A FUNCTIONAL ANALYSIS APPROACH TO L_∞ STABILITY AND ITS APPLICATIONS TO SYSTEMS WITH HYSTERESIS

BY

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Introduction

The functional analysis approach to the stability of nonlinear feedback systems has been developed recently. Papers by Desoer and Lee [1], Sandberg [3] and Zames [4] have pointed out how gain and positivity criteria yielded, in addition to the classical stability results, information on the relations between inputs and outputs and weaker requirements on the linear part of the system which can contain delays for instance. The system used in stability theory has usually been modelled by the system S.1 (Fig. 1). Zames [4] considered such a system and imposed the condition that the nonlinear element be such that $Nx = 0$ whenever $x = 0$. Such a restriction rules out an important class of nonlinearities such as backlash and hysteresis. These nonlinearities presenting memory and possibly having time varying characteristics are so complex that they cannot be modeled well. Usually the modeling is done for a class of input function (Chua and Stromsmoe [5]), or by replacing the hysteresis by a backlash and a nonlinear gain (Kodama and Shirakawa [6]). The Lyapunov theory allows one to define more loosely the nonlinear element in exchange for a stronger condition on the linear element and of a zero, or constant input (Weissenberger [7], [8], Walker and McClamroch [9], Yakubovich [10]). Since the properties of hysteresis type nonlinearities imply errors which do not go to zero it is often required of the linear element to have an integrator in order for a L_2 stability criterion to be used. However, the stability condition could very well be of a bounded input-bounded output type, i.e. L_∞ stability, with in addition the absence of sustained

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oscillations. Such a condition could be obtained through the L_1 stability of the derivatives of the functions appearing in the system.

In this paper the main condition on the nonlinearity will be that its slopes be bounded. The requirement on the linear element will be that it satisfies either a circle condition or a Popov type condition.

In the first section the use of a class of function

$$\{x(\cdot) | \quad \sigma > 0 \ , \ \ e^{\sigma t} \ x(t) \in L_2[0,\infty)\} \subset L_1[0,\infty) \cap L_2[0,\infty)$$

will lead to a L_1 stability criterion. In the second section this criterion will be used to find a L_∞ criterion. In the third section transformations will lead to a circle criterion and a Popov type criterion. Examples and experiments will be presented in the fourth section. The fifth section will consist of a short discussion on the problems of modelling hysteresis.

Definitions and Notations

The following definitions will be used through the paper. All the functions appearing in this paper will map the line $[0,\infty)$ on the real line

i.e. $X = \{x(\cdot) | x : [0,\infty) \to (-\infty,\infty)\}$ is the linear space of functions which will be considered.

Let $x \in X$, given $T > 0$, the truncation x_T of x at T is defined as a function belonging to X such that

$$\begin{cases} x_T(t) = x(t) & \forall t \ni 0 \leq t \leq T \\ x_T(t) = 0 & \forall t > T \end{cases}$$

Let $Y \subset X$ be a normed linear subspace of X and let $\|x\|_y$ be the norm in Y of $x \in Y$.

The extension Y_e of Y is defined as $Y_e = \{x(\cdot) | x \in X$ and $x_T \in Y \quad \forall$ finite $T > 0\}$.

The extended norm $\|x\|_{ye}$ of $x \in Y_e$ is defined as

$$\begin{cases} \|x\|_{ye} = \|x\|_y \text{ if } x \in Y \\ \|x\|_{ye} = \infty \text{ if } x \in Y_e \quad x \notin Y \end{cases}$$

Remark $\|x_T\|_y$ is a monotone nondecreasing function of T hence if Y is complete and if there exists a constant $A > 0$ such that

$$\forall T > 0 \; \|x_T\|_y \leq A, \text{ then } x \in Y \text{ and } \|x\|_y \leq A$$

