that Cases 1 and 2 are consistent.

Corollary:

If $g \in L_R^1$ and

$$\sum_{n=-\infty}^{\infty} |G(jn\omega_0)| < \infty ,$$

then

$$\sum_{n=-\infty}^{\infty} g(t + nT) = \sum_{n=-\infty}^{\infty} G(jn\omega_0)e^{jn\omega_0 t}$$

for almost all $t \in R$.

Proof:

Let

$$f(t) = \sum_{n=-\infty}^{\infty} g(t + nT)$$

and

$$h(t) = \sum_{n=-\infty}^{\infty} G(jn\omega_0 t)e^{jn\omega_0 t}$$

By Theorem A2, the nth Fourier coefficient of f is $G(jn\omega_0)$. By Theorem A3, the nth Fourier coefficient of h is $G(jn\omega_0)$. Therefore f = h almost everywhere.

This concludes the proof of the corollary.

APPENDIX B

If G(s) is a rational function of s, it may be easy to find

$$g^*(t) = \sum_{n=-\infty}^{\infty} G(jn\omega_0)e^{jn\omega_0 t} .$$

The technique is similar to that used in finding inverse Laplace transforms.

First expand G(s) in partial fractions, then replace s with $jn\omega_0$ getting

$$G(jn\omega_0) = \sum_{i=1}^{m} \sum_{\ell=1}^{\nu_i} \frac{a_{i\ell}}{(jn\omega_0 - \alpha_i)^{\ell}} \qquad (45)$$

Each term in (45) has an inverse transform of the form

$$g^*_{i\ell}(t) = \frac{a_{i\ell}}{T(\ell - 1)!} \frac{\partial^{\ell-1}}{\partial\alpha_i^{\ell-1}} \left[\frac{e^{\alpha_i T}}{1 - e^{\alpha_i T}}\right]$$

Finally,

$$g^*(t) = \sum_{i=1}^{m} \sum_{\ell=1}^{\nu_i} g^*_{i\ell}(t)$$

Notice that the assumptions on G require $(jn\omega_0 - \alpha_i) \neq 0$ for all integers n.

For convenience, some common special cases are given in Table 1. This table is a partial reproduction of Table 4.4 in Kaplan [8]. Due to a slight difference in definitions, he does not have the multiplicative factor T.

Table 1

$G_{i\ell}(jn\omega_0)$	$g^*_{i\ell}(t)$
$\dfrac{1}{jn\omega_0 - \alpha}$	$T\,\dfrac{e^{\alpha t}}{1 - e^{\alpha T}}$
$\dfrac{1}{(jn\omega_0 - \alpha)^2}$	$T\,\dfrac{e^{\alpha t}}{1 - e^{\alpha T}}\left[t + T\,\dfrac{e^{\alpha T}}{1 - e^{\alpha T}}\right]$
$\dfrac{1}{(jn\omega_0 - \alpha)^2 + \beta^2}\ ,\quad \beta \neq 0$	$\dfrac{T}{\beta}\,\dfrac{e^{\alpha t}\left[\sin \beta t - e^{\alpha T}\sin \beta(t - T)\right]}{1 + e^{2\alpha T} - 2e^{\alpha T}\cos \beta T}$

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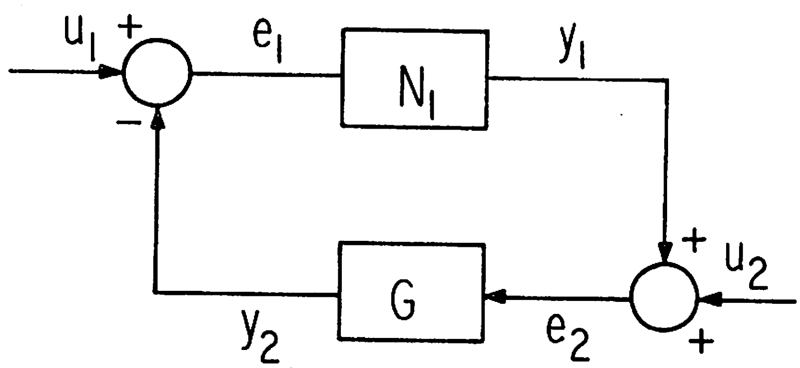
FIGURE CAPTIONS

Fig. 1. System (S)

Fig. 2. Testing Inequality (I)

