# A DECOMPOSITION ALGORITHM FOR SOLVING
# A CLASS OF OPTIMAL CONTROL PROBLEMS

by

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## I. INTRODUCTION.

The solution of complex optimal control problems is frequently facilitated by an imbedding or, to be more precise, by a decomposition into a family of simple optimal control problems. This paper presents a (single iterative) decompositon algorithm which shares a number of geometric ideas with algorithms given by Frank and Wolfe [1], Neustadt [2], Eaton [3], Gilbert [6], Barr and Gilbert [7], Polak [10] and Polak and Deparis [11].

In comparing the algorithm in this paper with the ones mentioned above, the reader will find that it applies to a larger class of problems and that it is made up of simpler subprocedures which often result in improved speed. In particular, it makes less stringent requirements of continutiy of the reachable sets than its predecessors, it only requires convexity rather than strict convexity of the state space constraints (a common feature in [2], [10], [11]), it is single rather than double iterative as in the case of the Barr-Gilbert algorithm (see [7]), and it does not require the search of a minimum along an arc (as is the case in [10] and [11]). The extent to which the algorithm presented in this paper differs from its predecessors is indicated to some extent by the fact that while all the algorithms presented in [2] to [11] construct a monotonically increasing sequence of reachable sets, the present algorithm produces a sequences of reachable sets which can oscillate.

Actually, to be exact, this paper presents not one, but five algorithms. The first algorithm solves a canonical geometric problem which is closely related to a large class of optimal control problems. This algorithm is then endowed with additional features to produce the others algorithms, each of which is directed towards different groups of optimal control problems.

The class of problems which can be treated by means of the algorithms to be presented includes minimum-energy, minimum-fuel and minimum-time discrete and continuous optimal control problems with linear dynamics and a finite number of convex state space constraints. As will be seen from the examples presented, the algorithms are fast enough to be usable for on line control in many practical situations.

## II. GEOMETRIC PROBLEM AND PRELIMINARIES.

It has already been demonstrated in [2 - 11], that a number of optimal control problems can easily be transcribed into a problem involving two convex compact sets: a target set T which is fixed and a reachable set, $R(\lambda)$, depending on a scalar parameter $\lambda$ which is usually the cost.

We shall now state this geometric problem and develop some of its properties which we shall need later.

Problem 1:

Given a convex compact subset T of a real Hilbert space $\mathcal{H}$[†] and a mapping $R(\cdot)$, from a compact subset $\Lambda$ of the reals into all the subsets of $\mathcal{H}$, satisfying

(i) $R(\lambda)$ is convex and compact for all $\lambda$ in $\Lambda$;

(ii) $R(\lambda') \subset R(\lambda'')$ for all $\lambda'$, $\lambda''$ in $\Lambda$ such that $\lambda' < \lambda''$;

(iii) $R(\cdot)$ is continuous on $\Lambda$;

Find a $\hat{\lambda}$ in $\Lambda$ and an $\hat{x}$ in T such that $\hat{\lambda} \leq \lambda$ for all $\lambda$ in $\{\lambda \in \Lambda \mid R(\lambda) \cap T \neq \emptyset\}$ and $\hat{x} \in R(\hat{\lambda})$.

Remark: By $R(\cdot)$ continuous on $\Lambda$ we mean that given any neighborhood $N(R(\lambda))$ of $R(\lambda)$ there exists a neighborhood $N(\lambda)$ of $\lambda$ in $\Lambda$ such that for all $\lambda' \in N(\lambda)$, $R(\lambda') \subset N(R(\lambda))$. Note that Problem 1 differs from the problems considered previously in the literature in two very important respects. In Problem 1, the set T is not required to be strictly convex and the mapping $R(\cdot)$ is not required to satisfy $R(\lambda)$ strictly convex for all $\lambda$ in $\Lambda$. We shall see later that this enables us to consider optimal

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[†]We use $\langle \cdot,\cdot \rangle$ to denote the inner product in $\mathcal{H}$.

control problems with polyhedral target sets, as well as optimal control problems with dynamics which are not completely controllable. It should also be noted that the continuity required of the mapping $R(\cdot)$ is weaker than the continuity usually required in the literature, as for example Hausdorff continuity. Consequently Problem 1 is extremely general and, as a result, a large number of problems can be transcribed into the form of Problem 1.

Theorem 1. The set $\{\lambda \in \Lambda | \; R(\lambda) \cap T \neq \emptyset\}$ is compact.

Proof: In order to prove this theorem, we shall show that the set $\tilde{\Lambda}=\{\lambda \in \Lambda | \; R(\lambda) \cap T = \emptyset\}$ is open. Suppose that $\tilde{\Lambda}$ is empty, then obviously $\tilde{\Lambda}$ is open; now suppose that $\tilde{\Lambda}$ is not empty and let $\lambda$ be in $\tilde{\Lambda}$. The definition of $\tilde{\Lambda}$ implies that $R(\lambda) \cap T = \emptyset$. $R(\lambda)$ and $T$ are closed subsets of a Hilbert space, then there exists $N(R(\lambda))$ and $N(T)$ disjoints neighborhoods of $R(\lambda)$ and $T$ respectively. The continuity of the mapping $R(\cdot)$ on $\Lambda$ implies that there exists $N(\lambda)$, a neighborhood of $\lambda$ in $\Lambda$ such that $R(\lambda') \subset N(R(\lambda))$ for all $\lambda'$ in $N(\lambda)$. It follows that $N(\lambda)$, neighborhood of $\lambda$ in $\Lambda$ belongs to $\tilde{\Lambda}$ i.e. $\tilde{\Lambda}$ is open. This implies that the set $\{\lambda \in \Lambda | \; R(\lambda) \cap T = \emptyset\}$ is a closed subset of $\Lambda$ which is compact and is therefore compact.

Remark: Theorem 1 shows that Problem 1 is well defined.

Definition 1. Let $P(\cdot,\cdot)$ be the mapping from $\mathcal{H} \times \mathcal{H}$ into all the subsets of $\mathcal{H}$ defined by

(1) $$P(v, s) = \{x \in \mathcal{H} \mid \; < s, x - v > = 0\}$$

Theorem 2. The set $\{\lambda \in \Lambda | \; R(\lambda) \cap P(v, s) \neq \emptyset\}$ is compact for all v and s in $\mathcal{H}$. The proof of Theorem 2 is basically the same as the proof of Theorem 1 and is therefore omitted.

Definition 2. Let $V(\cdot)$ be the mapping from $\mathcal{H}$ into all the subsets of T defined by

$$(2) \qquad V(s) = \{v \in T \mid \langle s, x - v \rangle \leq 0 \quad \text{for all } x \text{ in } T\}.$$

Remark: The compactness of T implies that the mapping $V(\cdot)$ is well defined.

