DESIGN OF LINEAR TIME-VARYING AND NONLINEAR TIME-INVARIANT NETWORKS

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ABSTRACT

This paper considers the design of linear time-varying networks and of nonlinear time-invariant networks, the latter being operated in the small signal mode. In the first part, the design example considered is a network whose time-delay is a prescribed function of time. A quadratic performance criterion is formulated and the design is obtained iteratively by steepest descent. The second part of the paper considers the design of a nonlinear time-invariant network with variable bias sources whose small signal equivalent network is identical with a given linear time-varying network. Explicit conditions are given under which this can be done.

I. INTRODUCTION

There are a number of technical applications where networks with delays that vary with time would be very useful [1]. A number of authors

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have suggested ways to design networks whose delays are discretely adjustable [1,2]. For these reasons we used as a design objective a network whose time delay must approximate a given function of time. Such networks might also find applications in FM and in systems with transportation lag.

One approach might be to attempt to design a nonlinear time-invariant network to be operated in the small-signal mode: by varying the operating point, the characteristics of the small-signal equivalent network about that operating point would vary and provide the desired variable delay. Such an approach was suggested in a previous paper [3]. From a design point of view, however, it raises many questions and poses problems yet unsolved. This approach, however, is based on the concept of a linear time-varying network supplying a prescribed time delay. For this reason we consider in the next three sections of this paper the problem of designing such a linear time-varying network. Since there are no known analytic definitions of delay for such time-varying networks, we define it in the time domain as is commonly done for pulse circuits (see Fig. 1). We use the speed of the computer to carry out an optimization using the steepest descent method and thus obtain an optimum linear time-varying network.

In Section V we consider the design of a nonlinear time-invariant network to be operated in the small-signal mode. We show that by using a single universal nonlinear characteristic for the nonlinear energy storing elements, we can choose the bias waveform so that the resulting small-signal equivalent network has state equations identical with those of the prescribed linear time-varying network. We note with interest that in the case of slowly varying bias only one bias source is required.

II. PROBLEM OF DESIGNING FOR PRESCRIBED TIME-DELAY USING LINEAR TIME-VARYING NETWORKS

1. Description of the Network

We consider linear time-varying networks with a single input u and single output v that are described by equations of the form

$$\dot{x}(t) = a(t)Ax(t) + bu(t) \quad , \quad x(0) = 0$$
$$v(t) = \frac{1}{C_1} a(t)x_1(t) \tag{1}$$

where the state $x(t) \in \mathbb{R}^n$; A is a real constant n×n matrix whose eigenvalues are in the open left half plane; b is a constant column vector and c_1 is a positive constant. The function $a(\cdot): \mathbb{R}_+ \to \mathbb{R}$ specifies the instantaneous characteristics of the network; clearly $a(t) > 0$ for all t. ($\mathbb{R}_+$ denotes the set of all nonnegative real numbers.) An example of a linear time-varying network described by equations such as (1) is a low-pass LC ladder terminated at both ends by time-invariant resistors (e.g. the network shown in Fig. 4). The components of x are capacitor charges and inductor fluxes. The scalar factor a(t) indicates that all reactive elements vary proportionately to one another.

We want to adjust the function $a(\cdot)$ so that the network has a prescribed time-delay characteristic. Typically we would have $a(\cdot)$ varying periodically. Consequently we propose to define the <u>time-delay at time t'</u> of the network described by (1) as follows: Consider the zero-state response to a unit step applied at time t', namely the function $t \mapsto v(t;t',\theta,1_{t'})$.

The second and third arguments, namely, "t', θ", indicate that the network starts from the zero-state at time t'. The fourth argument denotes the input which is $1_{t'}$, the unit step applied at time t'. This response will have the shape shown in Fig. 1: because the network is time-varying, the "steady state" oscillates between v_{max} and v_{min}. The delay at time t', denoted by $T_d(t')$, is defined by

$$v\left(t'+T_d(t');\ t',\ \theta,\ 1_{t'}\right) \qquad = \frac{1}{4}\left(v_{max} + v_{min}\right) . \tag{2}$$

2. Statement of the Problem

The topology and the element values of the network $\mathcal{N}$ under consideration have been selected by elementary considerations (e.g. a max. flat delay characteristic); thus, A, b, C_1 are given. We are also given a desired time-delay characteristic $T(\cdot)$, typically a periodic function. The problem is to choose $a(\cdot)$ so that $T_d(\cdot)$, the time-delay characteristic of $\mathcal{N}$, is as close as possible to $T(\cdot)$. For this purpose we propose the performance criterion

$$J(a) \triangleq \int_0^T \left[T(t') - T_d(t')\right]^2 dt' \tag{3}$$

where T includes one or more periods of the periodic variations of $T(\cdot)$.

3. Method of Solution

Clearly there is no hope to obtain a closed form solution; an iterative technique is proposed. Roughly it goes as follows:

(i) Pick a reasonable $a(\cdot)$ and use (1) and (2) to calculate the functions $x(\cdot)$, $v(\cdot)$, $T_d(\cdot)$ and the cost function $J(a)$ by (3).

(ii) Introduce a small perturbation δa; calculate δx, δv, δT_d and δJ. Use this analysis to obtain $\mathcal{L}(\cdot)$, the direction of steepest descent of J.

(iii) Calculate the optimum μ so that $J(a - \mu\mathcal{L})$ is minimum as a function of μ. Then $\delta a = -\mu\mathcal{L}$.

(iv) Obtain the new steepest descent direction at the new point, etc.

III. ANALYSIS

1. Assumptions

For reference purposes it is convenient to list all assumptions here, although the reason for some of them will be clear only later.

A1 $a(\cdot)$ is C^1 (continuous derivative); $a(\cdot)$ is positive and bounded away from zero.

A2 $\delta a(\cdot)$ is C^1. To express the condition that δa is "small" we use the following norm:

$$\|\delta a\|_w \triangleq \sup_{t\geq 0} |\delta a(t)| + \sup_{t\geq 0} |\delta \dot{a}(t)| \tag{4}$$

A3 $a(\cdot)$ and $\delta a(\cdot)$ are constrained so that the map $t \mapsto t + T_d(t) \triangleq T_d'(t)$ is strictly monotonically increasing. (Therefore it has an inverse.)

It is intuitively clear that if $a(\cdot)$ varies very slowly in comparison to the impulse response of $\mathcal{N}$ then A3 will be automatically satisfied.

2. Perturbation Calculations

If we change $a(\cdot)$ to $a(\cdot) + \delta a(\cdot)$, the resulting trajectory is labelled $x + \delta x$ and, from (1),

$$\delta\dot{x}(t) = a(t)A\delta x(t) + \delta a(t)Ax(t) + \delta a(t)A\delta x(t) \tag{5}$$

Since A is stable, $a(t) > 0$ for all t and bounded away from zero, and since the state transition matrix of (1) is

$$\Phi(t,\tau) = \exp[A \int_{\tau}^{t} a(t')dt'] , \tag{6}$$

it is easy to prove that the last term of (5) is of second order in

$$\|\delta a\|_{\infty} \triangleq \sup_{t\geq 0} |\delta a(t)| . \tag{7}$$

Therefore the variational equation is

$$\delta\dot{x}(t) = a(t)A\delta x(t) + \delta a(t)Ax(t) \tag{8}$$

with

$$\delta x(0) = 0$$

A similar reasoning shows that within second order terms in $\|\delta a\|_{\infty}$,

$$\delta v(t) = \frac{1}{C_1}\left[a(t)\delta x_1(t) + \delta a(t)x_1(t)\right] \tag{9}$$

Where $x_1(t)$ denotes the first component of the vector $x(t)$.

