INPUT-OUTPUT PROPERTIES OF MULTIPLE-INPUT MULTIPLE-OUTPUT DISCRETE SYSTEMS: PART I

by

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Memorandum No. ERL-M-259

1 May 1969

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Research sponsored by the National Aeronautics and Space Administration under Grant NGL-05-003-016(Sup 6) and the Joint Services Electronics Program under Grant AF-AFOSR-68-1488.

ABSTRACT

Part I of this paper presents the best results concerning the existence, uniqueness and stability of linear discrete feedback systems. Both the multiple-input, multiple-output and the single-input single-output cases are considered. The results presented extend previously known results and they constitute key tools for the study of nonlinear time-invariant and/or time-varying systems. The simplicity of the analytic methods used in the derivations is worth noting.

I. Introduction

Part I of this paper considers linear systems. Part II will consider nonlinear systems. The purpose of Part I is to present the best results concerning the existence, uniqueness and stability of linear discrete feedback systems. All results and derivations concern the multiple-input multiple-output case. The specialized and often sharper results of the single-input single-output case are stated in the corollaries which follow each theorem. Our results generalize well established facts found in the literature (1,2,3,4,6). We need not survey previous results since this has been done in a paper by Tsypkin and Jury (5).

Our first theorem gives necessary and sufficient conditions under which a class of linear time-varying discrete systems is determinate. This theorem when coupled with Mason's signal flow graph theory leads to conditions under which any interconnection of linear time-invariant discrete systems and time-varying gains is determinate. Our second and third theorems consider linear time-invariant systems and give stability results in the form of input-output properties. Conditions under which zero position error is achieved are emphasized. Our last theorem considers a linear time-invariant subsystem and a time-varying gain. It shows that if the time-varying gain may be written as a constant plus an ℓ^1 sequence the stability properties are qualitatively unchanged. This result corresponds to Chen's result for the single-input single-output continuous-time case (7).

The results presented in Part I are important not only as far as linear systems are concerned but also because they constitute key tools for the study of nonlinear, time-invariant or time-varying systems. For the convenience of the reader we first settle some notations, and define terms. After describing the system under consideration, we state under the section Main Results each theorem and corrollary which we follow by commentaries. All proofs are relegated to the Appendix.

II. Notations

In the following, J_+ denotes the set of all nonnegative integers; $\mathbb{R}(\mathbb{R}_+)$ the set of all (nonnegative) real numbers; $\mathbb{R}^n$ the set of all n-vectors with real elements and $\mathbb{R}^{n\times n}$ the set of all nxn matrices with real elements. We use boldface for vectors and matrices. Matrices are distinguished by capitals. The symbol $|\cdot|$ denotes the absolute value of a real (or complex) number, the norm of a vector in $\mathbb{R}^n$ or the corresponding induced norm of a matrix in $\mathbb{R}^{n\times n}$ (viz. $|z|$, $|\mathbf{e}|$, $|\mathbf{A}|$). Since all norms in $\mathbb{R}^n$ are equivalent, our results hold for any norm in $\mathbb{R}^n$ provided the corresponding induced norm is used for $\mathbb{R}^{n\times n}$. The symbol $\|\cdot\|$ is used to denote norms in sequence spaces. The number zero, the zero vector in $\mathbb{R}^n$ and the zero matrix in $\mathbb{R}^{n\times n}$ are denoted by 0, $\mathbf{0}$ and $\mathbf{O}$, respectively.

III. Basic Terms

1) Let $p \in [1,\infty)$. A sequence in $\mathbb{R}^n$, $\mathbf{g} \triangleq \{\mathbf{g}_i\}_0^\infty : J_+ \to \mathbb{R}^n$, is said to be in ℓ_n^p if $\sum_{i=0}^{\infty} |\mathbf{g}_i|^p < \infty$; the corresponding ℓ_n^p-norm is defined by

$$\|\mathbf{g}\|_p \triangleq \left[\sum_{i=0}^{\infty} |\mathbf{g}_i|^p\right]^{1/p} .$$

Similarly, $\underset{\sim}{g} : J_+ \to \mathbb{R}^n$ is said to be in ℓ_n^∞ if $\sup_{i\in J_+} |\underset{\sim}{g}_i| < \infty$; the corresponding ℓ_n^∞-norm is defined by

$$\|\underset{\sim}{g}\|_\infty \triangleq \sup_{i\in J_+} |\underset{\sim}{g}_i| .$$

If $n = 1$ (scalar case), we write ℓ^p for ℓ_n^p $(1 \leq p \leq \infty)$. It is easy to show that $\ell_n^1 \subset \ell_n^2 \subset \ell_n^\infty$.

2. A sequence in $\mathbb{R}^{n\times n}$, $\underset{\sim}{G} = \{\underset{\sim}{G}_i\}_0^\infty : J_+ \to \mathbb{R}^{n\times n}$, is said to be in $\ell_{n\times n}^1$ if $\sum_{i=0}^{\infty} |\underset{\sim}{G}_i| < \infty$; the corresponding norm is defined by

$$\|\underset{\sim}{G}\|_1 \triangleq \sum_{i=0}^{\infty} |\underset{\sim}{G}_i| .$$

Note that $|\underset{\sim}{G}_i|$ is the <u>induced norm</u> of the $n\times n$ matrix $\underset{\sim}{G}_i$ corresponding to the norm selected for the vectors in $\mathbb{R}^n$.

3. Given a sequence in $\mathbb{R}^n$, $\underset{\sim}{g} = \{\underset{\sim}{g}_i\}_0^\infty$, the z-transform of the sequence is the function of the complex variable z defined by

$$\underset{\sim}{\tilde{g}}(z) \triangleq \sum_{i=0}^{\infty} \underset{\sim}{g}_i \, z^{-i}$$

It is well known that $\underset{\sim}{\tilde{g}}(\cdot)$ is analytic for $|z| > \limsup_{k\to\infty} |\underset{\sim}{g}_k|^{1/k}$.

4. Let $\underset{\sim}{G} : J_+ \to \mathbb{R}^{n\times n}$ and $\underset{\sim}{f} : J_+ \to \mathbb{R}^n$. The convolution of the sequence $\underset{\sim}{f}$ by $\underset{\sim}{G}$, denoted as $\underset{\sim}{G} * \underset{\sim}{f}$, is the sequence which maps $J_+ \to \mathbb{R}^n$ and whose mth term is

$$(\underset{\sim}{G} * \underset{\sim}{f})_m \triangleq \sum_{i=0}^{m} \underset{\sim}{G}_{m-i} \, \underset{\sim}{f}_i .$$

IV. System Description

In this paper we consider the multiple-input, multiple-output, linear, discrete system $\underset{\sim}{S}$ shown in Fig. 1, where $\underset{\sim}{u}$, $\underset{\sim}{e}$ and $\underset{\sim}{y} : J_+ \to \mathbb{R}^n$, $\underset{\sim}{G} : J_+ \to \mathbb{R}^{n\times n}$ and $\underset{\sim}{K} : J_+ \to \mathbb{R}^{n\times n}$. The sequences of n-vectors $\underset{\sim}{u} \triangleq \{\underset{\sim}{u}_i\}_0^\infty$, $\underset{\sim}{e} \triangleq \{\underset{\sim}{e}_i\}_0^\infty$ and $\underset{\sim}{y} \triangleq \{\underset{\sim}{y}_i\}_0^\infty$, are the input, the error and the output, respectively. The sequence of n×n matrices $\underset{\sim}{G} \triangleq \{\underset{\sim}{G}_i\}_0^\infty$ specifies the impulse response matrix which characterizes the linear dynamical subsystem. The sequence of n×n matrices $\underset{\sim}{K} \triangleq \{\underset{\sim}{K}_i\}_0^\infty$ specifies a memoryless time-varying gain. In contrast to a number of previous papers, we do not assume that the $\underset{\sim}{K}_i$'s are diagonal, or symmetric. The system $\underset{\sim}{S}$ is described by the following system equations:

$$\underset{\sim}{y} = \underset{\sim}{G} * (\underset{\sim}{K}\,\underset{\sim}{e}) \tag{1}$$

$$\underset{\sim}{e} = \underset{\sim}{u} - \underset{\sim}{G} * (\underset{\sim}{K}\,\underset{\sim}{e}) \tag{2}$$