Definition 3. Let $\lambda_{min}$ and $\lambda_{max}$ be respectively defined by

$$\lambda_{min} = \min \{\lambda \in \Lambda\};$$

$$\lambda_{max} = \max \{\lambda \in \Lambda\}.$$

Definition 4. Let $Q(\cdot,\cdot)$ be the mapping from $\mathcal{H} \times \mathcal{H}$ into $\Lambda$ defined by

$$(3) \quad Q(v, s) = \begin{cases} \lambda_{max} \text{ when } P(v, s) \cap R(\lambda) = \emptyset \text{ for all } \lambda \in \Lambda; \\ \min \{\lambda \in \Lambda \mid P(v, s) \cap R(\lambda) \neq \emptyset\} \text{ otherwise.} \end{cases}$$

Definition 5. Let $z(., ., .)$ be the mapping from $\mathcal{H} \times \mathcal{H} \times \mathcal{H}$ into $\mathcal{H}$ defined by:

(i) $z(a, b, c) \in [b, c]^{\dagger}$;

(ii) $\| z(a, b, c) - a\| \leq \|z - a\| \quad$ for all $z \in [b, c]$.

It is not difficult to see that $z(., ., .)$ is jointly continuous in all its arguments.

Definition 6. Let $\psi(., ., .)$ be the mapping from $\mathcal{H} \times \mathcal{H} \times \mathcal{H}$ into $E^1$

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$^{\dagger}$Note: Given two points b and c in $\mathcal{H}$, the set $\{y \in \mathcal{H} \mid y = \nu b + (1-\nu) c,\ 0 \leq \nu \leq 1\}$ is denoted by $[b, c]$.

defined by:

(4) $$\psi(a, b, c) = \| b - a\|^2 - \| z(a, b, c) - a\|^2$$

Obviously, $\psi(., ., .)$ is jointly continuous in all its arguments.

Theorem 3. Let $a^*$, $b^*$, $c^*$ be points in $\mathcal{H}$ such that $\langle b^* - a^*, b^* - c^* \rangle > 0$. Then $\psi^* = \psi(a^*, b^*, c^*) > 0$ and there exists neighborhoods $N(a^*)$, $N(b^*)$ and $N(c^*)$ of $a^*$, $b^*$ and $c^*$, respectively, such that

$\psi(a, b, c) \geq \frac{\psi^*}{2} > 0$ for all a in $N(a^*)$, for all b in $N(b^*)$, for all c in $N(c^*)$.

Proof: It is easy to show that if $\langle b^* - a^*, b^* - c^* \rangle > 0$, then $\psi^* = \psi(a^*, b^*, c^*) > 0$. The existence of the neighborhoods $N(a^*)$, $N(b^*)$ and $N(c^*)$ now follows from the continuity of the mapping $\psi(., ., .)$.

## III. ALGORITHM FOR PROBLEM 1.

We shall now give an algorithm for solving Problem 1 . This algorithm requires the knowledge of two initial points, one in the set T and one in the set $R(\lambda_{min})$. We assume that we can compute exactly a $v_i$ in $V(s_i)$ and $Q(v_i, s_i)$ at each iteration, as required by the algorithm. We shall see later from the examples to be presented, that this assumption is entirely justified. Finally, in order to obtain meaningful results, we must suppose that Problem 1 has a solution, i.e. we shall assume that there exists a $\lambda$ in $\Lambda$ such that $R(\lambda) \cap T \neq \emptyset$.

Algorithm 1:

Step 0: Compute an $x_0 \in T$ and a $y_0 \in R(\lambda_{min})$. Let $i = 0$ and go to Step 1.

Step 1: If $y_i$ is in T set $x_i = y_i$ and go to Step 2, else go to Step 2.

Step 2: If $\|y_i - x_i\| = 0$ stop, else go to Step 3.

Step 3: Compute $s_i = y_i - x_i$ and a point $v_i$ in $V(s_i)$ as defined in (2).

Step 4: If $\langle s_i, y-v_i \rangle > 0$ for all y in $R(\lambda_{min})$, compute $Q(v_i, s_i)$ as defined in (3), set $\lambda_i = Q(v_i, s_i)$ and go to Step 5, else set $\lambda_i = \lambda_{min}$ and go to Step 5.

Step 5: Compute a $w_i$ in $R(\lambda_i)$ satisfying $\langle s_i, w_i - v_i \rangle \leq 0$.

Step 6: Compute $y_{i+1}$ in $[y_i, w_i]$ and $x_{i+1}$ in $[x_i, v_i]$ such that $\| y_{i+1} - x_{i+1} \| \leq \| y - x \|$ for all x in $[x_i, v_i]$ and for all y in $[y_i, w_i]$. Let $i = i + 1$ and go to Step 1.

Lemma 1. Consider the sequences $\{x_i\}$, $\{v_i\}$, $\{y_i\}$, $\{w_i\}$ and $\{\lambda_i\}$ generated by Algorithm 1 , then:

(i) the sequences $\{x_i\}$ and $\{v_i\}$ are in T;

(ii) the sequences $\{y_i\}$ and $\{w_i\}$ are in $R(\hat{\lambda})$;

(iii) $\langle y_i - x_i, v_i - w_i \rangle \geq 0$ for all i;

(iv) $\langle x_i - y_i, x_i - v_i \rangle \geq 0$ for all i;

(v) the sequence $\{\| y_i - x_i \|\}$ is monotonocally decreasing;

(vi) the sequence $\{\lambda_i\}$ satisfies: $\lambda_i \leq \hat{\lambda}$ for all i.

Proof:

(i), (iii), (iv) and (v) are self evident.

(vi) By assumption, Problem 1 has a solution $\hat{\lambda}$ in $\Lambda$. It follows that if $\lambda_i = \lambda_{min}$, then $\lambda_i \leq \hat{\lambda}$.

Hence the only interesting case to consider is when $\langle s_i, y - v_i \rangle > 0$ for all y in $R(\lambda_{min})$. Suppose that this is indeed the case and that $\lambda_i = Q(v_i, s_i) > \hat{\lambda}$. Then the definition of $Q(v_i, s_i)$ implies that $R(\hat{\lambda}) \cap P(v_i, s_i) = \phi$. But $R(\hat{\lambda})$ is convex and contains $R(\lambda_{min})$: therefore $\langle s_i, y - v_i \rangle > 0$ for all y in $R(\hat{\lambda})$. Now, by construction, $\langle s_i, x - v_i \rangle \leq 0$ for all x in T, and hence $R(\hat{\lambda})$ and T must be disjoint, which contradicts the definition of $\hat{\lambda}$. It follows that if $\langle s_i, y - v_i \rangle > 0$ for all y in $R(\lambda_{min})$ then $\lambda_i \leq \hat{\lambda}$.

(ii) is self evident in view of $(v_i)$.

Theorem 4. Any accumulation point $(x^*, v^*, y^*, w^*)$ of a sequence $\{x_i, v_i, y_i, w_i\}$ generated by Algorithm 1 satisfies:

$$\langle x^* - y^*, x^* - v^* \rangle = 0 \tag{5}$$

$$\langle y^* - x^*, y^* - w^* \rangle = 0 \tag{6}$$

Proof: Consider an infinite sequence $\{x_i, v_i, y_i, w_i\}$ generated by Algorithm 1 and let $\{x^*, v^*, y^*, w^*\}$ be an accumulation point of this sequence, then there exists K, a subset of the integers such that the subsequence $\{(x_i, v_i, y_i, w_i)\}_K$[†] converges to $(x^*, v^*, y^*, w^*)$.