Examples of normed spaces used in this paper:

$$L_1[0,\infty) = \{x(\cdot)|x \in X \text{ and } \int_0^\infty |x(t)|dt < \infty\}$$

$$L_2[0,\infty) = \{x(\cdot)|x \in X \text{ and } \int_0^\infty x^2(t)\, dt < \infty\}$$

$$L_\infty[0,\infty) = \{x(\cdot)|x \in X \text{ and } \sup_{t\geq 0} |x(t)| < \infty\}$$

given $\sigma > 0$

$$N_{2\sigma}[0,\infty) = \{x(\cdot)|x \in X \text{ and } \int_0^\infty (e^{\sigma t}\, x(t))^2\, dt < \infty\}$$

given $\sigma > 0$; given I a countable set, given $\tau_i \geq 0$

$$N_{3\sigma}[0,\infty) = \{x(\cdot)|x \in X \text{ and } x(t) = x_a(t) + \sum_{i \in I} x_i\, \delta(t - \tau_i)$$

$$\text{with } \int_0^\infty (e^{\sigma t}\, x_a(t))^2 dt + \sum_{i \in I} e^{\sigma \tau_i} |x_i| < \infty\}$$

Remark: given $\sigma_1 > 0$ $\forall \sigma > 0$ $\sigma < \sigma_1$ $N_{2\sigma_1}[0,\infty) \subset N_{2\sigma}[0,\infty)$

Two other spaces on which no norm will be defined are going to be used

$$N_2[0,\infty) = \bigcup_{\sigma > 0} N_{2\sigma}[0,\infty) = \{x(\cdot)\,|\,x \in X \text{ and } \exists\, \sigma > 0 \text{ such that } e^{\sigma t}x(t) \in L_2[0,\infty)\}$$

$$N_3[0,\infty) = \bigcup_{\sigma > 0} N_{3\sigma}[0,\infty)$$

by extension of the notion of extended spaces $N_{2ex}[0,\infty)$ and $N_{3ex}[0,\infty)$ are introduced as

$$N_{2ex}[0,\infty) \doteq \{x(\cdot)\,|\,x \in X \text{ and } x_T \in N_2[0,\infty) \quad \forall \text{ finite } T > 0\}$$

$$N_{3ex}[0,\infty) = \{x(\cdot)\,|\,x \in X \text{ and } x_T \in N_3[0,\infty) \quad \forall \text{ finite } T > 0\}$$

Even though it will not be possible to use directly the space $N_2[0,\infty)$, because of the lack of a convenient norm, the ties of $N_2[0,\infty)$ with $L_1[0,\infty)$ and $L_2[0,\infty)$ are very important. The two following lemmas are going to show these ties and their consequences.

<u>Lemma 0.1</u>

The space $N_2[0,\infty)$ has the following properties

a) $N_2[0,\infty) \subset L_1[0,\infty) \cap L_2[0,\infty)$

b) $N_{2ex}[0,\infty) = L_{2e}[0,\infty)$

c) $\forall x \in N_2[0,\infty)\ \exists\ \sigma > 0$ such that

$$(0.1) \qquad \|x\|_{L_1} \leq \frac{1}{\sqrt{2\sigma}} \|x\|_{N_{2\sigma}}$$

Proof:

a) assume $x \in N_2[0,\infty)$

then $\exists \ \sigma > 0 \quad e^{\sigma t} x(t) \in L_2[0,\infty)$

however

$$e^{\sigma t} \geq 1 \qquad \forall t \geq 0$$

then $\quad (x(t))^2 \leq (e^{\sigma t} x(t))^2 \qquad \forall t \geq 0$

and $\quad x \in L_2[0,\infty)$ with $\|x\|_{L_2} \leq \|e^{\sigma t} x(t)\|_{L_2}$

hence $\quad N_2[0,\infty) \subset L_2[0,\infty)$

$$|x(t)| = e^{-\sigma t}(e^{\sigma t} |x(t)|)$$

but $\quad e^{-\sigma t} \in L_2[0,\infty)$ as a matter of fact $\|e^{-\sigma t}\|_{L_2} = \frac{1}{\sqrt{2\sigma}}$

and $\quad (e^{\sigma t} |x(t)|) \in L_2[0,\infty)$ by hypothesis

then $\quad |x(t)|$ being the product of two L_2 functions is an L_1 function

in addition

$$\int_0^\infty |x(t)| dt \leq \|e^{-\sigma t}\|_{L_2} \cdot \|e^{\sigma t} x(t)\|_{L_2} \qquad \text{by the Schwartz inequality}$$

then $\quad \|x\|_{L_1} \leq \frac{1}{\sqrt{2\sigma}} \|x\|_{N_{2\sigma}}$ which proves (c)