Equivalently, (2) can be written as

$$\underset{\sim}{e}_m = \underset{\sim}{u}_m - \sum_{i=0}^{m} \underset{\sim}{G}_{m-i}\,\underset{\sim}{K}_i\,\underset{\sim}{e}_i \quad , \qquad \forall m \in J_+ \tag{3}$$

For the scalar case (n = 1), $u \triangleq \{u_i\}_0^\infty$, $e \triangleq \{e_i\}_0^\infty$, $y \triangleq \{y_i\}_0^\infty$, $g \triangleq \{g_i\}_0^\infty$, $k \triangleq \{k_i\}_0^\infty$, are the sequences which map $J_+ \to \mathbb{R}$ and the system equations which describe the single-input, single-output, linear time-varying discrete system S become

$$y = g * (k\,e) \tag{1a}$$

$$e = u - g * (k\,e) \tag{2a}$$

Similarly, we can rewrite (2a) as

$$e_m = u_m - \sum_{i=0}^{m} g_{m-i}\, k_i\, e_i \quad , \qquad \forall m \in J_+ \tag{3a}$$

As a special case, if the gain block is time-invariant, then $\underset{\sim}{K} = \overline{\underset{\sim}{K}} \in \mathbb{R}^{n\times n}$ is an nxn constant matrix and $k = \bar{k} \in \mathbb{R}$ is a constant. Consequently (3) and (3a) become respectively

$$\underset{\sim}{e}_m = \underset{\sim}{u}_m - \sum_{i=0}^{m} \underset{\sim}{G}_{m-i}\, \overline{\underset{\sim}{K}}\, \underset{\sim}{e}_i \tag{3)'}$$

$$e_m = u_m - k \sum_{i=0}^{m} g_{m-i}\, e_i \tag{3a)'}$$

Remarks:

1. Note that for the time-invariant scalar case the order of g and $\bar{k}$ in (3a)' is immaterial, in contrast to (3)' where the order of $\underset{\sim}{G}$ and $\underset{\sim}{K}$ is important. However, in the time-varying case, the order of $\underset{\sim}{K}$ (k) and $\underset{\sim}{G}$ (g) is important both in (3) and (3a).

2. There is no loss of generality in assuming that $\underset{\sim}{K}_i$'s and $\underset{\sim}{G}_i$'s are square nxn matrices: indeed if it were not so we could make them square by adding rows (or columns) of zeros. Therefore the theory described here applies also to the case where the number of inputs is different from the number of outputs, a common occurrence in practice.

V. Main Results

Theorem 1 specifies completely the conditions under which the linear time-varying system $\underset{\sim}{S}$ is determinate ($\underset{\sim}{13}$, p. 96).

Theorem 1

For any input sequence $\underset{\sim}{u} : J_+ \to \mathbb{R}^n$, the system equation (2) has a unique solution $\underset{\sim}{e} : J_+ \to \mathbb{R}^n$ if and only if the matrices $\underset{\sim}{I} + \underset{\sim}{G}_o \underset{\sim}{K}_i$ are nonsingular for all $i \in J_+$, i.e.,

$$\det (\underset{\sim}{I} + \underset{\sim}{G}_o \underset{\sim}{K}_i) \neq 0 \qquad \forall i \in J_+ \; . \tag{4}$$

For the time-invariant gain, $\underset{\sim}{K}_i = \overline{\underset{\sim}{K}}$, $\forall i \in J_+$, (4) reduces to the single condition

$$\det (\underset{\sim}{I} + \underset{\sim}{G}_o \overline{\underset{\sim}{K}}) \neq 0 \tag{4'}$$

Corollary 1

For any input sequence $u : J_+ \to \mathbb{R}$, the system equation (2a) has a unique solution $e : J_+ \to \mathbb{R}$ if and only if $1 + k_i g_o \neq 0$, $\forall i \in J_+$. In particular, if the gain k is constant, i.e. $k_i = \overline{k}$, $\forall i \in J_+$, then the above conditions become the single condition $1 + \overline{k} g_o \neq 0$.

Comment:

These results are important because they specify the necessary and sufficient conditions for an arbitrary interconnection of multiple-input, multiple-output, linear, discrete systems to be determinate: indeed it is well known that any such arbitrary interconnection can by Mason's signal flow graph theorem be reduced to a single feedback loop model, to which our theorem applies.

Theorem 2 below gives a stability criterion for the linear time-invariant discrete system $\underset{\sim}{S}$ shown in Fig. 1

Theorem 2

Consider the multiple-input, multiple-output, linear, time-invariant discrete system $\underset{\sim}{S}$ shown in Fig. 1 which is characterized by the system equations (1) and (2) with $\underset{\sim}{K} = \overline{\underset{\sim}{K}} \in \mathbb{R}^{n\times n}$ being a constant gain matrix. Let the open-loop z-transfer function of the linear subsystem $\underset{\sim}{G}$ be of the form

$$\tilde{\underset{\sim}{G}}(z) = \underset{\sim}{R}\left(1 - z^{-1}\right)^{-1} + \sum_{i=0}^{\infty} \underset{\sim}{G}_i z^{-i} \triangleq \underset{\sim}{R}\left(1 - z^{-1}\right)^{-1} + \tilde{\underset{\sim}{G}}_\ell(z) \qquad (5)$$

where $\underset{\sim}{R}$ is an n×n constant matrix and $\underset{\sim}{G}_\ell \triangleq \{\underset{\sim}{G}_i\}_0^\infty = \mathcal{Z}^{-1}\{\tilde{\underset{\sim}{G}}_\ell(z)\} \in \ell^1_{n\times n}$.
Under these conditions, if

$$\inf_{|z|\geq 1} \left|\det[\underset{\sim}{I} + \tilde{\underset{\sim}{G}}(z)\,\overline{\underset{\sim}{K}}]\right| > 0 \qquad (6)$$

and if either $\underset{\sim}{R} = \underset{\sim}{0}$ or $\underset{\sim}{R}\,\overline{\underset{\sim}{K}}$ is nonsingular, then

(a) the closed-loop z-transfer function (relating $\underset{\sim}{y}$ to $\underset{\sim}{u}$)

$$\tilde{\underset{\sim}{H}}(z) = [\underset{\sim}{I} + \tilde{\underset{\sim}{G}}(z)\,\overline{\underset{\sim}{K}}]^{-1}\,\tilde{\underset{\sim}{G}}(z)\,\overline{\underset{\sim}{K}} \qquad (7)$$

is analytic outside the unit disk $|z| \leq 1$, and the closed loop impulse response

$$\underset{\sim}{H} \triangleq \{\underset{\sim}{H}_i\}_0^\infty = \mathcal{Z}^{-1}\{\underset{\sim}{\tilde{H}}(z)\} \in \ell^1_{n\times n} \tag{8}$$

(b) For any fixed $p \in [1,\infty]$, $\underset{\sim}{u} \in \ell^p_n$ implies that $\underset{\sim}{y} \in \ell^p_n$.