From Lemma 1 we have:

$$\langle x_i - y_i, x_i - v_i \rangle \geq 0 \text{ for all } i.$$

It follows by continuity of the scalar product that

$$\langle x^* - y^*, x^* - v^* \rangle \geq 0.$$

Suppose that $\langle x^* - y^*, x^* - v^* \rangle > 0$, then theorem (3) implies that:

$$\psi^* = \psi(y^*, x^*, v^*) > 0$$

and that there exist neighborhoods $N(y^*)$, $N(x^*)$, $N(v^*)$ of $y^*$, $x^*$ and $v^*$ respectively such that:

$$\|x - y\|^2 - \|z(y, x, v) - y\|^2 > \frac{\psi^*}{2}$$

for all y in $N(y^*)$, for all x in $N(x^*)$, for all v in $N(v^*)$. This in turn implies that there exists a positive integer k such that:

$$\|x_i - y_i\|^2 - \|z(y_i, x_i, v_i) - y_i\|^2 \geq \frac{\psi^*}{2}$$

for all $i \geq k$, i in K.

Now by construction, $z(y_i, x_i, v_i) \in [x_i, v_i]$ and the definition of $y_{i+1}$

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[†] Note: Let $\{x_i\}$ be a sequence and K be a subset of the integers then we denote by $\{x_i\}_K$ the subsequence of $\{x_i\}$ consisting of all the $x_i$ such that i belongs to K.

and $x_{i+1}$ implies that

$$\|y_{i+1} - x_{i+1}\| \leq \|z(y_i, x_i, v_i) - y_i\|.$$

It follows that

$$\|y_i - x_i\|^2 - \|y_{i+1} - x_{i+1}\|^2 \geq \frac{\psi^*}{2}$$

for all $i \geq k$, $i$ in K. But this contradicts the fact that $\{(x_i, y_i)\}_K$ converges to $(x^*, y^*)$ and hence we must have $\langle x^* - y^*, x^* - v^* \rangle = 0$. The proof of (6) is done exactly in the same manner, replacing $x^*$, $y^*$ and $v^*$ by $y^*$, $x^*$ and $w^*$.

Theorem 5. Consider the monotonocally decreasing sequence $\{\|y_i - x_i\|\}$ generated by Algorithm 1. Then either the sequence is finite and its last element $\|y_k - x_k\|$ satisfies $\|y_k - x_k\| = 0$ or it is infinite and the sequence $\{\|y_i - x_i\|\}$ converges to zero.

Proof: First suppose that the sequences $\{x_i\}$ and $\{y_i\}$ generated by the algorithm are finite. Then since the only stop command of Algorithm 1 is in Step 2, $x_k$ and $y_k$ must satisfy $\|y_k - x_k\| = 0$. Now suppose that the sequences $\{x_i\}$ and $\{y_i\}$ are infinite. Consider the sequence $\{(x_i, v_i, y_i, w_i)\}$ in $\mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \mathcal{H}$. The points $x_i$ and $v_i$ are in T which is compact, the points $y_i$, $w_i$ are in $R(\hat{\lambda})$ which is also compact. In other words the sequence $\{(x_i, v_i, y_i, w_i)\}$ is in $T \times T \times R(\hat{\lambda}) \times R(\hat{\lambda})$ which is compact. It follows that there exists a subset K of the integers such that the subsequence $\{(x_i, v_i, y_i, w_i)\}_K$ converges to a point $\{(x^*, v^*, y^*, w^*)\}$ of $T \times T \times R(\hat{\lambda}) \times R(\hat{\lambda})$.

From Lemma 1 we have:

$\langle y_i - x_i, v_i - w_i \rangle \geq 0$ for all i.

It follows by continuity of the scalar product that:

$$(7) \qquad \langle y^* - x^*, v^* - w^* \rangle \geq 0.$$

The following equality is easy to establish:

$$\langle y^* - x^*, y^* - w^* \rangle = \|y^* - x^*\|^2 + \langle y^* - x^*, v^* - w^* \rangle + \langle y^* - x^*, x^* - v^* \rangle$$

It follows, using (5) and (7) that

$$\langle y^* - x^*, y^* - w^* \rangle \geq \|y^* - x^*\|^2.$$

Relation (6) shows that $\|y^* - x^*\|^2 = 0$. Since the decreasing sequence $\{\|y_i - x_i\|\}$ possess a subsequence which converges to zero, the sequence $\{\|y_i - x_i\|\}$ itself converges to zero, which completes our proof.

__Theorem 6__. If the sequence $\{\lambda_i\}$ generated by Algorithm 1 is finite then

$$\max_i \lambda_i = \lambda$$

and if the sequence $\{\lambda_i\}$ is infinite then

$$\sup_i \lambda_i = \hat{\lambda}$$

__Proof__: Let $\tilde{\lambda}$ be defined by

$\tilde{\lambda} = \max_i \lambda_i$ when the sequence is finite,

$\tilde{\lambda} = \sup_i \lambda_i$ when the sequence is infinite.

The compactness of $\Lambda$ implies that $\tilde{\lambda}$ is in $\Lambda$. By definition, $R(\tilde{\lambda}) \cap T \neq \phi$. Suppose that $\tilde{\lambda} < \hat{\lambda}$, then it follows that $R(\tilde{\lambda})$ and T are disjoint. Now T and $R(\tilde{\lambda})$ are convex, compact subsets of a Hilbert space $\mathcal{H}$ and therefore there exists an $\varepsilon > 0$ such that $\|x - y\| \geq \varepsilon > 0$ for all x in T and y in $R(\tilde{\lambda})$. The definition of $\tilde{\lambda}$ i.e. $\lambda_i \leq \tilde{\lambda}$ for all i, implies that $\|x - y\| \geq \varepsilon > 0$ for all x in T and y in $R(\lambda_i)$, i.e., $\|x_i - y_i\| \geq \varepsilon > 0$ for all i, contradicting Theorem 5 . Therefore $\tilde{\lambda} \geq \hat{\lambda}$, and since by Lemma 1 , $\lambda_i \leq \hat{\lambda}$ for all i, we conclude that $\tilde{\lambda} = \hat{\lambda}$, and the Theorem is proved.

Remark: The sequence $\{\lambda_i\}$ generated by Algorithm 1 does not necessarily converge to $\hat{\lambda}$. However Theorem 6 does show that the sequence $\{\lambda_i\}$ has a subsequence which converges to $\hat{\lambda}$. This fact can be incorporated in a heuristic stopping rule for Algorithm 1 , which could partly be based on the rate of increase of the sequence $\{\mu_i\}$ defined below.

Definition 7: Let $\{\lambda_i\}$ be a sequence computed by Algorithm 1 in the process of solving Problem 1 . We associate with this sequence a sequence $\{\mu_i\}$ defined as follows,

$\mu_i = \max \{\lambda_j \mid j \leq i\}$ for every i.

Lemma 2. Let $\{\lambda_i\}$ be a sequence computed by Algorithm 1 and let $\{\mu_i\}$ be the sequence obtained by using Definition 7 , then either the sequence $\{\mu_i\}$ is finite and its last element $\mu_k$ satisfies $\mu_k = \hat{\lambda}$ or it is infinite and the sequence $\{\mu_k\}$ converges to $\hat{\lambda}$.

