THE "FROZEN OPERATING POINT" METHOD OF SMALL SIGNAL ANALYSIS

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Abstract

The method of small-signal analysis of nonlinear time-invariant networks about a fixed operating point is well known. Desoer and Wong have suggested the use of a time-varying bias so that, as far as the small signal response is concerned, the network looks like a linear time-varying network [1]. In the present work, we show how calculations can be greatly simplified when the bias is slowly varying. Making use of recent results of stability theory, we show that small-signal analysis about the frozen operating point is correct within higher order terms in the small signal provided a correction term is inserted in the equation. An important feature of the theory is that its assumptions can often be checked by inspection because it involves only the properties of the frozen network. Section I gives a formal description of the method. In Section II, the method is rigorously analyzed and the results are stated in the form of two assertions.

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I. The Method

In this section we describe the procedure for writing out the equations for the small signal analysis about the "frozen operating point."

Suppose we have a nonlinear network or system described by

$$\dot{x} = f(x,\tilde{u}) \qquad x(0) = 0 \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $\tilde{u}(t) \in \mathbb{R}^m$ and $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. A circumstance often encountered in practice is that where the input $\tilde{u}$ can be written as

$$\tilde{u} = \overline{u} + u$$

where $\overline{u}$ is a "slowly varying" bias and u is "small signal." In the absence of the small signal u, we may use (1) to calculate (as in Ref. 1) the <u>variable operating point</u> $\overline{x}$ by solving

$$\dot{\overline{x}} = f(\overline{x},\overline{u}) \qquad \overline{x}(0) = 0 \tag{2}$$

For simplicity, we assume throughout that the network starts from zero initial conditions. The solution of (1) is then written as

$$x = \overline{x} + \xi \tag{3}$$

where $\overline{x}$ is presumably "slowly varing" and ξ is presumably "small." Substituting (3) in (1) and using (2) gives, after Taylor expansion,

$$\dot{\xi} = (D_1 f)_{(\overline{x},\overline{u})}\, \xi + (D_2 f)_{(\overline{x},\overline{u})}\, u + g(\xi,u,t) \tag{4}$$

where $D_i f$ denotes the derivative of the vector-valued function f with respect to its i^{th} argument. Given suitable restrictions on f, g is of second order in ξ and u, hence by neglecting it, we obtain the linearized equations

$$\dot{\xi}_0 = (D_1 f)_{(\bar{x},\bar{u})} \xi_0 + (D_2 f)_{(\bar{x},\bar{u})} u \qquad \xi_0(0) = 0 \tag{5}$$

In Ref. 1, conditions were obtained under which, for sufficiently small u, ξ_0 the solution of the approximate linear equations (5) was arbitrarily close the ξ the solution of the exact nonlinear equations (4). Equation (5) can be described as the linearized small-signal equations about the moving operating point $(\bar{x},\bar{u})$.

Let us sketch out another small-signal analysis which makes explicit use of the slowly varying character of u. For each t, calculate the frozen operating point $x_0(t)$ by solving

$$f(x_0(t), \bar{u}(t)) = 0 \tag{6}$$

For simplicity, we constrain $\bar{u}(0)$ to be such that $x_0(0) = 0$. Physically, $x_0(t)$ is the operating point that would exist if the bias were constant and had the value $\bar{u}(t)$. In many applications, the frozen operating point is easy to calculate. Indeed, in contrast to the differential equation (2), to solve equation (6) for $x_0(t)$ is a d.c. calculation, i.e. the inductors are replaced by short circuits and the capacitors are replaced by open circuits. For example, the circuit of Fig. 1, because of its nonlinear inductors and capacitors, is an adjustable low-pass filter when operated in the small signal mode. Clearly, whatever the value of the bias voltage $\bar{u}(t)$ is, the frozen operating point $x_0(t)$ is characterized by equal inductor currents and equal capacitor voltages which are easy to calculate by inspection. In contrast, the calculation of $\bar{x}(t)$ requires the solution of a system of nonlinear differential equations of order 7.