(c) $\underset{\sim}{u} \in \ell^\infty_n$ and $\lim_{m\to\infty} \underset{\sim}{u}_m = \underset{\sim}{0}$ implies that $\underset{\sim}{y} \in \ell^\infty_n$ and $\lim_{m\to\infty} \underset{\sim}{y}_m = \underset{\sim}{0}$.

(d) If, in particular, $\underset{\sim}{R}\,\underset{\sim}{\overline{K}}$ is nonsingular, then $\underset{\sim}{u} \in \ell^\infty_n$ and $\lim_{m\to\infty} \underset{\sim}{u}_m = \underset{\sim}{u}_\infty$ (a constant vector in $\mathbb{R}^n$) implies that $\lim_{m\to\infty} \underset{\sim}{e}_m = \underset{\sim}{0}$ and $\lim_{m\to\infty} \underset{\sim}{y}_m = \underset{\sim}{u}_\infty$.

Corollary 2

Consider the single-input, single-output, linear, time-invariant, discrete system S shown in Fig. 1, which is described by the system equations (1a) and (2a) with $k = \overline{k} \in \mathbb{R}$ being a constant gain. Let the open-loop z-transfer function of the linear subsystem G be of the form

$$\tilde{g}(z) = r\left(1 - z^{-1}\right)^{-1} + \sum_{i=0}^{\infty} g_i z^{-i} \triangleq r\left(1 - z^{-1}\right)^{-1} + \tilde{g}_\ell(z) \tag{5a}$$

where r is a real constant and $g_\ell \triangleq \{g_i\}_0^\infty = \mathcal{Z}^{-1}\{\tilde{g}_\ell(z)\} \in \ell^1$. Under these conditions, if

$$\inf_{|z|\geq 1} |1 + \overline{k}\,\tilde{g}(z)| > 0 \tag{6a}$$

and if either $r = 0$ or $r\overline{k} \neq 0$, then

(a) the closed-loop z-transfer function (between y and u)

$$\tilde{h}(z) = \frac{\bar{k}\,\tilde{g}(z)}{1 + \bar{k}\,\tilde{g}(z)} \tag{7a}$$

is analytic for $|z| > 1$ and the closed loop impulse response

$$h \triangleq \{h_i\}_0^{\infty} = \mathcal{Z}^{-1}\left\{\frac{\bar{k}\,\tilde{g}(z)}{1 + \bar{k}\,\tilde{g}(z)}\right\} \in \ell^1 \tag{8a}$$

(b) for any fixed $p \in [1,\infty]$, $u \in \ell^p$ implies that $y \in \ell^p$.

(c) $u \in \ell^{\infty}$ and $\lim_{m\to\infty} u_m = 0$ implies that $y \in \ell^{\infty}$ and $\lim_{m\to\infty} y_m = 0$.

(d) If, in particular, $r\bar{k} \neq 0$, then $u \in \ell^{\infty}$ and $\lim_{m\to\infty} u_m = u_{\infty}$ (a constant scalar) implies that $\lim_{m\to\infty} e_m = 0$ and $\lim_{m\to\infty} y_m = u_{\infty}$.

Comments:

1. The map $z \mapsto \tilde{\underset{\sim}{G}}(z)$ is analytic outside the unit disk $|z| \le 1$, consequently $\det[\underset{\sim}{I} + \tilde{\underset{\sim}{G}}(z)\,\bar{\underset{\sim}{K}}]$ is analytic for $|z| > 1$ (10, Chap. IX).

2. Since $\det[\underset{\sim}{I} + \tilde{\underset{\sim}{G}}(z)\,\bar{\underset{\sim}{K}}]$ is analytic for $|z| > 1$, note that z real and $z \to \infty$ in (6) implies that $\det[\underset{\sim}{I} + \underset{\sim}{G}_o\,\bar{\underset{\sim}{K}}] \neq 0$, therefore there is a <u>unique</u> sequence $\underset{\sim}{M} : J_+ \to \mathbb{R}^{n\times n}$ such that

$$\underset{\sim}{M} = \underset{\sim}{V} - \underset{\sim}{G} * (\bar{\underset{\sim}{K}}\,\underset{\sim}{M})$$

with $\underset{\sim}{V} = \underset{\sim}{I}, \underset{\sim}{0}, \underset{\sim}{0}, \ldots$. Furthermore, $\tilde{\underset{\sim}{M}}(z) = [\underset{\sim}{I} + \tilde{\underset{\sim}{G}}(z)\,\bar{\underset{\sim}{K}}]^{-1}$. By Lemma 2 in the Appendix, it follows that under the conditions of Theorem 2, $\underset{\sim}{M} \triangleq \{\underset{\sim}{M}_i\}_0^{\infty} = \mathcal{Z}^{-1}\left\{[\underset{\sim}{I} + \tilde{\underset{\sim}{G}}(z)\,\bar{\underset{\sim}{K}}]^{-1}\right\} \in \ell^1_{n\times n}$.

3. In case $\underset{\sim}{R} = \underset{\sim}{0}$, if we plot on the complex plane the graph of

$\phi \mapsto \det[\underset{\sim}{I} + \tilde{\underset{\sim}{G}}(\varepsilon^{j\phi})\ \bar{\underset{\sim}{K}}]$ for ϕ increasing from 0 to 2π, then the number of clockwise encirclements of the origin is equal to the number of zeros of $\det[\underset{\sim}{I} + \tilde{\underset{\sim}{G}}(z)\ \bar{\underset{\sim}{K}}]$ in the set $\{z \mid |z| \geq 1\}$. In case $\underset{\sim}{R} \neq \underset{\sim}{0}$, the usual indentation about the point $z = (1,0)$ must be used.

4. Part (d) of Theorem 2 states that provided that $\underset{\sim}{R}\ \bar{\underset{\sim}{K}}$ is nonsingular and $\lim_{m\to\infty} \underset{\sim}{u}_m = \underset{\sim}{u}_\infty$ (a constant vector in $\mathbb{R}^n$), the system has a zero steady-state error, i.e. $\lim_{m\to\infty} \underset{\sim}{e}_m = \underset{\sim}{0}$. Therefore the system $\underset{\sim}{S}$ behaves as a position servo should! Note that the zero-steady state error $\lim \underset{\sim}{e}_m = \underset{\sim}{0}$ means zero steady-state error at the sample points only.

5. If in all the statements of the multiple-input, multiple-output case, the matrices $\underset{\sim}{G}\ \bar{\underset{\sim}{K}}$, $\underset{\sim}{R}\ \bar{\underset{\sim}{K}}$ and $\underset{\sim}{G}_o\ \bar{\underset{\sim}{K}}$ are changed to $\bar{\underset{\sim}{K}}\ \underset{\sim}{G}$, $\bar{\underset{\sim}{K}}\ \underset{\sim}{R}$ and $\bar{\underset{\sim}{K}}\ \underset{\sim}{G}_o$ respectively, then all the results in the paper also hold for the system model $\underset{\sim}{S}_1$ shown in Fig. 2 with $\underset{\sim}{K} = \bar{\underset{\sim}{K}} \in \mathbb{R}^n$. In this connection, note that for any n×n matrices $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\det[\underset{\sim}{I} + \underset{\sim}{A}\ \underset{\sim}{B}] = \det[\underset{\sim}{I} + \underset{\sim}{B}\ \underset{\sim}{A}]$. This will be used below.

6. The results that are closest in spirit to our Theorem 2 are those of Sandberg (6).

Theorem 3 below allows $\tilde{\underset{\sim}{G}}(z)$, the z-transfer function of the linear subsystem, to have poles <u>outside</u> the unit disk $|z| \leq 1$.