Fig. 3. System of Example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+
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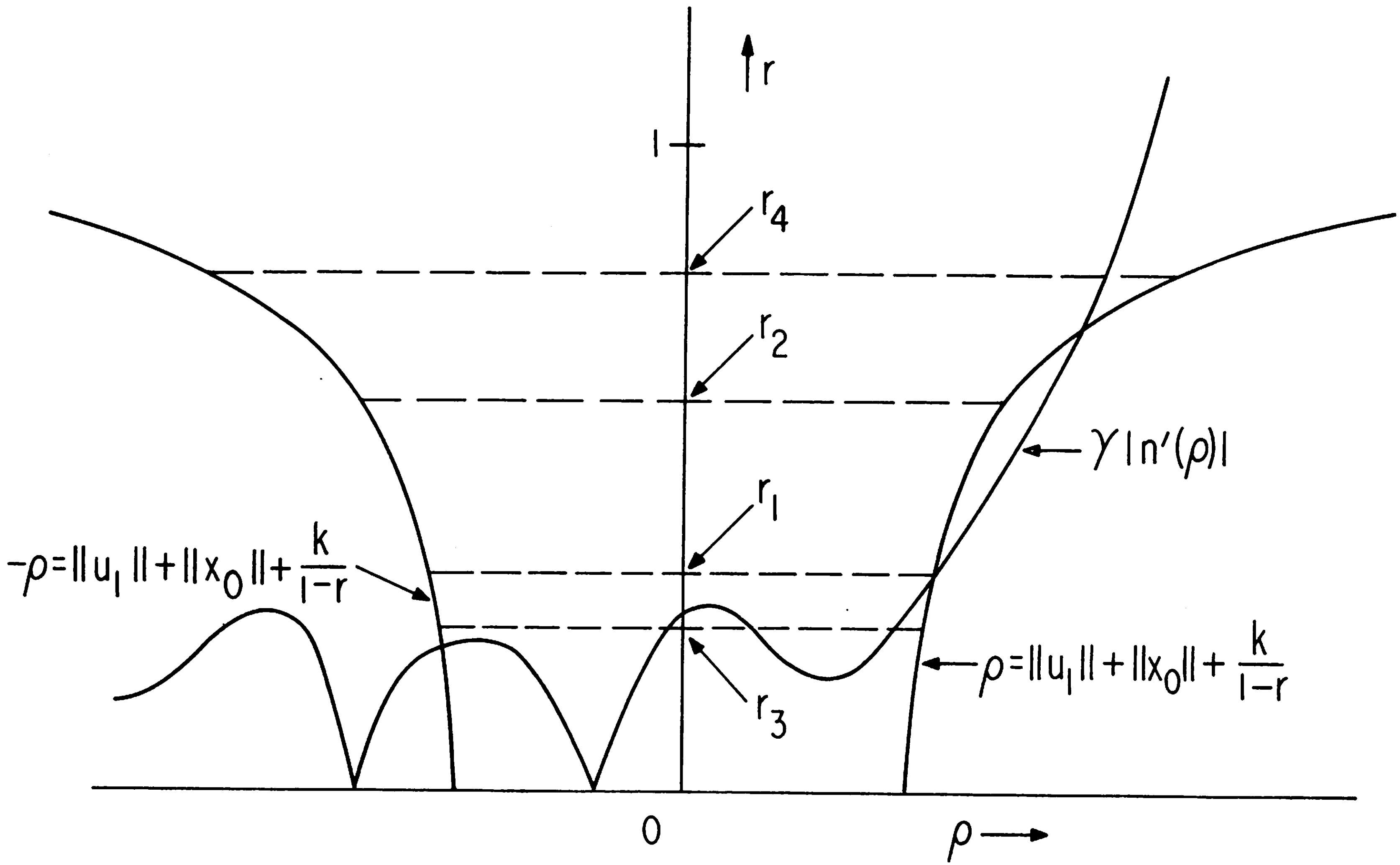
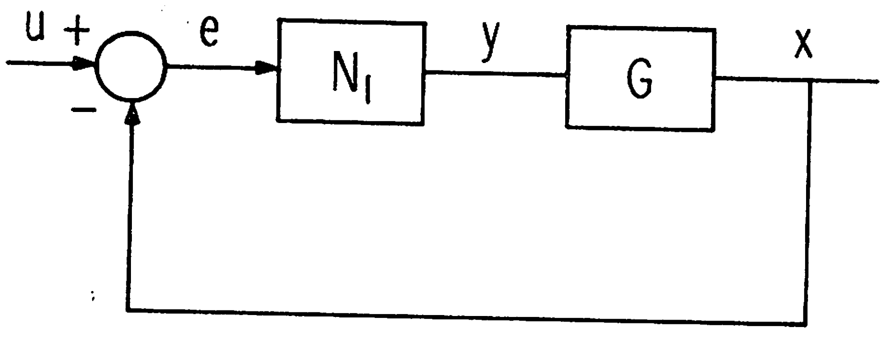

r
1
r_4
r_2
r_1
r_3
$\gamma|n'(\rho)|$
$-\rho=\|u_1\|+\|x_0\|+\frac{k}{1-r}$
$\rho=\|u_1\|+\|x_0\|+\frac{k}{1-r}$
0
ρ


u
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e
N_1
y
G
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