The perturbation $\delta a(\cdot)$ causes a perturbation in the output, $\delta v(\cdot)$, and hence in the delay, $\delta T_d(\cdot)$. Referring to Fig. 2 we have

$$v\left(T_d'(t') + \delta T_d(t')\right) + \delta v\left(T_d'(t') + \delta T_d(t')\right) = \frac{1}{4}\left(v_{max} + v_{min}\right) \tag{10}$$

Since $a(\cdot)$ and $x_1(\cdot) \in C^1$, it follows by (1) that $v(\cdot) \in C^1$, hence by Taylor expansion we obtain from (10) after discarding higher order terms in $\|\delta T_d\|_{\infty}$

$$\delta T_d(t') = -\frac{\delta v\left(T_d'(t')\right)}{\dot{v}\left(T_d'(t')\right) + \delta\dot{v}\left(T_d'(t')\right)} \tag{11}$$

Now from (8), as $\|\delta a\|_{\infty} \to 0$, $\|\delta x\|_{\infty}$ and $\|\delta\dot{x}\|_{\infty} \to 0$; consequently from (9), as $\|\delta a\|_{W} \to 0$, $\|\delta\dot{v}\|_{\infty} \to 0$. Hence in the denominator of (11), the term $\delta\dot{v}\left(T'_d(t')\right)$ is of higher order than the term $\dot{v}\left(T_d'(t')\right)$; consequently, within first order terms in δa, we have

$$\delta T_d(t') = -\frac{\delta v\left(T_d'(t')\right)}{\dot{v}\left(T_d'(t')\right)} \tag{12}$$

<u>Remarks</u>

I. To be completely unambiguous, we should use the notation of (2) to rewrite the numerator of (12) as

$$\delta v\left(T_d'(t');\ t',\ \theta,\ 1_{t'}\right)$$

and the denominator as

$$\dot{v}\left(T_d'(t');\ t',\ \theta,\ 1_{t'}\right)$$

II. From (9) and (8), as $\|\delta a\|_\infty \to 0$, $\|\delta v\|_\infty \to 0$. From (1) we see that $\dot{v}$ does not depend on δa. Hence as $\|\delta a\|_\infty \to 0$ (and, a fortiori, as $\|\delta a\|_w \to 0$), we have $\|\delta T_d\|_\infty \to 0$.

From (3), we obtain immediately (within first order terms)

$$\delta J = -2 \int_0^T \left[T(t') - T_d(t')\right] \delta T_d(t')dt' \tag{13}$$

Let

$$p\left(T_d'(t'),\ t'\right) \triangleq \frac{-2\left[T(t') - T_d(t')\right]}{\dot{v}\left(T_d'(t')\right) c_1} \tag{14}$$

hence using (9) and (12)

$$\delta J = \int_0^T p\left(T_d'(t'),\ t'\right) \left[a\left(T_d'(t')\right)\delta x_1\left(T_d'(t')\right) + \delta a\left(T_d'(t')\right)x_1\left(T_d'(t')\right)\right] dt' \tag{15}$$

Again if the argument of δx_1 and x_1 were written out in detail, it would be "$T_d'(t')$; t', θ, $1_{t'}$". Let e_1^T denote the row vector $(1, 0, \ldots, 0)$. Using (6) and (8) we can split δJ as

$$\delta J = \delta J_1 + \delta J_2 \tag{16}$$

where

$$\delta J_1 = \int_0^T p\Big(T_d'(t'),\ t'\Big)x_1\Big(T_d'(t')\Big)\,\delta a\Big(T_d'(t')\Big)\,dt' \tag{17}$$

and

$$\delta J_2 = \int_0^T p\Big(T_d'(t'),\ t'\Big)a\Big(T_d'(t')\Big)\int_{t'}^{T_d'(t')} e_1^T\Phi\Big(T_d'(t'),\ \alpha\Big)Ax(\alpha)\,\delta a(\alpha)\,d\alpha\,dt' \tag{18}$$

3. <u>Calculation of δJ_1 and δJ_2</u>

By definition, $T_d'(\cdot)$ maps $[0,T]$ onto $[T_d'(0),\ T_d'(T)]$ (see Fig. 3); by A3, $T_d'(\cdot)$ is <u>strictly</u> monotonically increasing. Consequently its inverse, R, maps $[T_d'(0),\ T_d'(T)]$ onto $[0,T]$. Hence let

$$T_d'(t') \triangleq \tau\ , \qquad \text{consequently} \qquad t' = R(\tau) \tag{19}$$

and (17) becomes

$$\delta J_1 = \int_0^{T_d'(T)} p\Big(\tau,\ R(\tau)\Big)x_1\Big(\tau;\ R(\tau),\ \theta,\ 1_{R(\tau)}\Big)\dot{R}(\tau)\,\delta a(\tau)\,d\tau \tag{20}$$

where the lower limit of the integral is zero (instead of $T_d'(0)$) because we extend the definition of R so that

$$R(\tau) = 0 \quad \text{and} \quad \dot{R}(\tau) = 0 \quad \forall \tau \in [0, T_d'(0)] \tag{21}$$

δJ_2 is given by (18) as an integral over the area shown in Fig. 3; (18) carries out the integration by vertical strips. Doing the integration by horizontal strips gives

$$\delta J_2 = \int_0^{T_d'(T)} d\alpha \int_{R(\alpha)}^{\alpha} dt' p\left(T_d'(t'), t'\right) a\left(T_d'(t')\right) e_1^T \phi\left(T_d'(t'), \alpha\right) Ax(\alpha)\delta a(\alpha) \tag{22}$$

Since α in (22) and τ in (20) are dummy variables we may write δJ as

$$\delta J = \int_0^{T_d'(T)} \mathcal{L}(\tau)\delta a(\tau)d\tau \tag{23}$$

where

$$\mathcal{L}(\tau) = \mathcal{L}_1(\tau) + \mathcal{L}_2(\tau) \tag{24-a}$$

and $\mathcal{L}_1(\tau)$ and $\mathcal{L}_2(\tau)$ are

$$\mathcal{L}_1(\tau) = p\left(\tau, R(\tau)\right) x_1(\tau)\dot{R}(\tau) \tag{24-b}$$

$$\mathcal{L}_2(\tau) = \int_{R(\tau)}^{T} p\left(T_d'(t'), t'\right) a\left(T_d'(t')\right) \lambda^T\left(\tau; T_d'(t'), e_1, \theta\right) Ax\left(\tau; t', \theta, 1_{t'}\right) dt' \tag{24-c}$$

In the expression for $\mathcal{L}_2(\tau)$ we set

$$\Phi\left(T_d'(t'),\ \tau\right)^T e_1 \quad = \quad \lambda\left(\tau;\ t',\ 0,\ 1_{t'}\right) \tag{25}$$

i.e. $\lambda(\cdot)$ is the solution of $\dot{z} = -a(t)A^T z$ subject to $\lambda\left(T_d'(t')\right) = e_1$.

4. Conclusion

These perturbation formulas allow us to calculate how to improve on the chosen $a(\cdot)$. We pick $\delta a(\tau)$ to be proportional to $\mathcal{L}(\tau)$:

$$\delta a(\tau) \quad = \quad -\mu\mathcal{L}(\tau) \tag{26}$$

where $\mu > 0$ and is small in order that the perturbation calculations be valid. It will be immediately recognized that this choice of $\delta a(\cdot)$ amounts to minimizing $J(a + \delta a)$ subject to the constraint

$$\int_0^{T_d'(T)} \left[\delta a(\tau)\right]^2 d\tau \quad = \quad \text{small constant.}$$

With δa given by (26) the iterative method sketched in Sec. II.3 above can be applied to minimize J.