## IV. DISCRETE MINIMUM - ENERGY OPTIMAL CONTROL PROBLEMS

We shall now show how a class of discrete, minimum-energy, optimal control problems can be transcribed into a slightly modified form of Problem 1. We shall also present a few specific problems in this class which were solved by means of Algorithm 1 in order to give the reader a feel for the numerical behavior of this algorithm.

The specific class of discrete minimum energy optimal control problems we shall consider is the following one,

<u>Problem 2:</u>

Minimize $\sum_{j=1}^{N} (u^j)^2$

subject to

(8) $$z_j = A_j z_{j-1} + b_j u^j \qquad j = 1, 2 \ldots N$$

(9) $$z_0 = \hat{z}_0;$$

(10) $$z_N \in T;$$

(11) $$|u^j| \leq 1 \qquad j = 1, 2 \ldots N;$$

where, for $j = 0, 1, 2 \ldots N$, $z_j \in E^n$ is the state of the system at time $j$ and, for $j = 1, 2 \ldots N$, $u^j \in E^1$ is the input at time $j$. The matrices $A_j$ and $b_j$ are real and are of dimensions $n \times n$ and $n \times 1$ respectively, for $j = 1, 2 \ldots N$. The set $T \subset E^n$ is assumed to be compact and convex.

As an intermediate step in transcribing Problem 2 into the form of Problem 1, it is convenient to rephrase this optimal control problem as a convex programming problem as follows. Let

(12) $$r_0 = A_N A_{N-1} \cdots A_1 \hat{z}_0;$$

(13) $$r_j = A_N A_{N-1} \cdots A_{j+1} b_j \qquad j = 1, 2 \ldots N-1;$$

(14) $$r_N = b_N.$$

then Problem 2 becomes

$$\text{Minimize} \sum_{j=1}^{N} (u^j)$$

subject to

(i) $$(r_0 + \sum_{j=1}^{N} r_j u^j) \in T$$

(ii) $$|u^j| \leq 1 \qquad j = 1, 2 \ldots N.$$

To complete the transcription in the form of Problem 1 , we define $\Lambda$ and $R(.)$ as follows:

<u>Definition 8</u>. Let $\Lambda$ be the subset of the reals defined by:

(15) $$\Lambda = \{\lambda \in E^1 \mid 0 \leq \lambda \leq N\}$$

<u>Definition 9</u>. Let $R(.)$ be the map from $\Lambda$ into all the subsets of $E^n$

defined by:

$$\text{(16)}\quad R(\lambda) = \{y \in E^N | y = r_0 + \sum_{j=1}^{N} r_j\, u^j,\ |u^j| \leq 1,\ \sum_{j=1}^{N} (u^j)^2 \leq \lambda\}$$

The following result is obvious.

Theorem 7. The mapping R(.) defined in (16) has the following properties:

(i) $R(\lambda)$ is convex and compact for all $\lambda$ in $\Lambda$,

(ii) $R(\lambda') \subset R(\lambda'')$ for all $\lambda'$, $\lambda''$ in $\Lambda$ such that $\lambda' < \lambda''$

(iii) $R(\lambda)$ is continuous on $\Lambda$.

We now see that Problem 2 can be restated as a sequence of two problems.

Problem 3. Minimize $\lambda$ subject to

$$\lambda \in \Lambda,\ R(\lambda) \cap T \neq \phi$$

where T, $\Lambda$ and R(.) are defined as in (10), (15) and (16) respectively.

Problem 4. Given that $\hat{\lambda}$ is the solution of Problem 3 find the sequence $\hat{u}^1, \ldots, \hat{u}^N$ with the properties that

$$\text{(i)}\quad \sum_{j=1}^{N} (\hat{u}^j)^2 = \hat{\lambda}$$

$$\text{(ii)}\quad (r_0 + \sum_{j=1}^{N} r_j\, \hat{u}^j) \in T$$

$$\text{(iii)}\quad |\hat{u}^j| \leq 1,\ j = 1, 2 \ldots N.$$

Consequently, if we wish to solve both problems simultaneously we must add a few operations to Algorithm 1 in order to take care of Problem 4. These will be stated in Algorithm 2 below. Since this is a specific algorithm, it contains exact instructions for carrying out the computations required by Algorithm 1.

<u>Notation:</u> We shall denote by u the sequence of controls $(u^1, u^2, \dots, u^N)$. Different control sequences will be denoted by $u_1$, $u_2$ etc.

<u>Remark</u>: Algorithm 1 may generate infinite sequences and therefore, from a practical point of view, some sort of truncation of the sequences must be included in the algorithm in order to obtain finite computational time. The positive scalar $\varepsilon$ introduced in Algorithm 2 fulfills this purpose.

<u>Algorithm 2</u>: Let $\varepsilon > 0$ be given.

<u>Step 0</u>: Compute $r_j$, $j = 0, 1, 2 \dots N$ using (12), (13) and (14). Compute an $x_0 \in T$, set $y_0 = r_0$, $u_0 = 0$, $i = 0$.

<u>Step 1</u>: If $y_i$ is in T let $x_i = y_i$ and go to Step 2, else go to Step 2.

<u>Step 2</u>: If $\|y_i - x_i\| \leq \varepsilon$ stop, else go to Step 3.

<u>Step 3</u>: Compute $s_i = y_i - x_i$ and a point $v_i \in V(s_i)$.

<u>Step 4</u>: If $\langle s_i, r_0 - v_i \rangle \leq 0$ set $\tilde{u}_i = 0$, $w_i = r_0$ and go to Step 6, else go to Step 5.

<u>Step 5</u>: Compute a scalor $\nu < 0$ satisfying

$$\sum_{j=1}^{N} \langle r_j, s_i \rangle \operatorname{sat} \langle \nu r_j, s_i \rangle = \langle v_i - r_0, s_i \rangle .$$

If no such $\nu$ exists, Problem 2 has no solution, stop,

else set

$$\tilde{u}_i^j = \text{sat}\,\langle \nu r_j, s_i \rangle^{\dagger}, \quad j = 1, 2, \ldots, N;$$

$$w_i = r_0 + \sum_{j=1}^{N} r_j \tilde{u}_i^j;$$

and go to Step 6.

<u>Step 6</u>: Compute $x_{i+1} \in [x_i, v_i]$

and $y_{i+1} \in [y_i, w_i]$

satisfying

$$\|y_{i+1} - x_{i+1}\| \leq \|y - x\|$$

for all $x \in [x_i, v_i]$, for all $y \in [y_i, w_i]$.

<u>Step 7</u>: Compute $\xi \in [0, 1]$ such that

$$y_{i+1} = (1-\xi)\, y_i + \xi\, w_i$$

and set $u_{i+1} = (1-\xi)\, u_i + \xi\, \tilde{u}_i$

<u>Step 8</u>: Let $\lambda_{i+1} = \sum_{j=1}^{N} (u_{i+1}^j)^2$

set i = i+1 and go to Step 1.

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†Note: The function sat (.) : $E^1 \to E^1$ is defined by

sat $(\alpha) = \alpha$ if $|\alpha| \leq 1$

sat $(\alpha) = 1$ if $\alpha > 1$

sat $(\alpha) = -1$ if $\alpha < -1$

In view of Lemma 1 and Theorem 5 , the following result is clear.