<u>Theorem 3</u>

Consider a multiple-input, multiple-output, linear, time-invariant discrete system such as shown in Fig. 1 and described by equations (1) and (2) with $\underset{\sim}{K} = \bar{\underset{\sim}{K}} \in \mathbb{R}^{n\times n}$ being a time-invariant gain matrix.

If

(i) $\det[\underset{\sim}{I} + \underset{\sim}{G}_o \, \underset{\sim}{\overline{K}}] \neq 0$

(ii) the open-loop impulse response sequence $\underset{\sim}{G} \triangleq \{\underset{\sim}{G}_i\}_0^\infty$ is exponentially bounded,

(iii) the closed-loop z-transfer function between $\underset{\sim}{u}$ and $\underset{\sim}{y}$

$$\underset{\sim}{\tilde{H}}(z) = [\underset{\sim}{I} + \underset{\sim}{\tilde{G}}(z) \, \underset{\sim}{\overline{K}}]^{-1} \, \underset{\sim}{\tilde{G}}(z) \, \underset{\sim}{\overline{K}} \tag{9}$$

is analytic for $|z| > \rho$ where $\rho < 1$,

then the closed-loop impulse response sequence $\underset{\sim}{H} \triangleq \{\underset{\sim}{H}_i\}_0^\infty = \mathcal{Z}^{-1}\{\underset{\sim}{\tilde{H}}(z)\}$ is bounded by a decaying exponential; more precisely, for any $\beta \in (\rho,1)$ there is a finite number b such that

$$|\underset{\sim}{H}_i| \leq b\beta^i \qquad \forall i \in J_+ \tag{10}$$

<u>Corollary 3</u>

Consider a single-input, single-output, linear, time-invariant discrete system such as shown in Fig. 1 and described by equations (1a) and (2a) with $k = \overline{k} \in \mathbb{R}$ being a constant gain. If

(i) $1 + \overline{k}\, g_o \neq 0$

(ii) the open-loop impulse response sequence $g \triangleq \{g_i\}_0^\infty$ is exponentially bounded,

(iii) the closed-loop z-transfer function between u and y

$$\tilde{h}(z) = \frac{\bar{k}\,\tilde{g}(z)}{1 + \bar{k}\,\tilde{g}(z)} \qquad (9a)$$

is analytic for $|z| > \rho_1$, where $\rho_1 < 1$,

then the closed-loop impulse response sequence $h \triangleq \{h_i\}_0^\infty = \mathcal{Z}^{-1}\{\tilde{h}(z)\}$ is bounded by a decaying exponential; more precisely, for any $\beta_1 \in (\rho_1, 1)$, there is a finite number b_1 such that

$$|h_i| \leq b_1 \beta_1^i \qquad \forall i \in J_+ \qquad (10a)$$

Comment:

The assumptions on the open-loop impulse response sequence $\underset{\sim}{G} \triangleq \{\underset{\sim}{G}_i\}_0^\infty$ are less restrictive than in the previous theorem: indeed $\tilde{\underset{\sim}{G}}(z)$ may have poles outside the unit disk $|z| \leq 1$. This is compensated by the requirement (iii) on $\tilde{\underset{\sim}{H}}(z)$. The conclusion is much stronger than previously: indeed $\{\underset{\sim}{H}_i\}_0^\infty$ is not only in $\ell^1_{n\times n}$ but decays exponentially.

Theorem 4 below gives a stability criterion for a class of multiple-input, multiple-output, linear, time-varying discrete systems.

Theorem 4

Consider the multiple-input, multiple-output, linear, time-varying, discrete system $\underset{\sim}{S}_1$ shown in Fig. 2. Let $\underset{\sim}{G}$ be the linear subsystem whose z-transfer function is given by

$$\tilde{\underset{\sim}{G}}(z) = \underset{\sim}{R}\left(1 - z^{-1}\right)^{-1} + \sum_{i=0}^{\infty} \underset{\sim}{G}_i\, z^{-i} \triangleq \underset{\sim}{R}\left(1 - z^{-1}\right)^{-1} + \tilde{\underset{\sim}{G}}_\ell(z) \qquad (11)$$

where $\underset{\sim}{R}$ is an n×n constant matrix and $\underset{\sim}{G}_\ell \triangleq \{\underset{\sim}{G}_i\}_0^\infty = \mathcal{Z}^{-1}\{\tilde{\underset{\sim}{G}}_\ell(z)\} \in \ell^1_{n\times n}$.
Let $\underset{\sim}{K} : J_+ \to R^{n\times n}$ be the memoryless, time-varying gain which is defined by

$$\underset{\sim}{K} \triangleq \overline{\underset{\sim}{K}} + \hat{\underset{\sim}{K}} = \overline{\underset{\sim}{K}} + \{\hat{\underset{\sim}{K}}_i\}_0^\infty \tag{12}$$

where $\overline{\underset{\sim}{K}}$ is an n×n constant matrix and $\hat{\underset{\sim}{K}} \triangleq \{\hat{\underset{\sim}{K}}_i\}_0^\infty \in \ell^1_{n\times n}$. Let

$$\det[\underset{\sim}{I} + (\underset{\sim}{R} + \underset{\sim}{G}_0)\overline{\underset{\sim}{K}} + (\underset{\sim}{R} + \underset{\sim}{G}_0)\hat{\underset{\sim}{K}}_i] \neq 0 \qquad \forall i \in J_+ \tag{13}$$

Under these conditions, if

$$\inf_{|z|\geq 1} \left|\det[\underset{\sim}{I} + \tilde{\underset{\sim}{G}}(z)\,\overline{\underset{\sim}{K}}]\right| > 0 \tag{14}$$

and if either $\underset{\sim}{R} = \underset{\sim}{0}$ or $\underset{\sim}{R}\,\overline{\underset{\sim}{K}}$ is nonsingular, then

(a) $\underset{\sim}{u} \in \ell^\infty_n$ implies that $\underset{\sim}{w} \in \ell^\infty_n$

(b) $\underset{\sim}{u} \in \ell^1_n$ implies that $\underset{\sim}{w} \in \ell^1_n$, consequently $\lim_{m\to\infty} \underset{\sim}{w}_m = \underset{\sim}{0}$.

The same results also hold for $\underset{\sim}{e}$ and $\underset{\sim}{y}$.

<u>Corollary 4</u>

Consider the single-input, single-output, linear, time-varying, discrete system S_1 shown in Fig. 2. Let G be the linear subsystem whose z-transfer function is given by

$$\tilde{g}(z) = r\left(1 - z^{-1}\right)^{-1} + \sum_{i=0}^{\infty} g_i z^{-i} \triangleq r\left(1 - z^{-1}\right)^{-1} + \tilde{g}_\ell(z) \tag{11a}$$

where r is a constant and $g_\ell \stackrel{\Delta}{=} \{g_i\}_0^\infty = \mathcal{Z}^{-1}\{\tilde{g}_\ell(z)\} \in \ell^1$. Let $k : J_+ \to \mathbb{R}$ be the memoryless time-varying gain which is defined by

$$k \stackrel{\Delta}{=} \bar{k} + \hat{k} = \bar{k} + \{\hat{k}_i\}_0^\infty \tag{12a}$$

where $\bar{k}$ is a constant and $\hat{k} \stackrel{\Delta}{=} \{\hat{k}_i\}_0^\infty \in \ell^1$. Let

$$1 + (r + g_o)\bar{k} + (r + g_o)\hat{k}_i \neq 0 \qquad \forall i \in J_+ \tag{13a}$$

Under these conditions, if

$$\inf_{|z|\geq 1} |1 + \bar{k}\,\tilde{g}(z)| > 0 \tag{14a}$$

and if either $r = 0$ or $\bar{k}r \neq 0$, then

(a) $u \in \ell^\infty$ implies that $w \in \ell^\infty$

(b) $u \in \ell^1$ implies that $w \in \ell^1$, consequently, $\lim_{m\to\infty} w_m = 0$.