IV. DESIGN EXAMPLE

The topology and the relative size of the elements are those of the seventh order max. flat delay network [4, p. 628] shown in Fig. 4 and whose transfer function is

$$G(s) = \frac{135135}{135135 + 135135s + 62370s^2 + 17325s^3 + 3150s^4 + 378s^5 + 28s^6 + s^7} \tag{27}$$

The corresponding zero frequency delay is 1 sec. and the 3 db bandwidth is 2.95 rad/sec. The state equations are

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{\phi}_5 \\ \dot{\phi}_6 \\ \dot{\phi}_7 \end{bmatrix} = \begin{bmatrix} \frac{-1}{C_1} & 0 & 0 & 0 & \frac{1}{L_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{L_5} & \frac{1}{L_6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{L_6} & \frac{1}{L_7} \\ 0 & 0 & 0 & \frac{-1}{C_4} & 0 & 0 & \frac{-1}{L_7} \\ \frac{-1}{C_1} & \frac{1}{C_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{C_2} & \frac{1}{C_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{C_3} & \frac{1}{C_4} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \tag{28}$$

with

$$\left.\begin{aligned} C_1 &= 0.0947 \\ C_2 &= 0.8829 \\ C_3 &= 0.2406 \\ C_4 &= 0.0564 \end{aligned}\right\} \text{farads}$$

$$\left.\begin{aligned} L_5 &= 0.2594 \\ L_6 &= 0.3042 \\ L_7 &= 0.1635 \end{aligned}\right\} \text{henries}$$

If we now let the elements be time-varying in the following manner

$$
\begin{aligned}
C_i(t) &= \frac{C_i}{a(t)} \qquad i = 1, 2, 3, 4 \\
L_i(t) &= \frac{L_i}{a(t)} \qquad i = 5, 6, 7
\end{aligned}
\tag{29}
$$

then the equations will have the form (1), with A given in (28).

For the design example, we pick for the desired time delay

$$T(t) = 1 + 0.5 \sin(0.1\pi t) \tag{30}$$

and the quadratic performance criterion (3) is calculated over [0, 20]. The time delay $T_d(t')$ is calculated by (2) with 0.25 being the value assigned to the right hand side. The function $T_d(\cdot)$ is obtained by calculating step responses at 0, 0.5, 1, 1.5, ..., 20; at intermediate values $T_d(t)$ is obtained by interpolation. To start the calculation we put

$$a_0(t) = \left[1 + 0.5 \sin(0.1\pi t)\right]^{-1}$$

The corresponding value of J is $J(a_0) = 0.2122$. Choosing $\mu_1 = 0.1$ and $\mu_2 = 0.4$ in (26) we obtained (calling $\mathcal{L}_0$ the corresponding value of $\mathcal{L}$)

$$J\left(a_0 - \mu_1 \mathcal{L}_0\right) = 0.162 \quad , \quad J\left(a_0 - \mu_2 \mathcal{L}_0\right) = 0.132$$

Quadratic interpolation through these three points gives for the minimizing

μ the value 0.3 (see Fig. 5) and

$$J\left(a_0 - \mu_{opt}\mathcal{L}_0\right) = 0.126$$

At the end of three successive iterations we obtain

$$J = 0.0937$$

At the point reached after the third iteration the gradient was found to be very small; this was interpreted to indicate that further iterations on $a(\cdot)$ will not reduce J appreciably (Fig. 6). The graph of $a(\cdot)$ corresponding to $J = 0.0937$ is plotted in Fig. 7. $T_d(\cdot)$ resulting from this choice of $a(t)$ is shown in Fig. 8; the desired delay $T(\cdot)$ is also plotted in the same figure for comparison. To observe the step response of the resulting network we applied unit steps at $t' = 0, 2, 4, \ldots, 18$. The corresponding step responses are shown in Fig. 9. This figure shows clearly how the delay $T_d(t)$ depends on t, the time of the application of the unit step.

V. IMPLEMENTATION BY NONLINEAR TIME-INVARIANT NETWORKS

1. Theory

We show below how by using a nonlinear time-invariant RLC network and operating it in the small signal mode [3], we can obtain by appropriate selection of bias-sources many desired time-varying networks. We assume for simplicity that (i) there are no loops of capacitors only, and no cut sets of inductors only; (ii) the resistors are linear; (iii) the inductors and capacitors are nonlinear and time-invariant with characteristics $\left[v_i = f_i(q_i) \text{ and } i_k = f_k(\phi_k)\right]$ such that $f_i \in C^2$ (twice continuously differentiable), $f_i''(\alpha) \neq 0 \ \forall \alpha$, and $f_i'(\cdot)$ is strictly monotonically increasing; (iv) the sources are either voltage sources in series with inductors or current sources in parallel with capacitors. Note that the characteristics given above imply that the inductors and the capacitors are uncoupled.

The equations of the nonlinear network are of the form

$$\dot{x} = Af(x) + \hat{u} \tag{31}$$

where x has capacitor charges and inductor fluxes as its components; the column vector $\hat{u}$ specifies the sources and the input; f(x) is a column vector whose ith element is $f_i(x_i)$ for $i = 1, 2, \ldots, n$. Using Kuh and Rohrer's notation [7], the matrix A is of the well-known form

$$A = \begin{bmatrix} -\mathcal{Y} & \mathcal{H} \\ -\mathcal{H}' & -\mathcal{Z} \end{bmatrix}$$

A is the hybrid matrix representing the resistive network which interconnects the inductors and capacitors.

Let $\hat{u}$ consist of a time-varying bias $\bar{u}$ and a small signal u (i.e. $\|u\| \triangleq \sup_{t\geq 0}|u(t)|$ is a small number),

$$\hat{u} = \bar{u} + u \tag{32}$$

The operating point $\bar{x}$, generated by $\bar{u}$, is defined by

$$\dot{\bar{x}} = Af(\bar{x}) + \bar{u} \tag{33}$$

with the initial condition $\bar{x}(0)$ specified below.

Write the solution of (31) as

$$x = \bar{x} + \xi \tag{34}$$

where $\bar{x}$ is the solution of (33) and ξ is the exact small-signal response. By throwing away second order terms, the approximate small-signal response ξ_0 about the operating point $\bar{x}(t)$ is given by

$$\dot{\xi}_0 = AJ_f\big(\bar{x}(t)\big)\xi_0 + u \tag{35}$$

where the Jacobian $J_f\big(\bar{x}(t)\big)$ is the diagonal matrix whose ith diagonal element is $f_i'\big(\bar{x}_i(t)\big)$. Conditions under which the approximate ξ_0 is close to the exact ξ are known [3]; of particular interest to circuit theorists are the results pertaining to the "slowly varying" case [5,6].

Suppose now that we are given a "desired" linear time-varying network $\mathcal{L}$ described by

$$\mathcal{L}: \qquad \dot{\eta} = AD(t)\eta + u \tag{36}$$

where A is the same matrix as in (31); D(t) is a diagonal matrix whose diagonal elements, $d_i(t)$, are known positive functions of time. Now ask the question: can we design a nonlinear network $\mathcal{N}$ (say, given by (31)) such that its (approximate) small-signal behavior (given by (35)) is identical with the behavior of the prescribed network $\mathcal{L}$ (given by (36))? The answer is yes. More precisely, we have the

Theorem. Let the nonlinear time-invariant network $\mathcal{N}$ be described by (31) and satisfy assumptions (i) to (iv) above. Let A be nonsingular. Let the linear time-varying network $\mathcal{L}$ be given by (36). Under these conditions, the prescribed network $\mathcal{L}$ is identical with the (approximate) small-signal equivalent network $\mathcal{N}_{\ddot{u}}$ described by (35) with $J_f\left(\overline{x}(0)\right) = D(0)$ if and only if the bias of $\mathcal{N}$ is given by

$$\overline{u}(t) = \frac{d}{dt}\left[J_f^{-1}\left(d(t)\right)\right] - Af\left[J_f^{-1}\left(d(t)\right)\right] \qquad \forall t \geq 0 \tag{37}$$

where $d(t) = \mathrm{col}[d_1(t), \ldots, d_n(t)]$.