and $\quad N_2[0,\infty) \subset L_1[0,\infty)$

b) trivially (a) implies $N_{2ex}[0,\infty) \subset L_{2e}[0,\infty)$

assume $x \in L_{2e}[0,\infty)$

then $x_T \in L_2[0,\infty) \quad \forall$ finite $T > 0$

but $e_T^{\sigma t} \in L_2[0,\infty) \quad \forall$ finite $T > 0$

using the Schwartz inequality

$$\int^T (e^{\sigma t}\ x(t))^2\ dt \leq e^{2\sigma T} \int_0^T (x(t))^2\ dt < \infty$$

hence $\|(x(t)e^{\sigma t})_T\|_{L_2} \leq e^{\sigma T}\ \|x_T\|_{L_2}$

and $x_T \in N_{2\sigma}[0,\infty)$ hence $x_T \in N_2[0,\infty)$

and $x \in N_{2ex}[0,\infty)$

Lemma 0.2

Let x be a function belonging to X and which has a derivative $\dot{x} \in X$.

a) if $\dot{x} \in N_2[0,\infty)$ then $x \in L_\infty[0,\infty)$ and there exists a constant A such that $x(t) \underset{t\to\infty}{\to} A$

b) if $\dot{x} \in N_3[0,\infty)$ then $x \in L_\infty[0,\infty)$ and there exists a constant B such that $x(t) \underset{t\to\infty}{\to} B$

Proof

a) If $\dot{x} \in N_2[0,\infty)$ then by Lemma 0.1 $\dot{x} \in L_1[0,\infty)$ hence $x \in L_\infty[0,\infty)$

also $\forall\ \varepsilon > 0\ \exists\ T_\varepsilon$ such that $\forall\ T_2 > T_1 > T_\varepsilon$

$$|x(T_2) - x(T_1)| = \left| \int_{T_1}^{T_2} \dot{x}(t)dt \right| \leq \int_{T_1}^{T_2} |\dot{x}(t)|dt < \varepsilon$$

hence $x(t)$ converges to a finite constant $A \leq \int_0^\infty |\dot{x}(t)|dt$ when $t \to \infty$

b) $\dot{x}(t) = x_a'(t) + \sum_{i \in I} x_i' \ \delta(t - \tau_i)$

with $x_a' \in N_2[0,\infty)$ and $\sum_{i \in I} |x_i'| < \infty$ by the definition of $N_3[0,\infty)$

then from part a) $\exists$ a finite constant $A \ni x_a'(t) \underset{t\to\infty}{\to} A$ and

$$\int_0^\infty \sum_{i \in I} x_i' \ \delta(t-\tau_i)dt = \sum_{i \in I} x_i' \int_0^\infty \delta(t-\tau_i)dt = \sum_{i \in I} x_i' \ 1(t-\tau_i)$$

where $1(t)$ is such that $\begin{cases} 1(t) = 0 & \forall\ t < 0 \\ 1(t) = 1 & \forall\ t \geq 0 \end{cases}$

since $\sum_{i \in I} |x_i'| < \infty$ there exists a finite constant A' such that $\sum_{i \in I} x_i' = A'$

then $x(t) \underset{t\to\infty}{\to} B = A + A'$

Some more definitions will be necessary.

A <u>relation</u> N with domain and range in X is a subset of the product space X x X.

If (x,y) is a pair belonging to the relation N y will be said to be the <u>image</u> of x under N.

A mapping M of X into X is a relation with domain and range in X such that no two members have the same first coordinate.
i.e. if $(x,y) \in M$ and $(x,z) \in M$ then $y = z$.

Let N be a relation with domain and range in a linear normed space Y.

The <u>gain</u> $g_y(N)$ of the relation N with respect to the norm on Y is defined as:

$$(0.2) \qquad g_y(N) = \sup \frac{\|z\|_y}{\|x\|_y}$$

where the supremum is taken over all the images z of x under N and over all x, $x \neq 0$ $x \in Do(N)$.

The system S.1 (Fig. 1) which is going to be studied is of a classical type. It has a linear time invariant element H and another element which behavior is only known through an input-output relationship: N.

This element N can be multivalued, time varying and nonlinear. A great number of practical systems can be modelled in this way.

The equations defining the system S.1 are:

$$\begin{cases} e_1(t) = u_1(t) - y_2(t) & \text{C.1.a} \\ e_2(t) = u_2(t) + y_1(t) & \text{C.1.b} \\ (e_1, y_1) \in N & \text{C.1.c} \\ y_2(t) = (He_2)(t) & \text{C.1.d} \end{cases}$$

where N is a relation with domain and range in X. And H is a linear mapping with domain and range in X.