The same results also hold for e and y.

Comment:

The condition on the time-varying gain that $\hat{\underset{\sim}{K}} \stackrel{\Delta}{=} \{\hat{\underset{\sim}{K}}_i\}_0^\infty \in \ell^1_{n\times n}$ ($\hat{k} = \{\hat{k}_i\}_0^\infty \in \ell^1$) corresponds to a similar result of Chen's $\underset{\sim}{(7)}$ for the single-input, single-output, continuous-time case.

Conclusion

The four theorems of Part I of this paper specify the input-output properties of large classes of multiple-input multiple-output linear discrete systems. Each one of them extends previously known results. We wish to draw attention to the simplicity of the analytical methods used in the derivations.

Appendix

To clarify the proofs of theorems, we state two lemmas whose proofs are straightforward and hence omitted.

Lemma 1

Let p be a fixed number in $[1,\infty]$, let $\underset{\sim}{G} \in \ell^1_{n\times n}$ and $\underset{\sim}{f} \in \ell^p_n$ and let $\underset{\sim}{h} = \underset{\sim}{G} * \underset{\sim}{f}$; then $\underset{\sim}{h} \in \ell^p_n$.

This lemma can easily be proved by the use of Hölder inequality applied to sequences (8).

Lemma 2

Let $\tilde{\underset{\sim}{A}}(z)$ be the z-transform of a sequence $\underset{\sim}{A} \triangleq \{\underset{\sim}{A}_i\}_0^\infty$ which maps $J_+ \to \mathbb{R}^{n\times n}$ and is in $\ell^1_{n\times n}$. If

$$\inf_{|z|\geq 1} |\det \tilde{\underset{\sim}{A}}(z)| > 0 \tag{15}$$

then $\mathcal{Z}^{-1}\{[\tilde{\underset{\sim}{A}}(z)]^{-1}\}$ is a sequence mapping $J_+ \to \mathbb{R}^{n\times n}$ and it is in $\ell^1_{n\times n}$.

For n = 1, this lemma follows from the observation that the inverse of a sequence $a = \{a_i\}_0^\infty$ with $a_o \neq 0$ (as a consequence of the assumption) is also a one-sided sequence mapping $J_+ \to \mathbb{R}$; furthermore, by Wiener's theorem (9), if $a \in \ell^1$ and $\tilde{a}(z) \neq 0$, $\forall |z| = 1$, its inverse $\mathcal{Z}^{-1}\{[\tilde{a}(z)]^{-1}\}$ is also in ℓ^1. For $n > 1$, the result follows by using Cramer's rule and the same reasoning.

Proof of Theorem 1

We shall prove it only for the time-varying case. The time-invariant case follows directly from the fact $\underset{\sim}{K}_i = \overline{\underset{\sim}{K}}$ (a constant matrix) for all $i \in J_+$.

First we prove the existence of a solution of (2). The sufficiency is proved by direct calculation. From (3) we have successively

$$
\begin{aligned}
\underset{\sim}{u}_o &= (\underset{\sim}{I} + \underset{\sim}{G}_o \underset{\sim}{K}_o)\, \underset{\sim}{e}_o \\
\underset{\sim}{u}_1 &= \underset{\sim}{G}_1 \underset{\sim}{K}_o \underset{\sim}{e}_o + (\underset{\sim}{I} + \underset{\sim}{G}_o \underset{\sim}{K}_1)\, \underset{\sim}{e}_1 \qquad (16)\\
&\vdots \\
\underset{\sim}{u}_i &= \sum_{j=0}^{i-1} \underset{\sim}{G}_{i-1-j} \underset{\sim}{K}_j \underset{\sim}{e}_j + (\underset{\sim}{I} + \underset{\sim}{G}_o \underset{\sim}{K}_i)\, \underset{\sim}{e}_i \\
&\vdots
\end{aligned}
$$

By assumption, matrices $(\underset{\sim}{I} + \underset{\sim}{G}_o \underset{\sim}{K}_i)$ are nonsingular for all $i \in J_+$, hence, starting from the top, each equation in (16) can be solved (uniquely) for $\underset{\sim}{e}_o, \underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_i, \ldots$. Therefore, $\det(\underset{\sim}{I} + \underset{\sim}{G}_o \underset{\sim}{K}_i) \neq 0, \forall i \in J_+$, implies that (2) has, for <u>any</u> $\underset{\sim}{u}$, at least one solution $\underset{\sim}{e} : J_+ \to \mathbb{R}^n$.

Necessity is proved by contradiction. Suppose $\det(\underset{\sim}{I} + \underset{\sim}{G}_o \underset{\sim}{K}_i) = 0$ for some $i \in J_+$, say m, then $(\underset{\sim}{I} + \underset{\sim}{G}_o \underset{\sim}{K}_m)$ has a nontrivial null space $\mathcal{N}_m$ and its range $\mathcal{R}_m$ has a dimension < n. If $\underset{\sim}{u}_m$ is such that

$$
\underset{\sim}{f}_m \triangleq \left(\underset{\sim}{u}_m - \sum_{j=0}^{m-1} \underset{\sim}{G}_{m-1-j} \underset{\sim}{K}_j \underset{\sim}{e}_j\right) \neq \underset{\sim}{0}
$$

but $\underset{\sim}{f}_m \notin \mathcal{R}_m$, then there is no vector $\underset{\sim}{e}_m \in \mathbb{R}^n$ which satisfies the mth equation in the family of equations in (16), i.e.

$$
\underset{\sim}{u}_m = \sum_{j=0}^{m-1} \underset{\sim}{G}_{m-1-j} \underset{\sim}{K}_j \underset{\sim}{e}_j + (\underset{\sim}{I} + \underset{\sim}{G}_o \underset{\sim}{K}_m)\, \underset{\sim}{e}_m \qquad (17)
$$

Therefore, in order that for <u>any</u> given input sequence $\underset{\sim}{u}$, the equation $\underset{\sim}{e} = \underset{\sim}{u} - \underset{\sim}{G} * (\underset{\sim}{K}\,\underset{\sim}{e})$ has a sequence $\underset{\sim}{e} : J_+ \to \mathbb{R}^n$ as a solution, it is necessary that $\det[\underset{\sim}{I} + \underset{\sim}{G}_o \underset{\sim}{K}_i] \neq 0$, $\forall i \in J_+$.

Next we show that the solution $\underset{\sim}{e}$ of (2) is unique. Suppose $\underset{\sim}{e}$ and $\hat{\underset{\sim}{e}}$ are two solutions of (2); then, by subtraction, we have

$$(\underset{\sim}{e} - \hat{\underset{\sim}{e}}) = -\underset{\sim}{G} * [\underset{\sim}{K}\,(\underset{\sim}{e} - \hat{\underset{\sim}{e}})] \qquad (18)$$

But referring to (16), it is seen that (18) can have a nontrivial solution only if $\det(\underset{\sim}{I} + \underset{\sim}{G}_o \underset{\sim}{K}_i) = 0$ for some $i \in J_+$; this is ruled out by assumption. Therefore uniqueness follows.