Proof. $\Longrightarrow$ Since (35) and (36) are identical by assumption and since A is nonsingular, we have, for all $t \geq 0$,

$$D(t) = J_f\left(\overline{x}(t)\right) \tag{38}$$

or, equivalently

$$d_i(t) = f_i'\left(\overline{x}_i(t)\right) \tag{39}$$

where d(t) is defined above; since f_i' has an inverse, (39) becomes

$$\bar{x}_i(t) = f_i'^{-1}\left(d_i(t)\right) \qquad i = 1, 2, \ldots, n \tag{40}$$

Eq. (37) follows immediately upon substituting (40) in (33).

$\Longleftarrow$ If we define y(t) by

$$y(t) \stackrel{\Delta}{=} J_f^{-1}\left(d(t)\right) \tag{41}$$

then substituting y(t) for $\bar{x}(t)$ in (33) shows that if $\bar{u}$ is given by (37) y(t) is a solution of (33). In other words, if $\bar{u}$ is given by (37), the solution of (33) subject to $J_f\left(\bar{x}(0)\right) = D(0)$ is given by (39) (or, equivalently, (41)). Now if (39) holds, so does (38), and upon multiplication on the left by A we have

$$AD(t) = AJ_f\left(\bar{x}(t)\right) \tag{42}$$

Hence (35) and (36) are identical when the bias is given by (37).

<u>Comment</u>. If A is singular, then $\bar{u}$ given by (37) still gives an $\mathcal{N}_{\bar{u}}$ identical with the prescribed $\mathcal{L}$. However, $\bar{u}$ is not unique in this case.

2. <u>Application to Time-Varying Delay Problem of Section II</u>

For the desired linear time-varying network $\mathcal{L}$ we choose the linear time-varying delay network discussed in Section IV. We write here its state equations in a slightly different form than in (28), namely,

$$\dot{x} = AD(t)x + bu \tag{43}$$

where

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{44}$$

and

$$D(t) = a(t) \begin{bmatrix} \frac{1}{C_1} & & & & & & \\ & \frac{1}{C_2} & & & 0 & & \\ & & \frac{1}{C_3} & & & & \\ & & & \frac{1}{C_4} & & & \\ & & & & \frac{1}{L_5} & & \\ & 0 & & & & \frac{1}{L_6} & \\ & & & & & & \frac{1}{L_7} \end{bmatrix}, \quad x = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{45}$$

For the nonlinear time-invariant network $\mathcal{N}$ we choose the one shown in Fig. 10, where

$$v_i = f_i(q_i) \qquad i = 1, 2, 3, 4 \qquad \text{and} \qquad i_k = f_k(\phi_k) \qquad k = 5, 6, 7;$$

$$f_j(x_j) = \gamma\left(\frac{x_j}{\alpha_j}\right) \qquad j = 1, 2, \ldots, 7 \tag{46}$$

where $\gamma(\cdot)$ is a function which satisfies the assumptions (i) to (iv) and the numbers α_j are related to the element values by

$$\alpha_j = \begin{cases} C_j & j = 1, 2, 3, 4 \\ L_j & j = 5, 6, 7 \end{cases} \tag{47}$$

Writing the state equations of $\mathcal{N}$ in the form (31) we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma\left(\frac{x_1}{\alpha_1}\right) \\ \gamma\left(\frac{x_2}{\alpha_2}\right) \\ \gamma\left(\frac{x_3}{\alpha_3}\right) \\ \gamma\left(\frac{x_4}{\alpha_4}\right) \\ \gamma\left(\frac{x_5}{\alpha_5}\right) \\ \gamma\left(\frac{x_6}{\alpha_6}\right) \\ \gamma\left(\frac{x_7}{\alpha_7}\right) \end{bmatrix} + \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 + u \\ \bar{u}_5 \\ \bar{u}_6 \\ \bar{u}_7 \end{bmatrix} \tag{48}$$

The operating point $\bar{x}$ is defined by

$$\dot{\bar{x}} = Af(\bar{x}) + \bar{u} \qquad \bar{x}_i(0) = \alpha_i \gamma'^{-1}\big(d_i(0)\big) \tag{49}$$

where A is given by (44) and $f(\cdot)$ by (46) and

$$\bar{u} = \mathrm{col}[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_7] \tag{50}$$

The equation of the small-signal equivalent network $\mathcal{N}_{\bar{u}}$ about the operating point $\bar{x}(t)$ is

$$\dot{\xi}_0 = AJ_f(\bar{x})\xi_0 + bu \tag{51}$$

where

$$J_f(\bar{x}) = \mathrm{diag}\left[\frac{1}{\alpha_1}\gamma'\left(\frac{\bar{x}_1}{\alpha_1}\right), \ldots, \frac{1}{\alpha_n}\gamma'\left(\frac{\bar{x}_n}{\alpha_n}\right)\right] \tag{52}$$

and b is given by (45).

Comparing (43) and (51) we obtain

$$f_i'\big(\bar{x}_i(t)\big) = \frac{a(t)}{\alpha_i} = \frac{1}{\alpha_i}\gamma'\left(\frac{\bar{x}_i(t)}{\alpha_i}\right) \qquad i = 1, 2, \ldots, 7; \quad t \geq 0$$

Inverting γ', we get

$$\frac{\bar{x}_i(t)}{\alpha_i} = \gamma'^{-1}\big(a(t)\big) \tag{53}$$

hence by (46)

$$f_i\big(\bar{x}_i(t)\big) = \gamma\left(\frac{\bar{x}_i(t)}{\alpha_i}\right) = \left(\gamma \circ \gamma'^{-1} \circ a\right)(t) \qquad i = 1, 2, \ldots, 7 \text{ and } t \geq 0 \tag{54}$$

where $\circ$ denotes the composition of functions.

Calculating the bias waveform by (37), we obtain

$$\overline{u}_i(t) = \frac{d}{dt}\left[\overline{x}_i(t)\right] = \alpha_i \frac{d}{dt}\left[\gamma'^{-1}\left(a(t)\right)\right] \qquad i = 1, 2, 3, 5, 6, 7 \tag{55}$$

and

$$\overline{u}_4(t) = \alpha_4 \frac{d}{dt}\left[\gamma'^{-1}\left(a(t)\right)\right] + 2\left(\gamma \circ \gamma'^{-1} \circ a\right)(t) \tag{56}$$

Thus, if the nonlinear time-invariant network $\mathcal{N}$ (described by (48)) starts from state $\overline{x}(0)$ (obtained from (53)) and is driven by the bias waveforms specified in (55) and (56), then the resulting small-signal equivalent network $\mathcal{N}_{\overline{u}}$ is identical with the desired linear time-varying network $\mathcal{L}$ specified by (43)-(45).

If the prescribed linear time-varying network $\mathcal{L}$ has sufficiently slowly varying characteristics, then from (55) and (56) we see that we need <u>only</u> <u>one</u> bias source because

$$\overline{u}_i(t) \simeq 0 \qquad i = 1, 2, 3, 5, 6, 7$$

$$\overline{u}_4(t) \simeq 2\left(\gamma \circ \gamma'^{-1} \circ a\right)(t)$$

The same result can be arrived at by using the "frozen operating point" method of analysis [8].