The following assumption on the linear element H will be used through the paper.

<u>Assumption 0.1</u>

Let H be a linear mapping of X into X such that there exists $h \in X$, $\sigma_0 > 0$ and a countable set I such that

$$(0.3) \qquad \begin{cases} h(t) = 0 \quad \forall\, t < 0 \\ h(t) = h_a(t) + \sum\limits_{i \in I} h_i\, \delta(t-t_i) \qquad \forall\, t \geq 0 \end{cases}$$

and $\quad e^{\sigma_0 t}\, h(t) \in L_1[0,\infty)^*$

defining H in the following way

$\forall\, x \in Do(H)$

$$(0.4) \qquad (Hx)(t) = (h*x)(t) = \int_0^t h(t-\tau)\, x(\tau) d\tau$$

The immediate consequences of assumption 0.1 are given in the

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* This means $e^{\sigma_0 t}\, h_a(t) \in L_1[0,\infty)$ and $\sum\limits_{i \in I} e^{\sigma_0 t_i} |h_i| < \infty$

following lemma.

Lemma 0.3

Let H be a linear mapping satisfying assumption 0.1 then

a) H maps $L_2[0,\infty)$ into $L_2[0,\infty)$

b) H maps $L_{2e}[0,\infty)$ into $L_{2e}[0,\infty)$

c) Let $g(t) = e^{\sigma t}$ and let $(g\cdot x)(t) = g(t)\cdot x(t)$ then for $\sigma \leq \sigma_0$ (σ_0 defined in assumption 0.1) and for all $x \in L_{2e}[0,\infty)$

(0.5) $\quad [g\cdot(h*x)](t) = [(g\cdot h)*(g\cdot x)](t)$

d) H maps $N_{2\sigma}[0,\infty)$ into $N_{2\sigma}[0,\infty)$ for $\sigma \leq \sigma_0$ and H maps $N_2[0,\infty)$ into $N_2[0,\infty)$.

Proof

It can be noticed at once that (a) implies (b) trivially

a) since $e^{\sigma_0 t}\ h(t) \in L_1[0,\infty)$ with $\sigma_0 > 0 \quad t \geq 0$

$h(t) \in L_1[0,\infty)$ it is a well known result that the convolution of a L_1 function with a L_2 function is a L_2 function.

c) $$[g\cdot(h*x)](t) = e^{\sigma t}\int_0^t h(t-\tau)x(\tau)d\tau = \int_0^t h(t-\tau)e^{\sigma(t-\tau)}e^{\sigma\tau}\ x(\tau)d\tau$$

$$= [(g\cdot h)*(g\cdot x)](t)$$

d) Let $x \in N_{2\sigma}[0,\infty)$ with $\sigma \leq \sigma_0$ and $y = h*x$

then $\quad e^{\sigma t} y(t) = e^{\sigma t}(h*x)(t)$

using part (c)

$$e^{\sigma t} y(t) = (e^{\sigma t}h)*(e^{\sigma t}x)$$

since $\quad \sigma \leq \sigma_0 \qquad e^{\sigma t}h(t) \in L_1[0,\infty)$

by hypothesis

$$e^{\sigma t} x(t) \in L_2[0,\infty)$$

then $\quad e^{\sigma t} y(t) \in L_2[0,\infty)$ and $y(t) \in N_{2\sigma}[0,\infty)$

let $\quad x \in N_2[0,\infty)$ then there exists a $\sigma \leq \sigma_0$ such that $x \in N_{2\sigma}[0,\infty)$ using the preceeding reasoning $y \in N_{2\sigma}[0,\infty)$ hence $y \in N_2[0,\infty)$.