<u>Proof of Theorem 2</u>

(a) By the definition of z-transform and the system equations (1) and (2) with $\underset{\sim}{K} = \overline{\underset{\sim}{K}} \in \mathbb{R}^{n\times n}$ being a constant gain matrix, we obtain immediately the closed-loop z-transfer function between $\underset{\sim}{u}$ and $\underset{\sim}{y}$:

$$\tilde{\underset{\sim}{H}}(z) = [\underset{\sim}{I} + \tilde{\underset{\sim}{G}}(z)\,\overline{\underset{\sim}{K}}]^{-1}\,\tilde{\underset{\sim}{G}}(z)\,\overline{\underset{\sim}{K}} \qquad (19)$$

Since $\tilde{\underset{\sim}{G}}(z)$ is analytic for $|z| > 1$ and hence $\det[\underset{\sim}{I} + \tilde{\underset{\sim}{G}}(z)\,\overline{\underset{\sim}{K}}]$ is also analytic for $|z| > 1$, by (6), $\tilde{\underset{\sim}{H}}(z)$ in (19) is analytic for $|z| > 1$. To show that $\underset{\sim}{H} \triangleq \{\underset{\sim}{H}_i\}_0^\infty = \mathcal{Z}^{-1}\{\tilde{\underset{\sim}{H}}(z)\} \in \ell^1_{n\times n}$, we consider two different cases.

<u>Case 1</u>: $\underset{\sim}{R} = \underset{\sim}{0}$. In this case $\tilde{\underset{\sim}{G}}(z) = \tilde{\underset{\sim}{G}}_\ell(z)$. Thus (19) becomes

$$\tilde{\underset{\sim}{H}}(z) = [\underset{\sim}{I} + \tilde{\underset{\sim}{G}}_\ell(z)\,\overline{\underset{\sim}{K}}]^{-1}\,\tilde{\underset{\sim}{G}}_\ell(z)\,\overline{\underset{\sim}{K}} \qquad (20)$$

By assumption, $\underset{\sim}{G}_\ell \in \ell^1_{n\times n}$, clearly $\mathcal{Z}^{-1}\{\underset{\sim}{\tilde{G}}_\ell(z)\ \underset{\sim}{K}\} \in \ell^1_{n\times n}$ and $\mathcal{Z}^{-1}\{[\underset{\sim}{I} + \underset{\sim}{\tilde{G}}_\ell(z)\ \underset{\sim}{\overline{K}}]\} \in \ell^1_{n\times n}$. It follows from assumption (6) and Lemma 2 that $\mathcal{Z}^{-1}\{[\underset{\sim}{I} + \underset{\sim}{\tilde{G}}_\ell(z)\ \underset{\sim}{\overline{K}}]^{-1}\} \in \ell^1_{n\times n}$. Consequently $\underset{\sim}{H} \triangleq \{\underset{\sim}{H}_i\}_0^\infty = \mathcal{Z}^{-1}\{\underset{\sim}{\tilde{H}}(z)\} \in \ell^1_{n\times n}$ because

$$\underset{\sim}{H} = \mathcal{Z}^{-1}\{[\underset{\sim}{I} + \underset{\sim}{\tilde{G}}_\ell(z)\ \underset{\sim}{\overline{K}}]^{-1}\} * \mathcal{Z}^{-1}\{\underset{\sim}{\tilde{G}}_\ell(z)\ \underset{\sim}{\overline{K}}\}$$

is the convolution of two $\ell^1_{n\times n}$ sequences, hence by Lemma 1 it is a $\ell^1_{n\times n}$ sequence.

<u>Case 2</u>: $\underset{\sim}{R} \neq \underset{\sim}{0}$ and $\underset{\sim}{R}\ \underset{\sim}{\overline{K}}$ is nonsingular. For this case $\underset{\sim}{\tilde{G}}(z) = \underset{\sim}{R}(1 - z^{-1})^{-1} + \underset{\sim}{\tilde{G}}_\ell(z)$. Now we introduce the factor $(1 - z^{-1})^{-1}(I - z^{-1})$ in (19) and rewrite $\underset{\sim}{\tilde{H}}(z)$ as

$$\begin{aligned}
\underset{\sim}{\tilde{H}}(z) &= \left\{(1 - z^{-1})[\underset{\sim}{I} + \underset{\sim}{\tilde{G}}(z)\ \underset{\sim}{\overline{K}}]\right\}^{-1} \left[(1 - z^{-1})\ \underset{\sim}{\tilde{G}}(z)\ \underset{\sim}{\overline{K}}\right] \\
&= \left[(\underset{\sim}{I} + \underset{\sim}{R}\ \underset{\sim}{\overline{K}} - z^{-1}\ \underset{\sim}{I}) + (1 - z^{-1})\ \underset{\sim}{\tilde{G}}_\ell(z)\ \underset{\sim}{\overline{K}}\right]^{-1}\left[\underset{\sim}{R}\ \underset{\sim}{\overline{K}} + (1 - z^{-1})\ \underset{\sim}{\tilde{G}}_\ell(z)\ \underset{\sim}{\overline{K}}\right] \\
&\triangleq \left[\underset{\sim}{\tilde{E}}_1(z)\right]^{-1} \underset{\sim}{\tilde{E}}_2(z) \qquad (21)
\end{aligned}$$

where $\underset{\sim}{\tilde{E}}_1(z)$ and $\underset{\sim}{\tilde{E}}_2(z)$ represent respectively the two expressions in the brackets of (21). By assumption $\underset{\sim}{G}_\ell \in \ell^1_{n\times n}$; therefore $\mathcal{Z}^{-1}\{\underset{\sim}{\tilde{E}}_1(z)\} \in \ell^1_{n\times n}$ and $\mathcal{Z}^{-1}\{\underset{\sim}{\tilde{E}}_2(z)\} \in \ell^1_{n\times n}$. The desired conclusion that $\underset{\sim}{H} \triangleq \mathcal{Z}^{-1}\{\underset{\sim}{\tilde{H}}(z)\} \in \ell^1_{n\times n}$ will follow if we show that $\mathcal{Z}^{-1}\{[\underset{\sim}{\tilde{E}}_1(z)]^{-1}\} \in \ell^1_{n\times n}$; by Lemma 2, we need only show that $\inf_{|z|\geq 1} |\det \underset{\sim}{\tilde{E}}_1(z)| > 0$. Now

$$\det \underset{\sim}{\tilde{E}}_1(z) = \det\left\{(1 - z^{-1})[\underset{\sim}{I} + \underset{\sim}{\tilde{G}}(z)\ \underset{\sim}{\overline{K}}]\right\}$$

(cont.)

$$= \left(1 - z^{-1}\right)^n \det[\underset{\sim}{I} + \tilde{\underset{\sim}{G}}(z)\ \overline{\underset{\sim}{K}}] \tag{22}$$

In view of assumption (6), the only possible difficulty is that det $\tilde{\underset{\sim}{E}}_1(z)$ might vanish at $z = 1$. To check its behavior at $z = 1$, we note from (21) that $\tilde{\underset{\sim}{E}}_1(1) = \underset{\sim}{R}\ \overline{\underset{\sim}{K}}$. By assumption $\underset{\sim}{R}\ \overline{\underset{\sim}{K}}$ is nonsingular, hence det $\tilde{\underset{\sim}{E}}_1(1) =$ det $\underset{\sim}{R}\ \overline{\underset{\sim}{K}} \neq 0$. Therefore $\inf_{|z|\geq 1} |\det \tilde{\underset{\sim}{E}}_1(z)| > 0$. This completes the proof for part (a).

(b) Since

$$\underset{\sim}{y} = \underset{\sim}{H} * \underset{\sim}{u} \tag{23}$$

and $\underset{\sim}{H} \in \ell^1_{n\times n}$, (b) follows directly from Lemma 1.