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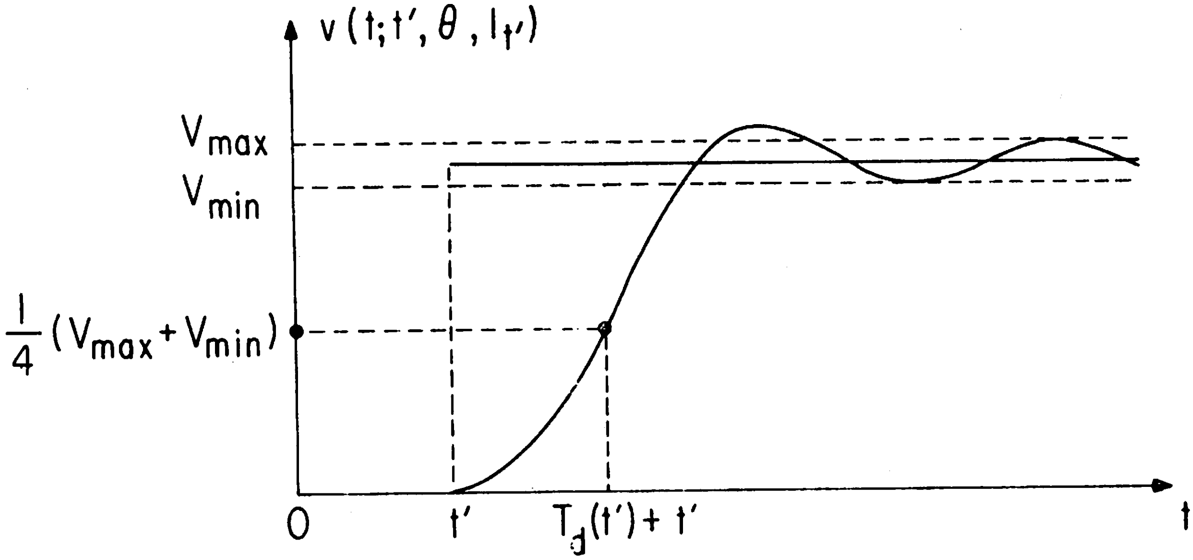

Fig. 1 Response of the linear time-varying network to a unit step applied at time t'. The delay of this network, $T_d(t')$, is defined by the time required for the step response of the network to reach to the halfway point of its steady state.

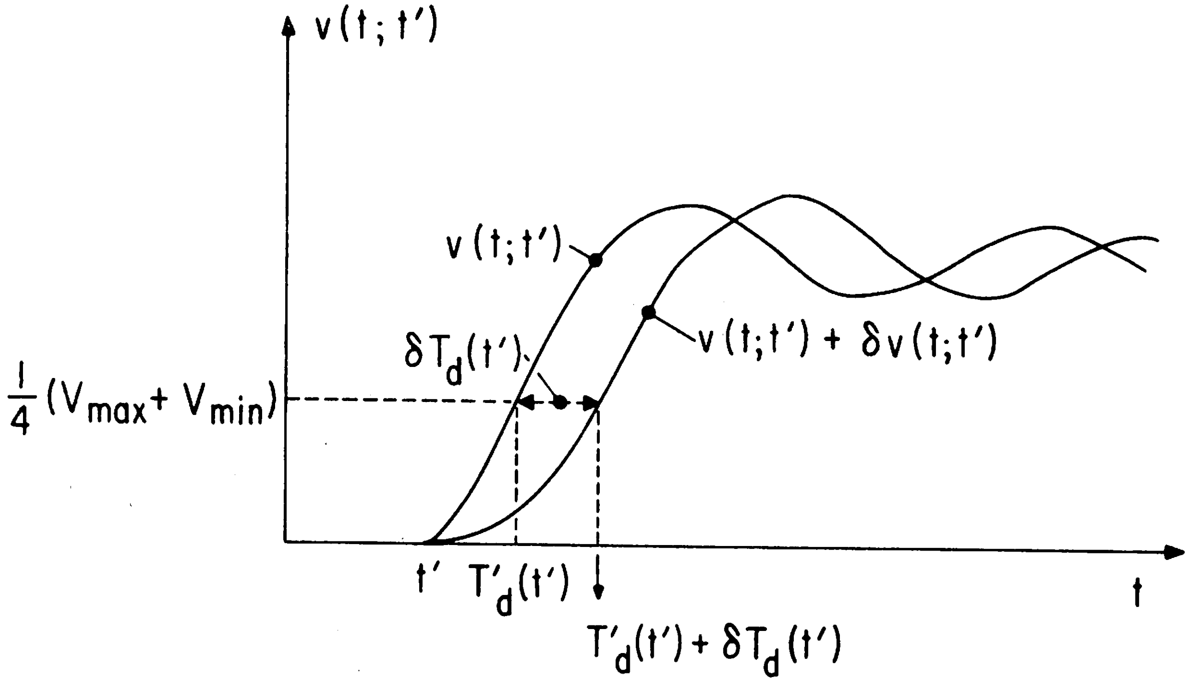

Fig. 2 Perturbation on the step response, $\delta v(\cdot, t')$, caused by a perturbation $\delta a(\cdot)$ on the nominal function $a(\cdot)$. As a result the delay at t', $T_d(t')$, is changed by $\delta T_d(t')$.

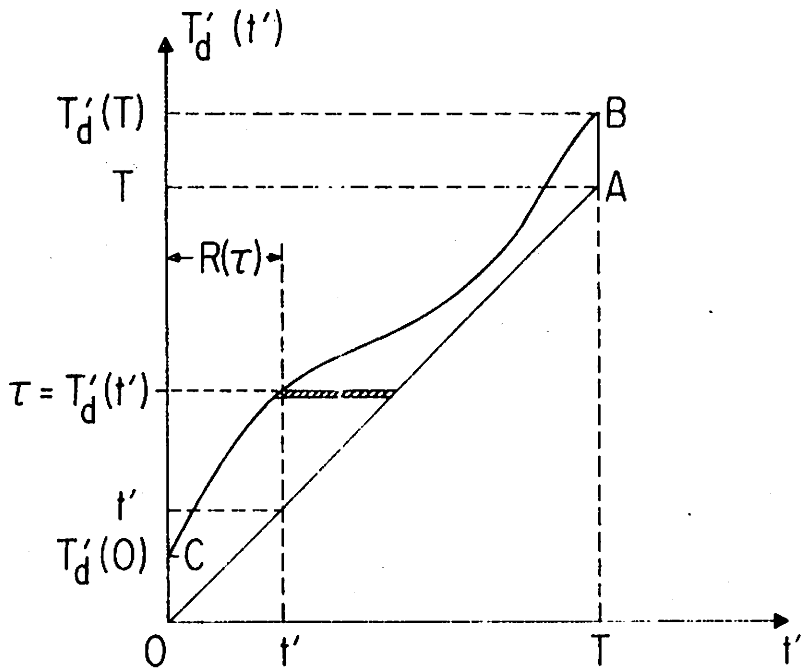

Fig. 3 In order to calculate the perturbation δJ_2, one has to carry out integration over the region OABC.