<u>Lemma 0.4</u>

Let H be a linear mapping satisfying assumption 0.1

If $\quad \sup |\hat{h}(j\omega)| < R \quad \forall\, \omega \in (-\infty,\infty)$

then $\quad \exists\, \sigma, \; 0 < \sigma \leq \sigma_0$ such that

$$|\hat{h}(\sigma + j\omega)| \leq R \quad \forall\, \omega \in (-\infty,\infty)$$

Proof

Since $h(t) = h_a(t) + \sum_{i \in I} h_i\, \delta(t - t_i)$

Let Δ_a and Δ_s be defined as follows:

$$\Delta_a = |\hat{h}_a(j\omega) - \hat{h}_a(\sigma + j\omega)|$$

$$\Delta_s = |\hat{h}_s(j\omega) - \hat{h}_s(\sigma + j\omega)|$$

then $\quad \Delta_a = \left| \int_0^\infty e^{j\omega t}(1 - e^{\sigma t})\, h_a(t)dt \right|$

$$\Delta_a \leq \left| \int_0^T e^{j\omega t}(1 - e^{\sigma t})\, h_a(t)dt \right| + \int_T^\infty |e^{j\omega t}(1 - e^{\sigma t})\, h_a(t)|dt$$

Note that for $0 \le t \le T$

$$|1 - e^{\sigma t}| \le \sigma T e^{\sigma t}$$

and that $\forall\, t \ge 0$

$$|1 - e^{\sigma t}| \le 2e^{\sigma t}$$

the inequality will then become

$$\Delta_a \le \sigma T \int_0^\infty e^{\sigma t}|h_a(t)|dt + 2\int_T^\infty e^{\sigma t}|h_a(t)|dt$$

given $\varepsilon > 0$ pick $T = T(\varepsilon)$ in such a way that

$$\int_{T_\varepsilon}^\infty e^{\sigma t}|h(t)|dt < \frac{\varepsilon}{8}$$

let $\qquad \delta_1(\varepsilon) = \dfrac{\varepsilon}{4T\|e^{\sigma_0 t}\, h_a(t)\|_{L_1}}$

then if $\sigma \le \delta_1(\varepsilon) \qquad \Delta_a \le \frac{\varepsilon}{2}$

$$\Delta_s = \Big|\sum_{i \in I} h_i(1 - e^{\sigma t_i})e^{j\omega t_i}\Big|$$

$$\Delta_s \le \sum_{i \in I_T} |h_i|\cdot|1 - e^{\sigma t_i}| + \sum_{i \in I - I_T} |h_i|\,|1 - e^{\sigma t_i}|$$

$$|1 - e^{\sigma t_i}| \le 2e^{\sigma t_i} \qquad \forall\, i \in I$$

$$|1 - e^{\sigma t_i}| \le \sigma T\, e^{\sigma t_i} \qquad \forall\, i \in I_T \text{ since } t_i < T \qquad \forall\, i \in I_T$$

$$\Delta_s \leq \sigma T \sum_{i \in I} |h_i| e^{\sigma t_i} + 2 \sum_{i \in I - I_T} |h_i| e^{\sigma t_i}$$

pick then $T' = T'(\varepsilon)$ in such a way that

$$\sum_{i \in I - I_{T'}} |h_i| e^{\sigma t_i} < \frac{\varepsilon}{8}$$

let $$\delta_2(\varepsilon) = \frac{\varepsilon}{4T \sum_{i \in I} |h_i| e^{\sigma_0 t_i}}$$

then if $\sigma \leq \delta_2(\varepsilon)$ $\quad \Delta_s \leq \frac{\varepsilon}{2}$

and if $\sigma \leq \min(\delta_1(\varepsilon), \delta_2(\varepsilon))$

$$|\hat{h}(j\omega) - \hat{h}(\sigma + j\omega)| \leq \varepsilon \qquad \forall\, \omega \in (-\infty,\infty)$$

letting $\sup_{\omega} |\hat{h}(j\omega)| = R_0$

pick $\frac{R - R_0}{2}$

<u>Assumption 0.2</u>

Let H satisfy assumption 0.1 and in addition $e^{\sigma_0 t} h_a(t) \in L_2[0,\infty)$

I. N_2 stability

This section is going to give the conditions on the elements of the system S.1 insuring N_2 and N_3 boundedness.

A "small gain theorem" using the close relationship between the space $N_2[0,\infty)$ and $L_2[0,\infty)$ will give these conditions. The fact that $N_2[0,\infty)$ is a linear subspace of $L_1[0,\infty)$ will be used in the next section when the results of this section are going to be applied to the behavior of the derivatives of the error and output functions.

A sector type condition is going to be imposed on the relation N.

Assumption 1.1

Let N be a relation with domain and range in X.

