REPRESENTING MARTINGALES AS STOCHASTIC INTEGRALS

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I. Introduction

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space and $\{\mathcal{A}_t, 0 \le t \le T\}$ an increasing family of sub-σ-algebras. We assume that $\mathcal{A}$ and every $\mathcal{A}_t$ is complete with respect to $\mathcal{P}$. Further, we assume that the family $\{\mathcal{A}_t, 0 \le t \le T\}$ is right-continuous, i.e.,

(1) $$\bigcap_{s>t} \mathcal{A}_s = \mathcal{A}_t.$$

Let $\{X_t, \mathcal{A}_t, 0 \le t \le T\}$ be a sample continuous second order martingale. Mayer [1] has shown that there exists a unique sample continuous increasing process $\{A_t, 0 \le t \le T\}$ such that

(2) $$X_t^2 - A_t = Y_t, \quad 0 \le t \le T$$

is a martingale. For simplicity we shall assume $X_t = 0$.

Let $\mathcal{B}$ denote the σ-algebra of Borel sets in $[0,T]$, and consider the measure

(3) $$\mu(d\omega dt) = \mathcal{P}(d\omega)A(\omega,dt)$$

defined on $\mathcal{A}_T \otimes \mathcal{B}$. If μ is equivalent to the product measure $\mathcal{P}(d\omega)\,dt$ then there exists a Brownian motion $\{W_t, \mathcal{A}_t, 0 \le t \le T\}$ such that

(4) $$X_t = \int_0^t \psi_s \, d\, W_s$$

with probability 1. The integrand $\{\psi_t, 0 \le t \le T\}$ is given by

(5) $$\psi_t(\omega) = \left[\frac{\mathcal{P}(d\omega)\ A(\omega,dt)}{\mathcal{P}(d\omega)\ dt}\right]^{1/2}.$$

Even if μ is merely absolutely continuous with respect to $d\mathcal{P}dt$, a representation of the form (4) still exists, but the adjunction of a Brownian motion may now be necessary [2, p. 71].

We note that μ is absolutely continuous with respect to $d\mathcal{P}dt$ measure if and only if $A_t(\omega)$ is absolutely continuous with respect to the Lebesgue measure for almost all ω. However, $\frac{dA_t(\omega)}{dt}$ is not automatically jointly measurable. Hence, (5) is a more natural choice for $\psi_t(\omega)$, although it may appear to be more complicated than necessary. The condition that almost sure absolute continuity of A_t with respect to the Lebesgue measure implies a representation of the form (4) was discovered by Fisk [3], who made use of an earlier condition of Doob [2, p. 449], namely, the existence of a $\psi_t(\omega)$ such that whenever $t > s$

(6) $$E^{\mathcal{a}_s}(X_t - X_s)^2 = \int_s^t E^{\mathcal{a}_s}\psi_t^2\, dt.$$

Neither Fisk's condition nor Doob's condition is easy to verify. We shall show that a sufficient condition for (4) is that there exist positive constants α and β such that

(7) $$\sup_{0<|t-s|\leq\beta} \frac{E|X_t - X_s|^{2+2\alpha}}{|t-s|^{1+\alpha}} < \infty.$$

In addition, we shall consider a stochastic integral representation which is more general than (4) but almost as useful. The sufficiency condition (7) will be modified accordingly.

II. A Stochastic Integral Representation

If X_t can be represented as in (4) then we have

$$E\, X_t^2 = \int_0^t E\psi_s^2\, ds. \tag{8}$$

Therefore, a simple necessary condition for (4) is that the increasing function

$$F(t) = E\, X_t^2 \tag{9}$$

should be absolutely continuous (with respect to the Lebesgue measure). Using this condition, we can generate at will examples of sample continuous second order martingales which cannot be represented as in (4). Take, for example, a Brownian motion $W_{F(t)}$ with a scale $F(t)$ which is continuous, but singular with respect to the Lebesgue measure. Clearly, $W_{F(t)}$ cannot be represented as in (4). An obvious modification of (4) is a representation of the form

$$X_t = \int_0^t \psi_s\, dW_{F(s)} \tag{10}$$

where the increasing function $F(t)$, $0 \le t \le T$, is defined by (9). If $F(t)$ is absolutely continuous with respect to the Lebesgue measure, then (10) can be rewritten in the form of (4). In a sense (10) is a more natural representation than (4), since the time scale $F(t)$ is now determined by the martingale itself.