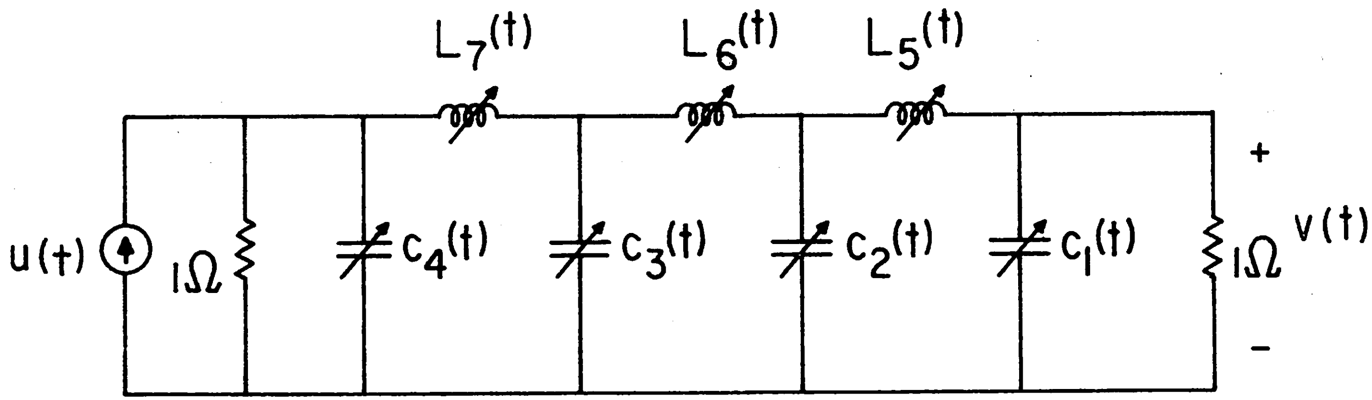

Fig. 4 Linear time-varying maximally flat low pass network whose reactive elements vary proportionally to one another. The current source $u(t)$ is the input and the voltage $v(t)$ across the capacitor $C_1(t)$ is the output.

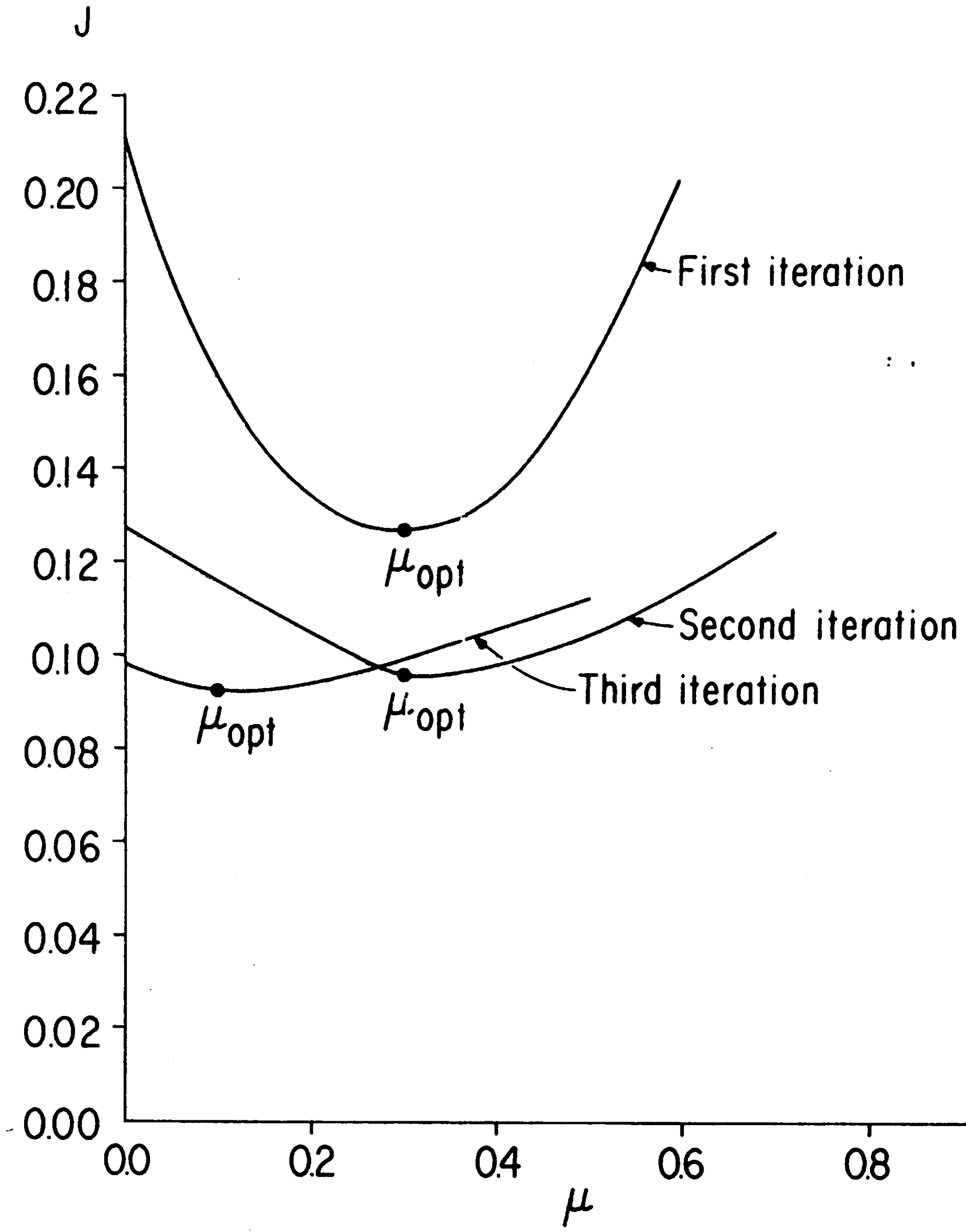

Fig. 5 Quadratic approximation of J for finding the step size that minimizes J along each steepest descent direction.