Differentiating (6) with respect to t and using the chain rule, we obtain

$$(D_1 f)_{(x_0,\bar{u})}\, \dot{x}_0 + (D_2 f)_{(x_0,\bar{u})}\, \dot{\bar{u}} = 0 \tag{7}$$

hence

$$\dot{x}_0 = -\,(D_1 f)^{-1}_{(x_0,\bar{u})}\,(D_2 f)_{(x_0,\bar{u})}\, \dot{\bar{u}} \triangleq v(t) \tag{8}$$

This last equation defines $v(t)$. To perform our small-signal analysis about the frozen operating point $x_0(t)$, we write

$$x = x_0 + \eta \tag{9}$$

Substituting (9) in (1), using (6) and (8) we obtain by Taylor expansion

$$\dot{\eta} = (D_1 f)_{(x_0,\bar{u})}\, \eta + (D_2 f)_{(x_0,\bar{u})}\, u - v(t) + \gamma(\eta,u,t) \tag{10}$$

where γ represents the higher order terms. Neglecting γ, we obtain the (linearized) equation

$$\dot{\eta}_0 = (D_1 f)_{(x_0,\bar{u})}\, \eta_0 + (D_2 f)_{(x_0,\bar{u})}\, u - v \tag{11}$$

Note that from (1), (6) and (8), η as well as η_0 have zero initial conditions.

This formal derivation leaves open two questions. Under what assumptions on the nonlinear circuit will the frozen operating point trajectory $x_0(\cdot)$ and the operating point trajectory $\bar{x}(\cdot)$ be close to one another? Similarly, what is the relation between the small signal $\xi(\cdot)$ calculated by (4) and $\eta_0(\cdot)$ calculated by (11)? The next section is devoted to a careful analysis whose conclusions are in the form of two assertions which answer these two questions.

II. Analysis

Let $|\cdot|$ be used to denote the absolute value of a number, a norm of a vector in $\mathbb{R}^n$ (selected once and for all), the corresponding induced norm of a matrix in $\mathbb{R}^{n\times n}$. All functions of time are defined on $\mathbb{R}_+ = \{t|t\geq 0\}$. For such functions taking values in $\mathbb{R}$, $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$, we use the sup. norm: $||x|| \triangleq \sup_{t\geq 0} |x(t)|$.

Assumptions.

A1. The function f in equation (1) has continuous second derivatives of all its arguments throughout $\mathbb{R}^n \times \mathbb{R}^m$.

A2. Let $(D_1 f)_{(x_0,\bar{u})} = A(t)$, $(D_2 f)_{(x_0,\bar{u})} = B(t)$. We assume that

$$\text{(a)} \quad ||A|| < \infty \quad \text{and} \quad ||B|| < \infty \tag{12}$$

(b) for each $t \in \mathbb{R}_+$ all eigenvalues of A(t) are in the left half plane and are bounded away from the jω-axis: more precisely, there is a $\sigma_0 > 0$ such that

$$\mathrm{Re}[\lambda_i(t)] \leq 2\sigma_0 < 0 \qquad \forall t \geq 0, \forall i \tag{13}$$

A direct consequence of this assumption is that for all $t \in R_+$, A(t) is invertible and

$$||A^{-1}|| < \infty \tag{14}$$

A3. The second derivatives of f $\frac{\partial^2 f_i}{\partial x_k \partial x_\ell}$ $(i,k,\ell = 1,2,..,n)$ and $\frac{\partial^2 f_i}{\partial x_k \partial u_j}$ $(i,k = 1,2,...,n;\ j = 1,2,...,m)$ are bounded along the frozen operating point trajectory $(x_0(t),\bar{u}(t))$.

A4. The input signal u and the first derivative of the bias $\bar{u}$ (i.e. $\dot{\bar{u}}$) must be regulated functions on $\mathbb{R}_+$ [2].