Now, define the function

$$F^{-1}(t) = \inf\ \{s:\ F(s) = t\} \tag{11}$$

and define a new process

$$\tilde{X}_t = X_{F^{-1}(t)}, \qquad 0 \le t \le F(t) \tag{12}$$

The process $\{\tilde{X}_t,\ 0 \le t \le F(t)\}$ is a martingale with respect to the family of σ-algebras $\{\tilde{\mathcal{A}}_t = \mathcal{A}_{F^{-1}(t)},\ 0 \le t \le F(t)\}$. Although F^{-1} may be discontinuous, $\{\tilde{X}_t,\ 0 \le t \le F(t)\}$ is sample continuous with probability 1, and $\{\tilde{\mathcal{A}}_t,\ 0 \le t \le F(t)\}$ is right continuous. Hence, $\{\tilde{X}_t, \tilde{\mathcal{A}}_t,\ 0 \le t \le F(t)\}$ is a sample continuous second order martingale with

$$E\tilde{X}_t^2 = F\left(F^{-1}(t)\right) = t. \tag{13}$$

Now set

$$\hat{t} = \inf\ \{s:\ F(s) = F(t)\} = F^{-1}\left(F(t)\right).$$

Then for every $t \in [0,T]$

$$X_t = X_{\hat{t}} = \tilde{X}_{F(t)} \tag{14}$$

with probability 1. It follows that $\{X_t,\ 0 \le t \le T\}$ has a representation given by (4) if and only if $\{\tilde{X}_t,\ 0 \le t \le F(t)\}$ can be represented as

$$\tilde{X}_t = \int_0^t \tilde{\psi}_s\, d\, W_s. \tag{15}$$

III. A Sufficient Condition

Let $\{X_t, \mathcal{A}_t,\ 0 \le t \le T\}$ be a second order sample continuous martingale and let $F(t)$ be defined by (9). Let $\{A_t,\ 0 \le t \le T\}$ be the increasing process defined by the Meyer decomposition (2). By virtue of the correspondence between X_t and $\tilde{X}_t$, it follows that if A_t is almost surely absolutely continuous with respect to $F(t)$, then X_t has a representation given by (10). A more easily verifiable sufficient condition is given as follows:

Theorem. Let $\{X_t, \mathcal{A}_t,\ 0 \le t \le T\}$ be a sample continuous second order martingale and let $F(t)$ be defined by (9). Suppose that for some finite positive constants α, β and κ

$$(16) \qquad \sup_{0<|F(t)-F(s)|\le\beta} \frac{E|X_t - X_s|^{2+2\alpha}}{|F(t) - F(s)|^{1+\alpha}} = \kappa < \infty.$$

Then X_t admits a representation of the form given by (10).

Proof: By virtue of the correspondence given by (12) and (14), we can assume

$$(17) \qquad F(t) = t.$$

Let $\{A_t\}$ be the increasing process defined by the decomposition (2). By the Lebesgue decomposition and Radon-Nikodym theorems we can write

$$(18) \qquad A_t(\omega) = \int_0^t \psi_s(\omega)ds + B_t(\omega).$$

We assume that $\{\psi_s(\omega), s\epsilon\ [0,T], \omega\epsilon\ \Omega\}$ has been chosen to be jointly measurable. This can always be done, if necessary, by defining ψ_s as in (5). In (18) $B_t(\omega)$ is almost surely singular with respect to the Lebesgue measure.

Now, from (2) we have

$$(19) \qquad EA_t = EX_t^2 = t.$$

Therefore, if $E\psi_t = 1$ for almost all t in $[0,T]$, then B_t is almost surely equal to zero and A_t is absolutely continuous (Lebesgue measure) with probability 1. Conversely, if A_t is absolutely continuous with probability 1, then $E\psi_t = 1$ for almost all t. Next, define $\psi_n(\omega,t)$ as follows:

$$(20) \qquad \psi_n(\omega,t) = \frac{2^n}{T}\ A_{\frac{\nu T}{2^n}}(\omega) - A_{\frac{(\nu-1)T}{2^n}}(\omega)$$

$$\text{for } \frac{(\nu-1)T}{2^n} \le t < \frac{\nu T}{2^n} \qquad \nu = 1, 2, \ldots, 2^n$$

By an application of the martingale convergence theorem, Doob [2, p. 346] has shown that for each ω $\psi_n(\omega,t)$ converges to $\frac{dA_t(\omega)}{dt}$ for almost all t. Hence for almost all (ω,t) ($d\mathcal{P}dt$ - measure)

$$\lim_{n\to\infty} \psi_n(\omega,t) = \psi_t(\omega). \tag{21}$$

Since $E\psi_n(\omega,t) = 1$ for all t and n, $E\psi_t = 1$ provided that

$$\sup_n \int_{\psi_n(\omega,t)>N} \psi_n(\omega,t)\mathcal{P}(d\omega) \xrightarrow[N\to\infty]{} 0. \tag{22}$$

By assumption we have

$$\sup_{0<|t-s|\leq\beta} \frac{E|X_t - X_s|^{2+2\alpha}}{|t - s|^{1+\alpha}} = \kappa < \infty \tag{23}$$

for some $\alpha, \beta > 0$. We now proceed to prove that (23) implies (22).