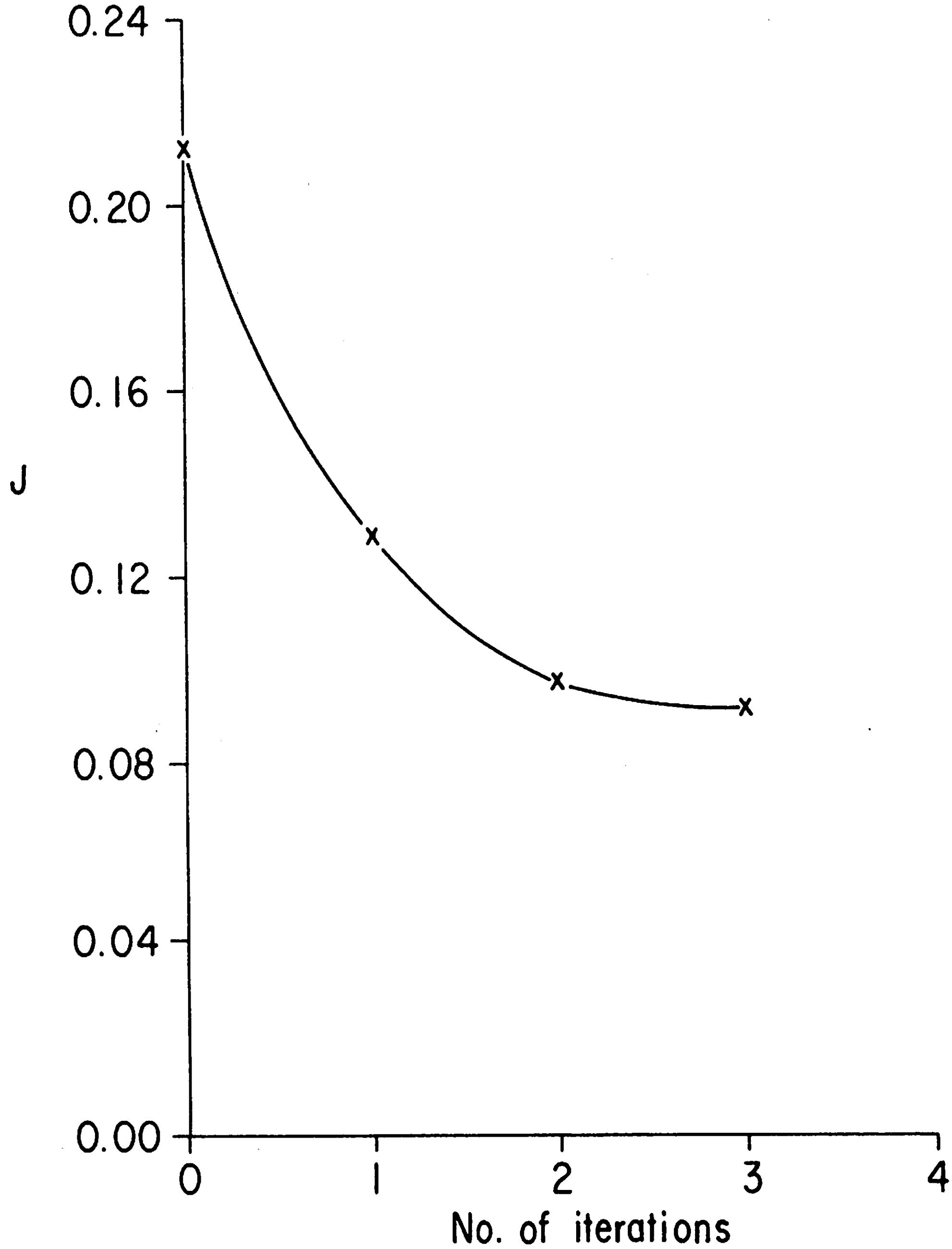

Fig. 6 Plot of the cost function J versus number of iterations. Using the steepest descent method, it takes three iterations to achieve the approximate minimum of J.

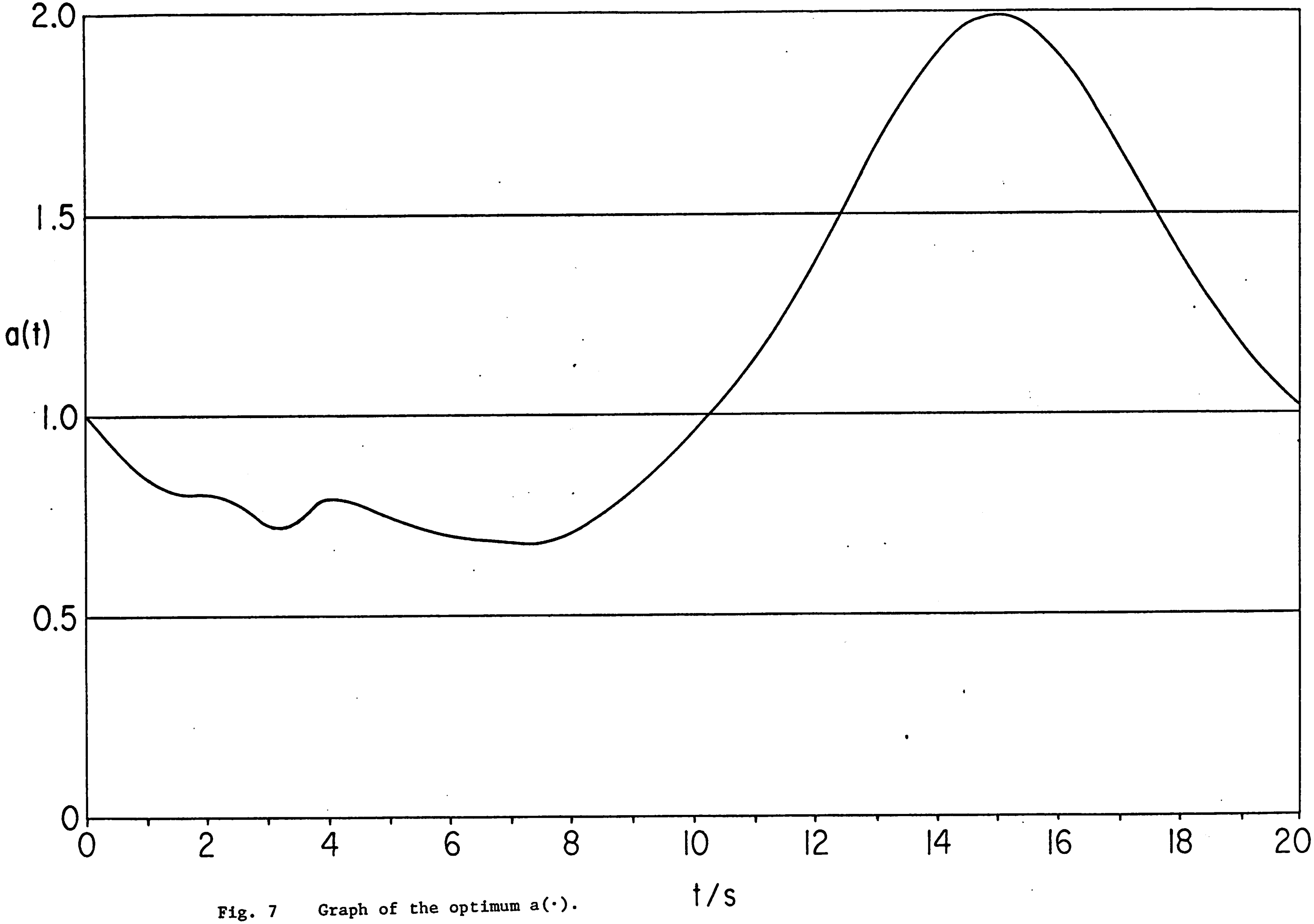

Fig. 7 Graph of the optimum $a(\cdot)$.