(c) $\underset{\sim}{u} \in \ell^\infty_n$ implies $\underset{\sim}{y} \in \ell^\infty_n$ is a special case of (b) with $p = \infty$. Therefore we need only to show that $\lim_{m\to\infty} \underset{\sim}{u}_m = \underset{\sim}{0}$ implies $\lim_{m\to\infty} \underset{\sim}{y}_m = \underset{\sim}{0}$. Now from (23), we have

$$\underset{\sim}{y}_m = \sum_{i=0}^{m} \underset{\sim}{H}_{m-i}\ \underset{\sim}{u}_i \tag{24}$$

Since $\lim_{m\to\infty} \underset{\sim}{u}_m = \underset{\sim}{0}$ (by assumption), for any $\varepsilon > 0$ there is an integer $N_u(\varepsilon)$ such that $m \geq N_u(\varepsilon)$ implies

$$|\underset{\sim}{u}_m| < \varepsilon \tag{25}$$

Recall that $\underset{\sim}{H} \in \ell^1_{n\times n}$, hence for any $\varepsilon > 0$, there is an integer $N_H(\varepsilon)$ such that for $m \geq N_H(\varepsilon)$

$$\sum_{i=m}^{\infty} |H_i| < \varepsilon \qquad (26)$$

Therefore for any $\varepsilon > 0$, $m > N_u(\varepsilon) + N_H(\varepsilon)$, we have

$$|y_m| \leq \sum_{i=0}^{m-N_H} |H_{m-i}|\, |u_i| + \sum_{i=m-N_H+1}^{m} |H_{m-i}|\, |u_i| \qquad (27)$$

Note that in the first summation, the argument of H varies from N_H to m and in the second summation, the argument of u varies from $m - N_H = N_u$ to m, thus by (25), (26), (27) and the facts that $u \in \ell_n^{\infty}$ and $H \in \ell_{n\times n}^{1}$, we have for $m > N_u(\varepsilon) + N_H(\varepsilon)$

$$|y_m| \leq \varepsilon\left(\|u\|_{\infty} + \|H\|_1\right) \qquad (28)$$

Since $\varepsilon > 0$ can be arbitrarily small, this shows that $\lim_{m\to\infty} y_m = 0$.

(d) Since $e_m = u_m - y_m$ and $\lim_{m\to\infty} u_m = u_{\infty}$, to show that $\lim_{m\to\infty} e_m = 0$ it suffices to show that $\lim_{m\to\infty} y_m = u_{\infty}$. Now let $u_m = (u_m - u_{\infty}) + u_{\infty} \triangleq \bar{u}_m + u_{\infty}$. By linearity of H and (23), we have

$$y_m = \left(H * \bar{u}\right)_m + \left(H * u_{\infty}\right)_m$$

$$= \sum_{i=0}^{m} H_{m-i}\, \bar{u}_i + \sum_{i=0}^{m} H_i\, u_{\infty} \qquad (29)$$

Since $\bar{u} \in \ell_n^{\infty}$ and $\lim_{m\to\infty} \bar{u}_m = 0$, it has been shown in (c) that $\lim_{m\to\infty} \left(H * \bar{u}\right)_m = 0$.

Consequently, we have

$$\lim_{m\to\infty} \underset{\sim}{y}_m = \lim_{m\to\infty} (\underset{\sim}{H} * \underset{\sim}{u})_m = \sum_{i=0}^{\infty} \underset{\sim}{H}_i \underset{\sim}{u}_{\infty} = \underset{\sim}{u}_{\infty} \tag{30}$$

because $\sum_{i=0}^{\infty} \underset{\sim}{H}_i = \underset{\sim}{\hat{H}}(1) = \underset{\sim}{I}$ as can be seen from (21).

Proof of Theorem 3

Assumption (ii) guarantees that $\underset{\sim}{\hat{G}}(z)$ is well defined and analytic outside some sufficiently large disk. Assumption (i) guarantees that the closed-loop impulse response is uniquely defined. From (iii) and the Laurent series expansion theorem applied to the matrix-valued function $\underset{\sim}{\hat{H}}(z)$ $(\underset{\sim}{10})$, for any z such that $|z| > \rho$, $\underset{\sim}{\hat{H}}(z)$ is represented by

$$\underset{\sim}{\hat{H}}(z) = \sum_{i=0}^{\infty} \underset{\sim}{H}_i z^{-i} \tag{31}$$

where the power series converges absolutely. Hence, for any $\beta \in (\rho,1)$,

$$\sum_{i=0}^{\infty} |\underset{\sim}{H}_i| \beta^{-i} < \infty \tag{32}$$

Since each term of this series is positive, it follows from inequality (32) that each term is smaller than some number b, and inequality (10) follows.

Proof of Theorem 4

Transform the given system $\underset{\sim}{S}_1$ (Fig. 2) into the system $\overline{\underset{\sim}{S}}_1$ shown in

Fig. 3, where

$$\underset{\sim}{H} \triangleq \{\underset{\sim i}{H}\}_0^\infty = \mathcal{Z}^{-1}\left\{\underset{\sim}{\tilde{G}}(z)[\underset{\sim}{I} + \underset{\sim}{\tilde{G}}(z)\,\underset{\sim}{\overline{K}}]^{-1}\right\} \quad (33)$$

$$\underset{\sim}{\hat{K}} \triangleq \{\underset{\sim i}{\hat{K}}\}_0^\infty = \underset{\sim}{K} - \underset{\sim}{\overline{K}} \quad (34)$$

$$\underset{\sim}{\overline{e}} = \underset{\sim}{e} + \underset{\sim}{\overline{K}}\,\underset{\sim}{w} \quad (35)$$

$$\underset{\sim}{\overline{y}} = \underset{\sim}{y} - \underset{\sim}{\overline{K}}\,\underset{\sim}{w} \quad (36)$$

and $\underset{\sim}{u}$, $\underset{\sim}{w}$ remain unchanged. It is easy to see that system $\underset{\sim 1}{S}$ and system $\underset{\sim 1}{\overline{S}}$ have the same input-output pairs $(\underset{\sim}{u},\underset{\sim}{w})$. Hence the solution of the system $\underset{\sim 1}{\overline{S}}$ is well defined if and only if the solution of the system $\underset{\sim 1}{S}$ is well defined. Therefore by Theorem 1, assumption (13) implies that $\det[\underset{\sim}{I} + \underset{\sim o}{H}\,\underset{\sim i}{\hat{K}}] \neq 0$, $\forall i \in J_+$. Since $\underset{\sim}{\hat{K}} \triangleq \{\underset{\sim i}{\hat{K}}\}_0^\infty \in \ell^1_{n\times n}$, $\lim_{i\to\infty} \underset{\sim i}{\hat{K}} = \underset{\sim}{0}$ and $\lim_{i\to\infty} \det[\underset{\sim}{I} + \underset{\sim o}{H}\,\underset{\sim i}{\hat{K}}] = 1$. Hence $\inf_{i\in J_+} |\det[\underset{\sim}{I} + \underset{\sim o}{H}\,\underset{\sim i}{\hat{K}}]| > 0$. Therefore there is a positive number λ such that

$$\sup_{i\in J_+} \|[\underset{\sim}{I} + \underset{\sim o}{H}\,\underset{\sim i}{\hat{K}}]^{-1}\| \triangleq \lambda < \infty \quad (37)$$