Mayer [1, p. 118] has constructed approximations to A_t of the form

$$\begin{aligned} A_t^h &= \int_0^t \frac{E^{\mathcal{a}_s}(X_{s+h}^2 - X_s^2)}{h} ds \\ &= \int_0^t \frac{E^{\mathcal{a}_s}(X_{s+h} - X_s)^2}{h} ds \end{aligned} \tag{24}$$

and has shown that $E|A_t^h - A_t| \xrightarrow[h\downarrow 0]{} 0$. Now set

$$\Delta_t A = A_{\left(\frac{\nu}{2^n}\right)T} - A_{\left(\frac{\nu-1}{2^n}\right)T}$$

$$\Delta_t A^h = A^h_{\left(\frac{\nu}{2^n}\right)T} - A^h_{\left(\frac{\nu-1}{2^n}\right)T}$$

where ν is the smallest integer greater than $2^n(t/T)$. Then, from (20) we can write

$$\begin{aligned} \psi_n(t) &= \frac{2^n}{T} \Delta_t A \\ &\leq \frac{2^n}{T}\{\Delta_t^h A + |\Delta_t A - \Delta_t^h A|\}. \end{aligned}$$

Now, if B, C and D are positive random variables and

(26) $$B \le C + D$$

then

(27) $$\int_{B \ge N} B\,d\mathcal{P} \le \int_{B \ge N} (C+D)\,d\mathcal{P}$$

$$\le \int_{\max(C,D) \ge N/2} (C+D)\,d\mathcal{P}$$

$$\le 2 \int_{(C \ge D,\, C \ge N/2)} C\,d\mathcal{P} + 2 \int_{(D \ge C,\, D \ge N/2)} D\,d\mathcal{P}$$

$$\le 2 \int_{C \ge N/2} C\,d\mathcal{P} + 2 \int_{D \ge N/2} D\,d\mathcal{P}.$$

Since $E|A_t^h - A_t| \xrightarrow[h \downarrow 0]{} 0$, we can find $h(n,N,t)$ such that $h \le h(n,N,t)$ implies:

(28) $$\frac{2^n}{T} E|\Delta_t A - \Delta_t^h A| \le \frac{1}{N} .$$

On the other hand, from (24) and using the Hölder inequality, we get

(29) $$E\left(\frac{2^n}{T} \Delta_t^h A\right)^{1+\alpha} \le \frac{2^n}{T} \int_{\left(\frac{\nu-1}{2^n}\right)T}^{\left(\frac{\nu}{2^n}\right)T} \frac{E|X_{s+h} - X_s|^{2+2\alpha}}{h^{1+\alpha}}\,ds.$$

Therefore, for all $h \le \beta$

(30) $$E\left(\frac{2^n}{T} \Delta_t^h A\right)^{1+\alpha} \le \kappa$$

where κ is defined by (23). The Markov inequality now immediately implies that for $0 < h \le \beta$

(31) $$\int_{\frac{2^n}{T} \Delta_t^h A \ge N/2} \frac{2^n}{T} \Delta_t^h A\,d\mathcal{P} \le \frac{2^\alpha}{N^\alpha} \kappa$$

Combining (25), (27), (28), and (31) and choosing h sufficiently small so that both (28) and (31) are satisfied, we get

$$\int_{\psi_n(t)\geq N} \psi_n(t)\,d\mathcal{P} \leq \frac{2^\alpha}{N^\alpha}\kappa + \frac{1}{N} \tag{32}$$

which verifies (22) and completes the proof for the theorem.

IV. An Example

Suppose that $F_1(t)$ and $F_2(t)$ are two continuous and increasing functions on $[0,1]$ such that as Borel measures they are mutually singular. Let W_t and V_t be two independent standard separable Brownian motions. Define

$$X_t = aW_{F_1(t)} + bW_{F_2(t)} \tag{33}$$

where a and b are two bounded random variables independent of the Brownian motions but not necessarily independent of each other. If we denote by $\mathcal{A}_t$ the completed σ-algebra generated by $\{W_{F_1(s)}, W_{F_2(s)}, 0 \leq s \leq t\}$, then $\{X_t, \mathcal{A}_t, 0 \leq t \leq 1\}$ is a sample continuous second martingale with

$$EX_t^2 = F(t) = (Ea^2)F_1(t) + (Eb^2)F_2(t).$$

It is by no means obvious that X_t admits a representation of the form

$$X_t = \int_0^t \psi_s \, dF(s).$$

However, by direct computation we find

$$E(X_t - X_s)^4 = 3\{(Ea^4)[F_1(t) - F_1(s)]^2 + (Eb^4)[F_2(t) - F_2(s)]^2 + 2Ea^2b^2[F_1(t) - F_1(s)][F_2(t) - F_2(s)]\}.$$

By the Schwarz inequality we get

$$E(X_t - X_s)^4 \leq 3[F(t) - F(s)]^2$$

so that

$$\sup \frac{E(X_t - X_s)^4}{[F(t) - F(s)]^2} \leq 3$$

and condition (16) is satisfied. Hence, X_t can indeed be represented as in (10).

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