Theorem 8. When Problem 2 has a solution, Algorithm 2 generates finite sequences

$$\{x_i\}_{i=0}^k,\ \{y_i\}_{i=0}^k,\ \{u_i\}_{i=0}^k \text{ and } \{\lambda_i\}_{i=0}^k$$

such that

(i) $y_k = r_0 + \sum_{j=1}^{N} r_j u_k^j \in R(\lambda_k)$;

(ii) $x_k \in T$;

(iii) $\|y_k - x_k\| \leq \varepsilon$;

(iv) $\lambda_k = \sum_{j=1}^{N} (u_k^j)^2 \leq \hat{\lambda}$, where $\hat{\lambda}$ is the optimal cost for Problem 2;

(v) $|u_k^j| \leq 1 \quad j = 1, 2 \ldots N.$

Remark: The definition of $R(\lambda_k)$ implies that

$$y_k = r_0 + \sum_{j=1}^{N} r_j u_k^j \in R(\lambda_k).$$

By (ii) in Theorem 7, $R(\lambda_k) \subset R(\hat{\lambda})$. Together these two facts imply that $y_k \in R(\hat{\lambda})$.

The reader can see easily that if $w_i$ is determined in Step 4 of Algorithm 2 then $\langle s_i, w_i - v_i \rangle \leq 0$, and if $w_i$ is computed in Step 5 of Algorithm 2 then $\langle s_i, w_i - v_i \rangle = 0$. In either case $w_i$ satisfies

$\langle s_i, w_i - v_i \rangle \leq 0$ as required in Step 5 of Algorithm 1.

<u>Computational results</u>

In order to obtain an idea of the computational behavior of Algorithm 2, Problem 2 has been solved for:

(17) $$n = 10$$

(18) $$N = 50$$

(19) $$A_j = \begin{bmatrix} 0.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9 \end{bmatrix}$$

for $j = 1, 2 \ldots 50$

(20) $$b_j = (7.6, 7.6, .1, .1, 15.2, 15.2, 0.1, 0.1, 7.6, 7.6)$$

for $j = 1, 2 \ldots 50$

(21) $$\hat{z}_0 = (3000, 3000, 1000, 1000, 6000, 6000, 1000, 1000, 3000, 3000)$$

Algorithm 2 was programmed in Fortran IV and the computations were carried out on a CDC 6400 computer. The value of $\varepsilon$ used in the algorithm was taken to be 0.2. The problem was solved for 3 different target sets T: a point, a ball and a cube in $E^n$.

<u>Case 1</u>: $T = \{t\}$, i.e., T is a point in $E^n$, with

(22) $$t = (0.6,\ 0.6,\ -0.07,\ -0.07,\ 1.2,\ 1.2,\ -0.08,\ -0.08,\ 0.6,\ 0.6)$$

The problem was solved in 2 iterations, the computation time being 1.21 second. The terminal cost was found to be 0.727.

<u>Remark</u>: The system defined in Problem 2 with $A_j$ and $b_j$ as given in (19) and (20) is not completely controllable and hence R(.) is not strictly convex. Incidentally this fact also made it somewhat difficult to construct a target set for which a solution exists.

<u>Case 2</u>: T is a ball in $E^n$, i.e.,

(23) $$T = \{x \mid x^T P\, x + p^T\, x + \pi \leq 0\}$$

where $P = I$, the $n \times n$ identity matrix

$$p = (0.8,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)$$

$$\pi = -9$$

The origin in $E^n$ was used as the initial point $x_0 \in T$ for Step 0 of Algorithm 2 . The problem was solved in 15 iterations, the computation time being 1.68 seconds. The terminal cost was found to be 0.725.

Case 3: T is a unit cube in $E^n$, i.e.,

(24) $$T = \{x \in E^n \mid |x^i| \leq 1 \quad i = 1, 2 \ldots n\}$$

The origin in $E^n$ was used as the initial point $x_0 \in T$ in Step 0 of Algorithm 2 . The problem was solved in 52 iterations, the computation time being 7.68 seconds. The terminal cost was found to be 0.734.

Remark: Because of lack of space we do not give the values of the sequence $\{x_i\}$, $\{v_i\}$, $\{y_i\}$, $\{w_i\}$ and $\{u_i\}$ generated by Algorithm 2 for the cases 1, 2 and 3. However, we think that an indication of the number of iterations and of the computation times should be sufficient to give an idea of the behavior of Algorithm 2 in the three specific cases considered. We stated the final value of the cost in each case in order to indicate that the three cases considered are in some sense "comparable".

Remark: At this point, it must be obvious to the reader that Algorithm 2 can be modified easily to solve problems which differ from Problem 2 only in that they use the cost function $\sum_{j=1}^{N} |u^j|$ instead of $\sum_{j=1}^{N} (u^j)^2$.

Problem 5: Minimize $\sum_{j=1}^{N} |u^j|$ subject to (8), (9), (10) and (11).

To solve Problem 5 we use Algorithm 2 with the Step 5' below replacing Step 5.

Step 5': Compute a scalar $\nu < 0$ satisfying

$$\sum_{j=1}^{N} \langle r_j, s_i \rangle \, \overline{\mathrm{dez}} \, (\langle \nu r_j, s_i \rangle) \geq \langle v_i - r_0, s_i \rangle$$

and

$$\sum_{j=1}^{N} \langle r_j, s_i \rangle \,\overline{\overline{\mathrm{dez}}}\,(\langle \nu r_j, s_i \rangle) \leq \langle v_i - r_0, s_i \rangle$$

where $\overline{\mathrm{dez}}(.)$ and $\overline{\overline{\mathrm{dez}}}(.)$ are functions from $E^1$ into $E^1$ defined by

$$\overline{\mathrm{dez}}(\alpha) = \begin{cases} 0 \text{ if } |\alpha| \leq 1 \\ 1 \text{ if } \alpha > 1 \\ -1 \text{ if } \alpha < -1 \end{cases}$$

$$\overline{\overline{\mathrm{dez}}}(\alpha) = \begin{cases} 0 \text{ if } |\alpha| < 1 \\ 1 \text{ if } \alpha \geq 1 \\ -1 \text{ if } \alpha \leq -1 \end{cases}$$

If no such $\nu$ exists, Problem 5 has no solution, stop, else set

$\tilde{u}_i^j = 1$ if $\langle \nu r_j, s_i \rangle > 1$

$\tilde{u}_i^j = 0$ if $|\langle \nu r_j, s_i \rangle| < 1$

$\tilde{u}_i^j = 1$ if $\langle \nu r_j, s_i \rangle < -1$

let $J = \{j \mid |\langle \nu r_j, s_i \rangle| \neq 1\}$

$J_+ = \{j \mid \langle \nu r_j, s_i \rangle = 1\}$

$J_- = \{j \mid \langle \nu r_j, s_i \rangle = -1\}$

then the $\tilde{u}_i^j$ , $j \in J_+ \cup J_-$ are determined by solving the following trivial problem:

find scalars $\tilde{u}_i^j$, $j \in J_+ \cup J_-$, satisfying

(a) $0 \leq \tilde{u}_i^j \leq 1 \qquad j \in J_+;$

(b) $-1 \leq \tilde{u}_i^j \leq 0 \qquad j \in J_-;$

(c) $$\sum_{J_+ \cup J_-} \langle r_j, s_i \rangle \tilde{u}_i^j = \langle v_i - r_0, s_i \rangle - \sum_{j \in J} \langle r_j, s_i \rangle \tilde{u}_i^j$$

Then let $w_i = r_0 + \sum_{j=1}^{N} r_j \tilde{u}_i^j$ and go to step 6.