Assumption A1 is required to justify the Taylor expansions [2]. Assumption A2 is quite natural if one has in mind the design of adjustable filters and equalizers [3]; A2 requires that for all times, the frozen small-signal equivalent circuit (about the frozen operating point) be stable. Assumptions A3 and A4 are required to control the behavior of A(t) as a function of time. In contrast to the assumptions made in [1] (Theorem I and Corollary III, in particular), the present assumptions are easy to test: they do not require knowledge of state transition matrices of time-varying networks, but only the knowledge of A(t) for each t. For example, if, in the network shown in Fig. 1, the small-signal inductances and capacitances belong, for all operating points, to the interval [a,b], with $0 < a < b < \infty$, then the assumptions A2 are satisfied.

Let

$$\delta x(t) = \bar{x}(t) - x_0(t) \qquad \forall\, t \geq 0 \tag{15}$$

<u>Assertion 1.</u> Suppose that A1, A2, A3 and A4 hold and that $||\dot{\bar{u}}||$ is small, then $||\delta x||$ is also small and of the same order as $||\dot{\bar{u}}||$. To show that we differentiate (15), use (8), (6) and Taylor's expansion:

$$\delta\dot{x}(t) = A(t)\delta x(t) - v(t) + \tilde{g}(\delta x,t) \tag{16}$$

where $\tilde{g}$ represents second order terms. If $||\dot{\bar{u}}||$ is small, then by eq. (8), (12) and (14) of A2, so is $||\dot{x}_0||$: indeed

$$||v|| = ||\dot{x}_0|| \leq ||A^{-1}|| \cdot ||B|| \cdot ||\dot{\bar{u}}|| \tag{17}$$

This implies that the elements of A(•) are slowly varying: indeed $a_{ij}(t) = (\partial f_i/\partial x_j)_{(x_0,\bar{u})}$ and

$$\dot{a}_{ij}(t) = \sum_{k=1}^{n} \frac{\partial^2 f_i}{\partial x_j \partial x_k} \dot{x}_{0k} + \sum_{\ell=1}^{m} \frac{\partial^2 f_i}{\partial x_j \partial u_\ell} \dot{\bar{u}}_\ell \tag{18}$$

From (17), (18) and A3, we conclude that $A(\cdot)$ is slowly varying in the sense that by taking $||\dot{\bar{u}}||$ sufficiently small we can make $||\dot{A}||$ arbitrarily small. Now using a continuity argument, Rosenbrock [4] has shown and Desoer has given explicit bounds on $||\dot{A}||$ [5] to prove the fact that if $||\dot{A}||$ is small then, under the assumptions A2 (a) and (b) and A4, the state transition matrix of $\dot{x} = A(t)x$ is uniformly exponentially bounded; more precisely for some positive constants m and σ_1,

$$|\Phi(t,\tau)| \leq m \in^{-\sigma_1(t-\tau)} \qquad \forall\, t \geq \tau,\ \tau \geq 0 \tag{19}$$

This fact together with Theorem I of [1] establishes that if we throw away the second order terms of (16), and define δx_0 by

$$\delta \dot{x}_0 = A(t)\ \delta x_0(t) - v(t) \tag{20}$$

we have

$$\frac{||\delta x - \delta x_0||}{||\dot{\bar{u}}||} \to 0 \quad \text{as } ||\dot{\bar{u}}|| \to 0 \tag{21}$$

Furthermore, from (17), (19) and (20), it follows immediately that

$$||\dot{\bar{u}}|| \to 0 \text{ implies that } ||\delta x_0|| \to 0 \tag{22}$$

Assertion 1 follows directly from (21) and (22).

Thus under the smoothness conditions stated by A1 to A4, provided the bias $\bar{u}$ varies sufficiently slowly, first (17) shows that $\dot{x}_0$, the velocity of the frozen operating point, is small; second, $\delta x(t)$ the difference between the operating point $\bar{x}(t)$ and the frozen operating point $x_0(t)$ is uniformly small on $\mathbb{R}_+$.

Assertion 2. Suppose that A1, A2, A3 and A4 hold, and that $||u||$ and $||\dot{u}||$ are small. Let ζ be defined by

$$\dot{\zeta} = A(t)\zeta + B(t)u \qquad \zeta(0) = 0 \tag{23}$$

Then ξ, the (exact) small-signal response about the operating point $\bar{x}$ (see eq. (4)), and ζ just defined are related by

$$\xi = \zeta + o(||u|| + ||\dot{u}||) \tag{24}$$

i.e. ξ and ζ are equal except for higher order terms in $||u|| + ||\dot{u}||$.