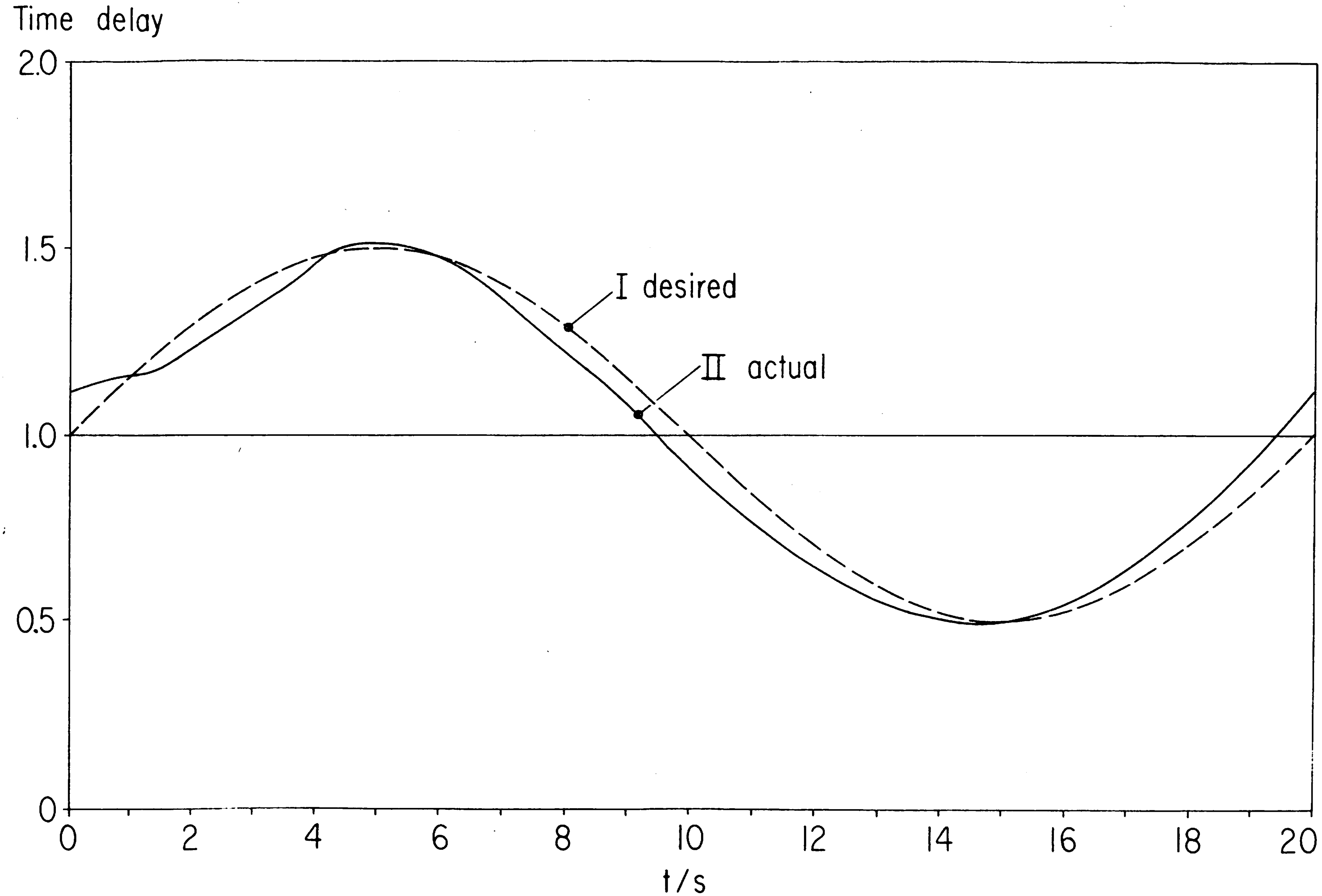

Fig. 8 **Actual and desired time delay of the linear time varying network under consideration.**

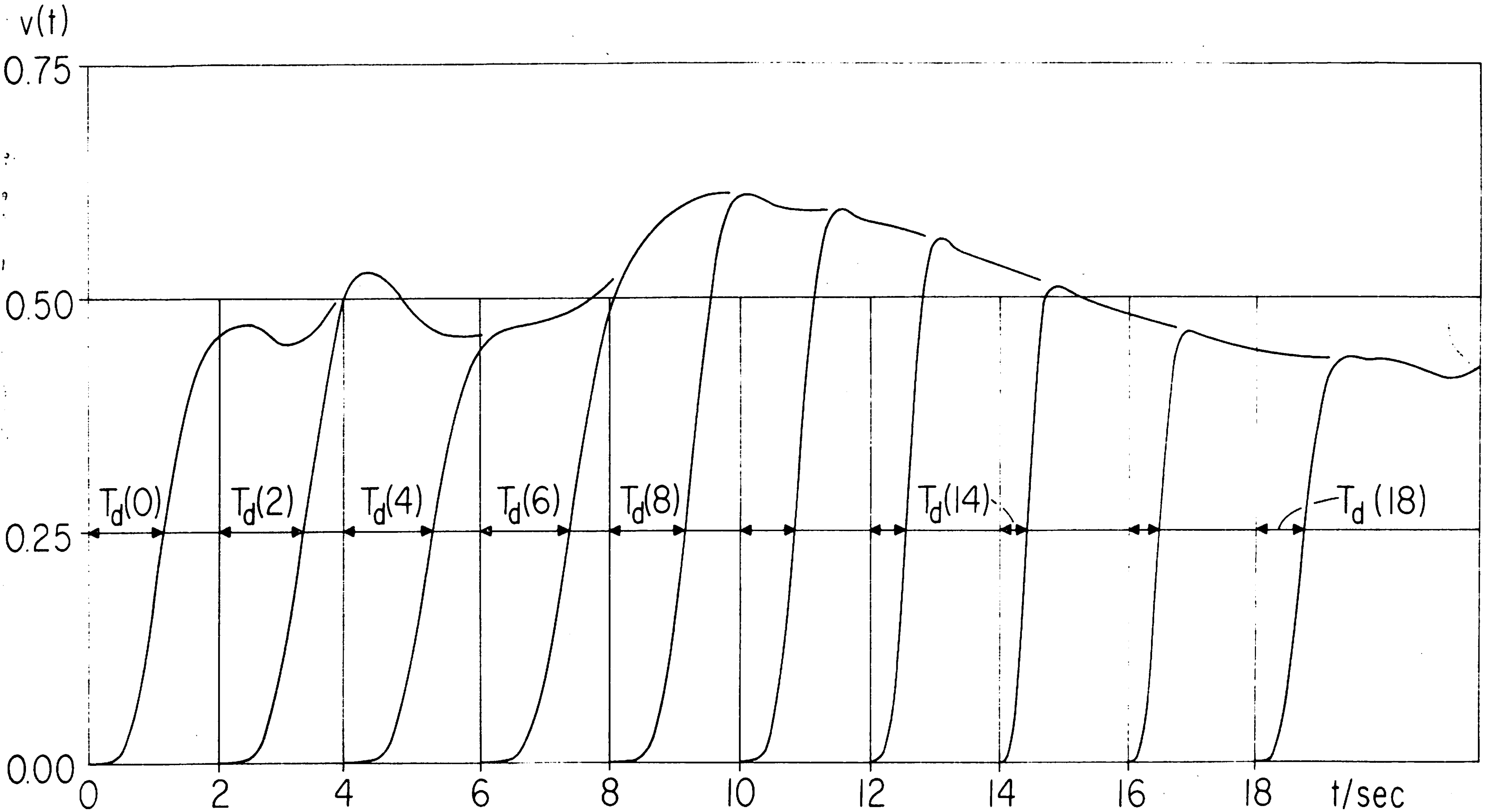

Fig. 9 Responses of the actual time-varying network to unit steps applied at t = 0, 2, ..., 18. The dependence of the delay T on time is quite visible.

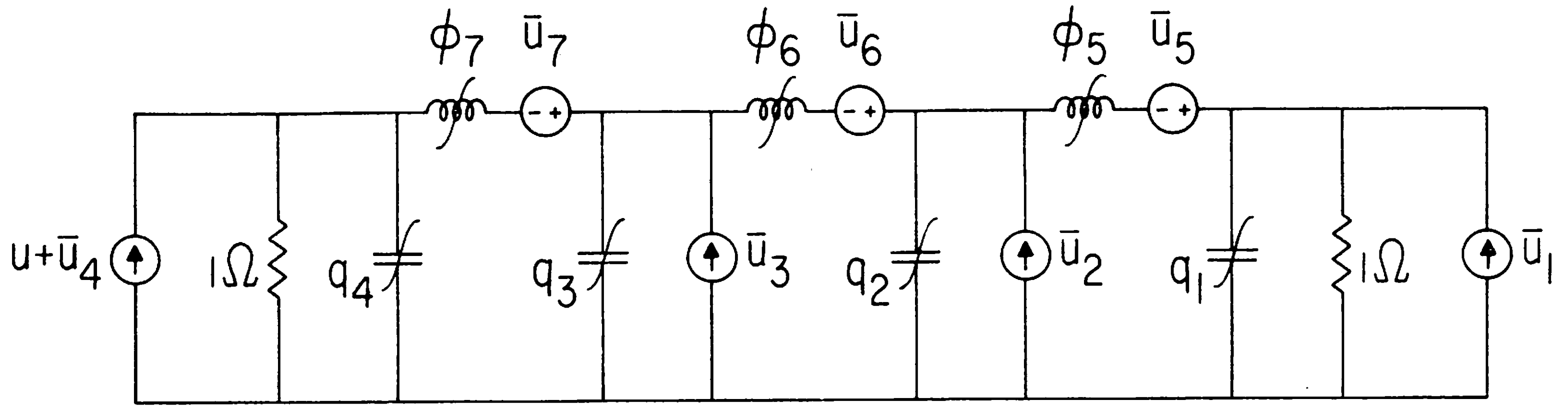

Fig. 10 Nonlinear time-invariant network $\mathcal{N}$. The bias sources are labelled $\bar{u}_1$, $\bar{u}_2$, ..., $\bar{u}_7$.