In view of Lemma 1 and Theorem 5 the following result is clear.

<u>Theorem 9</u>. When Problem 5 has a solution, Algorithm 2, with Step 5' replacing Step 5, generates finite sequences $\{x_i\}_{i=0}^k$, $\{y_i\}_{i=0}^k$, $\{u_i\}_{i=0}^k$ and $\{\lambda_i\}_{i=0}^k$ such that

(i) $y_k = r_0 + \sum_{j=1}^{N} r_j u_k^j \in R(\lambda_k);$

(ii) $x_k \in T;$

(iii) $\|y_k - x_k\| \leq \varepsilon;$

(iv) $\lambda_k = \sum_{j=1}^{N} |u_k^j| \leq \hat{\lambda}$, where $\hat{\lambda}$ is the optimal cost for Problem 5;

(v) $|u_k^j| \leq 1 \qquad j = 1, 2 \ldots N.$

## V. A SPECIAL CASE OF THE GEOMETRIC PROBLEM:

So far, we have always considered Problem 1 in its most general form. We shall now consider a special case of Problem 1 and obtain for it a specialized form of Algorithm 1 . As we shall later see, discrete minimum time optimal control problems reduce to this special case of Problem 1.

<u>Problem 6</u>: Given two convex compact subsets T and R of a real Hilbert space $\mathcal{H}$, find a point in $T \cap R$. Problem 6 is a problem of the form of Problem 1 , with $\Lambda$ containing only one point.

<u>Algorithm 3</u>:

<u>Step 0</u>: Compute an $x_0 \in T$, a $y_0 \in R$ and set $i = 0$.

<u>Step 1</u>: If $y_i$ is in T set $x_i = y_i$ and go to Step 2, else go to Step 2.

<u>Step 2</u>: If $\|y_i - x_i\| = 0$ stop, else go to Step 3.

<u>Step 3</u>: Compute $s_i = y_i - x_i$.

<u>Step 4</u>: Compute $v_i \in T$ satisfying $\langle s_i, v_i \rangle \geq \langle s_i, v \rangle$ for all v in T.

<u>Step 5</u>: Compute $\overline{w}_i \in R$ satisfying $\langle s_i, \overline{w}_i \rangle \leq \langle s_i, w \rangle$ for all w in R.

<u>Step 6</u>: If $\langle s_i, \overline{w}_i - v_i \rangle > 0$, stop, Problem (6) has no solution, else go to Step 7.

<u>Step 7</u>: Compute $y_{i+1}$ in $[y_i, \overline{w}_i]$ and $x_{i+1}$ in $[x_i, v_i]$ such that $\|y_{i+1} - x_{i+1}\| \leq \|y - x\|$ for all x in $[x_i, v_i]$ and for all y in $[y_i, \overline{w}_i]$. Let $i = i+1$ and go to Step 1.

In view of Lemma 1 and Theorem 5 the following result is clear.

<u>Theorem 10</u>. When Problem 6 has a solution i.e. when $T \cap R \neq 0$, Algorithm 3 generates sequences $\{x_i\}$ and $\{y_i\}$ in T and R respectively such that:

(i) the sequence $\{\|y_i - x_i\|\}$ is monotonocally decreasing;

(ii) when the sequences $\{x_i\}$ and $\{y_i\}$ are finite, their last element satisfies $\|y_k - x_k\| = 0$;

(iii) when the sequences $\{x_i\}$ and $\{y_i\}$ are infinite, the sequence $\{\|y_i - x_i\|\}$ converges to zero.

<u>Theorem 11</u>. When Problem 6 has no solution i.e. $R \cap T = \phi$, Algorithm 3 stops in Step 6 after a finite number of iterations.

<u>Proof</u>: Let $R \cap T = \phi$, then Algorithm 3 cannot stop in Step 2. Suppose that Algorithm 3 does not stop in Step 6 after a finite member of iterations, i.e. it generates an infinite sequence $\{(x_i, v_i, y_i, \bar{w}_i)\}$. The compactness of R and T implies that there exists an infinite subset of the integers K such that the subsequence $\{(x_i, v_i, y_i, \bar{w}_i)\}_K$ converges to some point, say $(x^*, v^*, y^*, \bar{w}^*)$. From Theorem 4 we get:

$$\langle x^* - y^*, x^* - v^* \rangle = 0$$

and

$$\langle y^* - x^*, y^* - \bar{w}^* \rangle = 0.$$

This implies that $\langle y^* - x^*, \bar{w}^* - v^* \rangle = \|y^* - x^*\|^2$. By assumption R and T are convex, compact and disjoint, therefore $\|y^* - x^*\|^2 > 0$. It follows by continuity of the scalar product that there exists a finite integer $k \in K$ such that

$$\langle y_i - x_i, \bar{w}_i - v_i \rangle > 0 \text{ for all } i \geq k,\ k \in K$$

i.e. $\langle s_i, \bar{w}_i - v_i \rangle > 0$ for all $i \geq k$, $k \in K$. This contradicts the hypothesis that the algorithm doesn't stop in Step 6 after a finite number of iterations and therefore the theorem is proved.

Remark: Theorem 11 is extremely important in that if Problem 6 has no solution, then Algorithm 3 will indicate this fact in a finite number of iterations i.e. in a finite time.

The author's computational experience leads them to suspect that the following conjecture is true.

Conjecture 1. If (int T) $\cap$ R $\neq \phi$, then Algorithm 3 generates finite sequences $\{x_i\}_{i=1}^{k}$ and $\{y_i\}_{i=1}^{k}$ such that $\|y_k - x_k\| = 0$ i.e., the solution of Problem 6 is obtained in a finite number of steps.

## VI. DISCRETE MINIMUM-TIME OPTIMAL CONTROL PROBLEMS.

In order to show the versatility of Algorithm 1, we have adapted and used it to solve a few discrete minimum-time optimal control problems which we shall now describe.

Problem 7: Minimize the integer N subject to

$$z_j = A_j z_{j-1} + b_j u^j \qquad j = 1, 2 \ldots, N \tag{25}$$

$$z_0 = \hat{z}_0; \tag{26}$$

$$z_N \in T; \tag{27}$$

$$|u^j| \leq 1 \qquad j = 1, 2 \ldots N; \tag{28}$$

where, for $j = 0, 1, 2 \ldots$, $z_j \in E^n$ is the state of the system at time j, and, for $j = 1, 2 \ldots$, $u^j \in E^1$ is the input at time j. The matrices $A_j$ and $b_j$ are real and are of dimension $n \times n$ and $n \times 1$ respectively for $j = 1, 2 \ldots$ . The set $T \subset E^n$ is assumed to be compact and convex. As an intermediate step in transcribing Problem 7 into the form of Problem 1 we rephrase this optimal control problem as a convex programming problem as follows. For $N = 1, 2 \ldots$, let

$$r_0(N) = A_N A_{N-1} \cdots A_1 \hat{z}_0; \tag{29}$$

$$r_j(N) = A_N A_{N-1} \cdots A_{j+1} b_j \qquad j = 1, 2 \ldots N-1; \tag{30}$$

$$r_N(N) = b_N. \tag{31}$$

then Problem 7 becomes: minimize the integer N subject to

$$(32) \qquad (r_0(N) + \sum_{j=1}^{N} r_j(N)u^j) \in T$$

$$(33) \qquad |u^j| \leq 1 \qquad j = 1, 2 \dots N.$$

The transcription in the form of Problem 1 is completed by defining $\Lambda$ and R(.) as follows:

<u>Definition 10</u>. Let $\Lambda$ be the subset of the reals consisting of the positive integers.