To establish this fact is easy: from (3), (9) and (15)

$$\xi = \eta - \delta x \tag{25}$$

From (21), we may replace δx by δx_0 at the cost of a term $o(||\dot{u}||)$. In view of assumptions A1, A2, A3 and A4 and the results of [1], we may replace η by η_0 at the cost of a term $o(||u||)$. Thus

$$\xi = \eta_0 - \delta x_0 + o(||\dot{u}|| + ||u||)$$

Comparing the differential equations (11) and (20), we see that if we put

$$\zeta = \eta_0 - \delta x_0 \tag{26}$$

then ζ satisfies eq. (23). Therefore Assertion 2 is established.

Conclusion

To calculate the (exact) response x (defined by (1)) we have two methods. First, the conventional method of small-signal analysis: we write $x = \bar{x} + \xi$, solve the differential equation (2) for the motion of the operating point $\bar{x}$, calculate ξ_0 by the linear equations (5). The smallness of $||u||$ guarantees that the error caused by calculating ξ_0 instead of ξ is of higher order in $||u||$. Provided both $||u||$ and $||\dot{u}||$ are small, then ξ can be calculated approximately (see eq. (24)) by solving Eq. (23) for ζ: the equation for ζ is the linearized equation about the frozem operating point. The second method, the frozen operating point method, is as follows: write $x = x_0 + \eta$, calculate by (6) the motion of the "frozen operating point" x_0, calculate η_0 by the linear equation (11). Again, the error caused by calculating η_0 instead of η is of higher order in $||u||$. Note that (11) includes a correction term $-v$ which is due to the fact that the expansion has been done about the frozen operating point. Inequality (17) gives a way to estimate when this correction term may be dropped.

References

[1] Desoer, C. A. and K. K. Wong, "Small-Signal Behavior of Nonlinear Lumped Networks," Proc. IEEE 56, 1, p. 14-22, January 1968.

[2] Dieudonné, J. "Foundations of Modern Analysis," Academic Press, New York, 1960.

[3] Desoer, C. A. and B. Peikari, "Design of Nonlinear Networks to be Operated in the Small Signal Mode," (Report in preparation).

[4] Rosenbrock, H. H., "The Stability of Linear Time-Dependent Control Systems," Jour. of Elect. and Control, Vol. 15, 1, p. 73-80, July 1963.

[5] Desoer, C. A., "Slowly Varying System $\dot{x} = A(t)x$," Trans. IEEE, Automatic Control, (To be published).

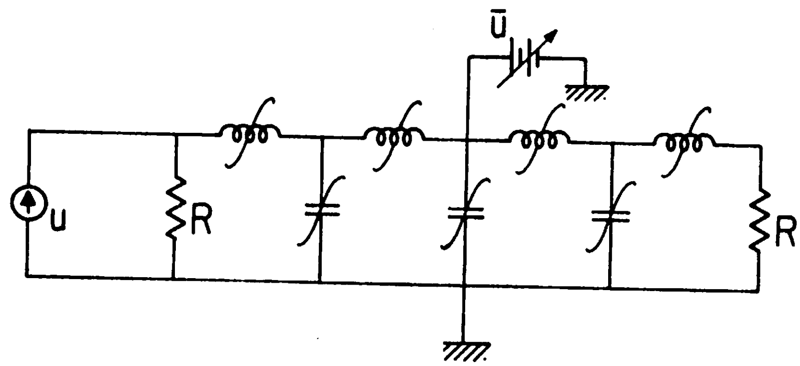

Fig. 1. In the time-invariant network shown above, the four inductors and the three capacitors are nonlinear. If the resistors R_1 and R_2 are equal, the "frozen" operating point corresponds to an equal current flowing through the four inductors and an equal voltage across the three capacitors. Clearly the frozen operating point $x_0(t)$ can be obtained by inspection.