From assumption (14) and Theorem 2, $\underset{\sim}{H} \in \ell^1_{n\times n}$, hence $\underset{\sim}{H} \in \ell^\infty_{n\times n}$, consequently $h_M \triangleq \sup_{i\in J_+} |\underset{\sim i}{H}| < \infty$. Now from Fig. 3, we have the system equation

$$\underset{\sim}{w} = \underset{\sim}{H} * \left(\underset{\sim}{u} - \underset{\sim}{\hat{K}}\,\underset{\sim}{w}\right) \triangleq \underset{\sim}{v} - \underset{\sim}{H} * \left(\underset{\sim}{\hat{K}}\,\underset{\sim}{w}\right) \quad (38)$$

or equivalently

$$w_m = v_m - \sum_{i=0}^{m} H_{m-i} \hat{K}_i w_i \tag{39}$$

where $v \overset{\Delta}{=} H * u$. Rewrite (39) as

$$\left(I + H_o \hat{K}_m\right) w_m = v_m - \sum_{i=0}^{m-1} H_{m-i} \hat{K}_i w_i \tag{40}$$

or

$$w_m = \left(I + H_o \hat{K}_m\right)^{-1} \left[v_m - \sum_{i=0}^{m-1} H_{m-i} \hat{K}_i w_i \right] \tag{41}$$

Taking the norm of both sides of (41), using (37) and the fact that the H_i's are bounded by h_M, we obtain

$$|w_m| \leq \lambda |v_m| + \lambda h_M \sum_{i=0}^{m-1} |\hat{K}_i| \; |w_i| \tag{42}$$

From a Bellman-Gronwall lemma type result for the discrete case (11, lemma 2), which is easily proven by induction, we have

$$|w_m| \leq \lambda |v_m| + \sum_{i=0}^{m-1} \left[\prod_{j=i+1}^{m-1} \left(1 + \lambda h_M |\hat{K}_j|\right) \lambda^2 h_M |\hat{K}_i| \; |v_i| \right] \tag{43}$$

Now using the following inequality (Ref. 12), which follows immediately from the power series expansion of the exponential,

$$\prod_{i=0}^{m} \left(1 + |x_i|\right) \leq \exp\left(\sum_{i=0}^{m} |x_i| \right) \qquad \forall |x_i| \in R_+ , \quad \forall m \in J_+$$

we obtain

$$|\underset{\sim}{w}_m| \leq \lambda|\underset{\sim}{v}_m| + \sum_{i=0}^{m-1}\left[\exp\left(\lambda h_M \sum_{j=0}^{\infty} |\hat{\underset{\sim}{K}}_j|\right)\right]\lambda^2 h_M |\hat{\underset{\sim}{K}}_i|\ |\underset{\sim}{v}_i| \qquad (44)$$

By assumption $\hat{\underset{\sim}{K}} \triangleq \{\hat{\underset{\sim}{K}}_i\} \in \ell^1_{n\times n}$, thus the exponential term inside the bracket in (44) is a finite number; call it α. Then (44) becomes

$$|\underset{\sim}{w}_m| \leq \lambda|\underset{\sim}{v}_m| + \lambda^2 h_M \alpha \sum_{i=0}^{m-1} |\hat{\underset{\sim}{K}}_i|\ |\underset{\sim}{v}_i| \qquad (45)$$

Now we shall consider two different classes of inputs:

(a) If $\underset{\sim}{u} \in \ell^\infty_n$, then $\underset{\sim}{v} \triangleq (\underset{\sim}{H} * \underset{\sim}{u}) \in \ell^\infty_n$; consequently $v_M \triangleq \sup_{i\in J_+} |\underset{\sim}{v}_i| < \infty$. Therefore for any $m \in J_+$, we have from (45)

$$|\underset{\sim}{w}_m| \leq \lambda v_M + \lambda^2 h_M \alpha v_M \sum_{i=0}^{\infty} |\hat{\underset{\sim}{K}}_i| < \infty \qquad (46)$$

because $\hat{K} \in \ell^1_{n\times n}$. Clearly (46) implies that $\underset{\sim}{w} \in \ell^\infty_n$.

(b) If $\underset{\sim}{u} \in \ell^1_n$, then $\underset{\sim}{u} \in \ell^\infty_n$. It follows immediately from (a) above that $\underset{\sim}{w} \in \ell^\infty_n$. Now go back to (38), since $\hat{\underset{\sim}{K}} \in \ell^1_{n\times n}$ and $\underset{\sim}{w} \in \ell^\infty_n$, hence $(\hat{\underset{\sim}{K}}\ \underset{\sim}{w}) \in \ell^1_n$. Recall that $\underset{\sim}{H} \in \ell^1_{n\times n}$ and $\underset{\sim}{v} = (\underset{\sim}{H} * \underset{\sim}{u}) \in \ell^1_n$; therefore both terms on the right side of (38) are in ℓ^1_n. Consequently $\underset{\sim}{w} \in \ell^1_n$ and $\lim_{m\to\infty} \underset{\sim}{w}_m = \underset{\sim}{0}$.

Finally, we shall show that $\underset{\sim}{e}$ and $\underset{\sim}{y}$ also have the same properties. Now from Fig. 3 we have $\overline{\underset{\sim}{y}} = \hat{\underset{\sim}{K}}\ \underset{\sim}{w}$. It is easy to see that $\underset{\sim}{w} \in \ell^\infty_n$ implies $\overline{\underset{\sim}{y}} \in \ell^\infty_n$ and $\underset{\sim}{w} \in \ell^1_n$ implies $\overline{\underset{\sim}{y}} \in \ell^1_n$ because $\hat{\underset{\sim}{K}} \in \ell^1_{n\times n} \cap \ell^\infty_{n\times n}$. We can conclude the same results for $\underset{\sim}{y}$ from (36) and for $\underset{\sim}{e}$ from the relation $\underset{\sim}{e} = \underset{\sim}{u} - \underset{\sim}{y}$ (Fig. 2).

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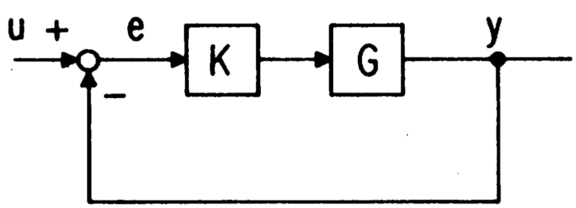

Fig. 1. Linear discrete feedback system $\underset{\sim}{S}$ under consideration. $\underset{\sim}{G}$ is a convolution-type linear time-invariant subsystem and $\underset{\sim}{K}$ is a (memoryless) time-varying gain. $\underset{\sim}{u}$ is the input and $\underset{\sim}{y}$ is the output.

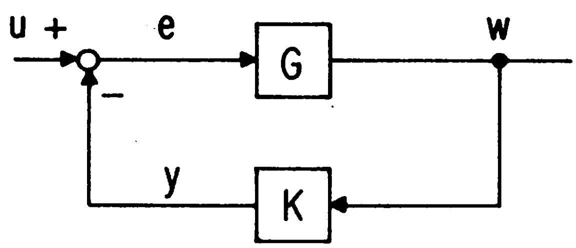

Fig. 2 System S_1 differs from system S by the interchange in the order of the subsystems G and K. u is the input and w is the output. (u,y) is an input-output pair of S if and only if $(K\,u,\ y)$ is an input-output pair of S_1.

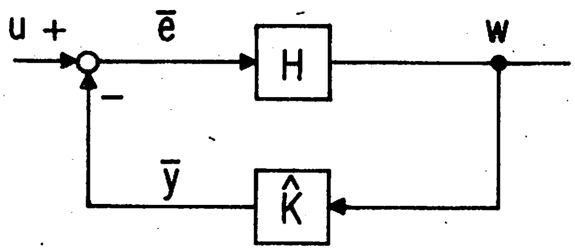

Fig. 3 $\overline{S}_1$ is obtained from S_1 by a standard loop transformation. S_1 and $\overline{S}_1$ have the same input-output pairs.