<u>Definition 11</u>. Let R(.) be the map from $\Lambda$ into all the subsets of $E^n$ defined by:

$$(34) \quad R(N) = \{y \in E^n | y = r_0(N) + \sum_{j=1}^{N} r_j u^j, \ |u^j| \leq 1 \quad j = 1, \dots N\}$$

Problem 7 can now be seen to be equivalent to the following one.

<u>Problem 8</u>: Find the smallest positive integer N such that $R(N) \cap T \neq \phi$ where R(N) is defined by (34).

At this point it must be obvious to the reader that for a fixed positive integer N, the sets R(N) and T are convex and compact, it follows that the solution of Problem 8 can be obtained by trying to solve a sequence of problems of the form of Problem 6 with N = 0, 1, 2 ...., Theorem 11 shows that Algorithm 3 will indicate that $R(N) \cap T$ is empty in a finite number of steps if $N < \hat{N}$. We note that the solution of Problem 7 consists of $\hat{N}$, the optimal number of steps and of $\hat{u} = (\hat{u}^1, \hat{u}^2 \dots \hat{u}^{\tilde{N}})$, a sequence of scalars satisfying (32) and (33). The algorithm we are about to describe includes a feature which generates automatically this control sequence.

Algorithm 4: $\varepsilon > 0$ is given.

Step 0: If $\hat{z}_0 \in T$, stop, else set $N = 1$ and go to Step 1.

Step 1: Compute $r_j(N)$, $j = 0, 1, 2 \dots N$ using (29), (30), (31) and an $x_0 \in T$; set $y_0 = r_0(N)$, $u_0 = 0$ and $i = 0$.

Step 2: If $y_i \in T$, set $x_i = y_i$ and go to Step 3, else go to Step 3.

Step 3: If $\|y_i - x_i\| \leq \varepsilon$ stop, else go to Step 4.

Step 4: Compute $s_i = y_i - x_i$ and a point $v_i \in V(s_i)$.

Step 5: If $\langle s_i, r_0(N) - v_i \rangle \leq 0$, set $\tilde{u}_i = 0$, $\overline{w}_i = r_0(N)$ and go to Step 7. If $\langle s_i, r_0(N) - v_i \rangle > 0$ set $\tilde{u}_i^j = - \operatorname{sgn} \langle s_i, r_j(N) \rangle$, $j = 1, 2 \dots N$;

$$\overline{w}_i = r_0 + \sum_{j=1}^{N} r_j(N)\tilde{u}_i^j,$$

and go to Step 6.

Step 6: If $\langle s_i, \overline{w}_i - v_i \rangle > 0$, set $N = N + 1$ and go to Step 1, else go to Step 7.

Step 7: Compute $x_{i+1} \in [x_i, v_i]$ and $y_{i+1} \in [y_i, \overline{w}_i]$ satisfying:

$$\|y_{i+1} - x_{i+1}\| \leq \|y - x\| \text{ for all } y \in [y_i, \overline{w}_i],$$

for all $x \in [x_i, v_i]$.

Step 8: Compute $\xi \in [0,1]$ such that

$y_{i+1} = (1-\xi)\, y_i + \xi\, \bar{w}_i$;

set $u_{i+1} = (1-\xi)\, u_i + \xi\, \tilde{u}_i$;

set $i = i + 1$ and go to Step 2.

In view of Lemma 1 and Theorem 5, we now get the following result:

Theorem 12. If Problem 7 has a solution, then Algorithm 4 generates finite sequences $\{x_i\}$, $\{y_i\}$, $\{u_i\}$ such that their last term satisfies:

(i) $y_k = r_0(N) + \sum_{j=1}^{N} r_j(N) u_k^j$;

(ii) $\|y_k - x_k\| \leq \varepsilon$, where $x_k \in T$;

(iii) $N \leq \hat{N}$ the solution of Problem 7;

(iv) $|u_k^j| \leq 1$, $j = 1, 2 \ldots N$.

Computational results:

We use again the system described in section IV i.e. $n$, $A_j$, $b_j$ and $\hat{z}_0$ are given the values defined in (17), (19), (20) and (21). The value of $\varepsilon$ used in the algorithm was taken to be 0.2. Again the Minimum Time Optimal Control Problem was solved with the target sets: a point, a ball and a cube.

Algorithm 4 was programmed in Fortran IV and the computations were carried out on a CDC 6400 computer.

Case 4: $T = \{t\}$ where t is defined by (22). The solution of the problem was obtained in 2.52 seconds. The minimum number of steps necessary to reach the prescribed neighborhood of T was found to be 37.

Case 5: T is a ball in $E^n$ defined as in (23). The origin in $E^n$ was used as the initial point $x_0 \in T$. The problem was solved in 2.33 seconds. The minimum number of steps being 36.

Remark: Due to the particular structure of Algorithm 4 and the fact that in this experiment the target set T has an interior, the algorithm generated a point in the interior of T in a finite number of steps, which support Conjecture 1.

Case 6: T is a unit cube in $E^n$ defined as in (24). Again, the origin in $E^n$ was used as the initial point $x_0 \in T$.

The problem was solved in 2.18 seconds, the minimum number of steps necessary to reach T was found to be 36. As in the preceding case, the algorithm generated a point in the interior of T in a finite number of steps.

## VII. CONTINUOUS OPTIMAL CONTROL PROBLEMS.

A large number of continuous optimal control problems can be cast into the form of Problem 1 . However, when the approach defined in section III is applied to continuous optimal control problems, the computational difficulties encountered can (but need not) be considerably greater than in the discrete case. In order to show how computational difficulties arise, we shall examine a specific continuous minimal-time optimal control problem.

Problem 8:

Consider the system described by the differential equation

$$\dot{z}(t) = A(t)\ z(t) + b(t)\ u(t) \tag{35}$$

where $z(t) \in E^n$ is the state of the system at time t, $u(t) \in E^1$ is the input at time t and A(.) and b(.) are continuous matrix valued functions of dimensions $n \times n$ and $n \times 1$, respectively. Let $U[t_0,\infty)$ be the set of all Lebesgue-measurable functions u(.) from $[t_0,\infty)$ into $E^1$ satisfying:

$|u(t)| \leq 1$ for almost all t in $[t_0,\infty)$.

Given the initial state $z(t_0) = 0$ at time $t_0$ and a convex compact target set T in $E^n$ find the smallest time $\hat{t}$ in which the system (35) can be taken from $z(t_0) = 0$ to T by a control function $\hat{u}(.)$ in $U[t_0, \hat{t}]$.

We shall suppose that a solution to Problem 8 exists and that we know a $t_{max} < \infty$ satisfying

$$t_{max} \geq \hat{t}$$

Definition 12. Let $\Lambda$ be the set

$$\Lambda = \{t \in E^1 | t_0 \leq t \leq t_{max}\}$$

Definition 13. Let $\phi(.,.)$ be the state transition matrix of system (35) i.e.

(i) $\phi(t_0, t_0) = I$;

(ii) $\frac{d}{dt} \phi(t, t_0) = A(t)\, \phi(t, t_0)$.

Then, since $z(t_0) = 0$,

$$z(t) = \int_{t_0}^{t} \phi(t, \tau)b(\tau)u(\tau)d\tau. \tag{36}$$

Definition 14. Let $R(.)$ be the mapping from $\Lambda$ into all the subsets of $E^n$ defined by:

$$R(t) = \{y | y = \int_{t_0}^{t} \phi(t, \tau)b(\tau)u(\tau)d\tau;\ u(.) \in U[t_0, t]\} \tag{37}$$

The following result is classical and is given without proof (see [2]).

Theorem 13. The mapping $R(.)$ as defined in (37) satisfy,

(i) $R(t)$ is convex and compact for all $t$ in $\Lambda$;

(ii) $R(t') \subset R(t'')$ for all $t'$, $t''$ in $\Lambda$ such that $t' < t''$;

(iii) $R(.)$ is continuous on $\Lambda$.

It follows that Problem 8 can be rewritten in the following form

Problem 9. Given a convex compact subset $T$ of $E^n$ and a mapping $R(.)$ from

$\Lambda$ into $E^n$ defined by (37), find $\hat{t}$ in $\Lambda$ satisfying

$R(\hat{t}) \cap T \neq \phi$;

$\hat{t} \leq t$ for all $t$ such that $R(t) \cap T \neq \phi$.

It must be obvious at this point that Algorithm 1 can be applied to Problem 8 . We note that in this case, $Q(v, s)$ is defined by:

$$(38)\quad Q(v, s) = \begin{cases} t_{max} \text{ when } P(v, s) \cap R(\lambda) = \phi \text{ for all } \lambda \in \Lambda; \\ \min \{t \in \Lambda | P(v, s) \cap R(t) \neq \phi\} \text{ otherwise.} \end{cases}$$

The quantity $Q(v, s)$ can be characterized in a different way.

<u>Definition 15</u>. Let $f$ be a mapping from $\Lambda \times E^n \times E^n$ into $E^1$ defined by

$$(39)\quad f(t, v, s) = - \langle s, v \rangle - \int_{t_0}^{t} \langle \phi(t, \tau)b(\tau), s \rangle \operatorname{sgn} \langle \phi(t, \tau)b(\tau), s \rangle \, d\tau$$

<u>Lemma 2</u>. When the set $\{t \in \Lambda | P(v, s) \cap R\ t) \neq \phi\}$ is not empty then $Q(v, s)$ satisfies:

(i) $f(Q(v, s), v, s) \leq 0$

(ii) $Q(v, s) \leq t$ for all $t$ in $\Lambda$ such that

$f(t, v, s) \leq 0$.

The algorithm needed to solve Problem 8 is now given.

Algorithm 5:

Step 0: Compute an $x_0 \in T$ and set $y_0 = 0$, $u_0(.) = 0$, $i = 0$.

Step 1: If $y_i \in T$, set $x_i = y_i$ and go to Step 2, else go to Step 2.

Step 2: If $\|y_i - x_i\| \leq \varepsilon$ stop, else go to Step 3.

Step 3: Compute $s_i = y_i - x_i$ and a point $v_i \in V(s_i)$.

Step 4: If $\langle s_i, v_i \rangle \geq 0$ set $\tilde{u}_i(.) = 0$, $w_i = 0$ and go to Step 6, else go to Step 5.

Step 5: Compute $t_i$ satisfying:

$$t_i = \min \{t \geq t_0 | \; f(t, v_i, s_i) \leq 0\}$$

then compute $\tilde{u}_i(.) \triangleq - \operatorname{sgn} \langle \phi(t_i, .)b(.), s_i \rangle$

$$w_i = \int_{t_0}^{t_i} \phi(t_i, \tau)b(\tau)\tilde{u}_i(\tau)d\tau$$

and go to Step 6.

Step 6: Compute $x_{i+1} \in [x_i, v_i]$ and $y_{i+1} \in [y_i, w_i]$ satisfying:

$\|y_{i+1} - x_{i+1}\| \leq \|y - x\|$ for all x in $[x_i, v_i]$, for all y in $[y_i, w_i]$.

Let $i = i+1$ and go to Step 1.

In view of Lemma 1 and Theorem 5 the following theorem is clear.

Theorem 14. When Problem 8 has a solution, Algorithm 5 generates finite sequences $\{x_i\}_{i=1}^k$, $\{y_i\}_{i=0}^k$, $\{u_i\}_{i=0}^k$, and $\{t_i\}_{i=0}^k$ such that

(i) $y_k = \int_{t_0}^{t_k} \phi(t_k, \tau)b(\tau)u_k(\tau)d\tau$;

(ii) $x_k \in T$;

(iii) $\|y_k - x_k\| \leq \varepsilon$;

(iv) $t_k \leq \hat{t}$, where $\hat{t}$ is the solution of the minimum-time optimal control Problem 7;

(v) $|u_k(t)| \leq 1$ for all t in $[t_0, \hat{t}]$.

A close examination of Algorithm 5 reveals several difficulties. To compute $t_i$ in Step 5, we must solve the nonlinear programming problem: minimize t subject to $f(t, v_i, s_i) \leq 0$. This problem is not amenable to finite step solution. While procedures such as the Fibonacci search (see [12]) can be used to obtain an arbitrarily good approximation in a finite number of steps, the calculations can become quite time consuming because of the need to integrate in calculating $f(t, v_i, s_i)$ as defined in (39). In addition, one may have some difficulty in ensuring that the integration subroutines used do not lead to an accumulation of excessive errors.

Because of the above mentioned difficulties, for efficient implementation on computer, the Algorithm 5 must be modified by the inclusion of $\varepsilon$-procedures, analogous to the ones outlined in [10]. Since one also has to use $\varepsilon$-procedures when one cannot compute exactly a $v \in V(s)$ in a finite number of steps (as in the case when T is strictly convex and has edges), and since these $\varepsilon$-procedures are quite complex and difficult to describe, the authors work on the use of $\varepsilon$-procedures in solving Problem 1 will be presented in a separate paper.

## VIII. CONCLUSION.

This paper presented five closely related decomposition algorithms for the solution of optimal control problems. The examples given, as well as other experimental evidence available, indicated that these algorithms are very efficient and that they do not suffer from undue ill-conditioning effects.

Preliminary work indicates that the range of applicability of these algorithms can be considerably extended by the addition of so called $\varepsilon$-procedures. These procedures are used to obtain various approximations in a finite number of steps while preserving the convergence properties of the algorithms. The authors will present their work in $\varepsilon$-procedures in a separate paper.

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