There exists a constant $k > 0$ such that

$$\text{(1.1)}\quad \left.\begin{array}{l} |y(t)| < k|x(t)| \\ y = 0 \text{ whenever } x = 0 \end{array}\right\} \begin{array}{l} \forall\, t \geq 0 \\ \forall \text{ images } y \text{ of } x \text{ under } N \\ \forall\, x \in L_{2e}[0,\infty) \cap Do(N) \end{array}$$

Lemma 1.1

Let N be a relation satisfying assumption 1.1 then:

a) If the domain of N is restricted to $Do(N) \cap L_2[0,\infty)$ then its range is in $L_2[0,\infty)$ and the L_2 gain of N $g_{L_2}(N)$ satisfies the relation:

$$g_{L_2}(N) < k$$

b) If the domain of N is restricted to $Do(N) \cap L_{2e}[0,\infty)$ its range is in $L_{2e}[0,\infty)$.

c) If the domain of N is restricted to $Do(N) \cap N_{2\sigma}[0,\infty)$ with $\sigma > 0$ then its range is in $N_{2\sigma}[0,\infty)$ and the $N_{2\sigma}$ gain of N : $g_{N_{2\sigma}}(N)$ satisfies the relation:

$$g_{N_{2\sigma}}(N) < k$$

d) If the domain of N is restricted to $Do(N) \cap N_2[0,\infty)$ then its range is in $N_2[0,\infty)$.

Proof

It can be seen at once that (a) implies (b) and that (c) implies

(d) trivially.

a) assume $x \in L_2[0,\infty) \cap Do(N)$ i.e. $\int_0^\infty x^2(t)dt < \infty$ then equation (1.1)

implies $\int_0^T y^2(t)dt < k^2 \int_0^T x^2(t)dt \leq k^2 \|x\|_{L_2}^2 < \infty$

since the right hand side does not depend on T

$$\int_0^\infty y^2(t)dt \text{ exists and is finite.}$$

hence

(1.2) $\|y\|_{L_2} < k \|x\|_{L_2}$

c) assume $x \in N_{2\sigma}[0,\infty)$ then $e^{\sigma t}\, x(t) \in L_2[0,\infty)$

since $\sigma > 0$

$$e^{\sigma t} \geq 1 \qquad \forall\ t \geq 0$$

and

(1.3) $e^{\sigma t}|y(t)| < k\ e^{\sigma t}|x(t)|$

and it suffices to apply part (a) to (1.3).

In the same way

$$g_{N_{2\sigma}}(N) < k.$$

So far the $N_{2\sigma}$ gain of the relation N has been obtained. The $N_{2\sigma}$ gain of the linear element must now be obtained. Since by far the most convenient way to deal with a linear element is to study its properties in the frequency domain the link between the space $N_2\ [0,\infty)$ and $L_2[0,\infty)$ is going to prove specially helpful in the following lemma.

Lemma 1.2

The $N_{2\sigma}$ gain of the linear element $g_{N_{2\sigma}}(H)$ is bounded in the following way for $\sigma > 0 \quad \sigma \leq \sigma_0$

(1.4) $$g_{N_{2\sigma}}(H) \leq \sup_{\omega \in (-\infty,\infty)} |h(\sigma + j\omega)|$$

Proof

Assume that $x \in N_{2\sigma}[0,\infty)$ with $\sigma \leq \sigma_0 \quad \sigma > 0$

using equation (0.5):

$$[g\cdot(h*x)](t) = [(g\cdot h)*(g\cdot x)](t) \qquad \text{with } g(t) = e^{\sigma t}$$

then

(1.5) $$\|e^{\sigma t}\, y(t)\|_{L_2} = \|e^{\sigma t}(h*x)(t)\|_{L_2} = \|(e^{\sigma t}\, h(t))*(e^{\sigma t}\, x(t))\|_{L_2}$$

using Parseval formula

$$\|e^{\sigma t}\, y(t)\|_{L_2} = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{h}(\sigma + j\omega)|^2\, |\hat{x}(\sigma + j\omega)|^2\, d\omega\right]^{1/2}$$

hence

(1.6) $$\|e^{\sigma t}\, y(t)\|_{L_2} \leq \left[\left(\sup_{\omega \in (-\infty,\infty)} |\hat{h}(\sigma + j\omega)|\right)^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{x}(\sigma + j\omega)|^2\, d\omega\right]^{1/2}$$

$$= \left(\sup_{\omega \in (-\infty,\infty)} |\hat{h}(\sigma + j\omega)|\right) \|e^{\sigma t}\, x(t)\|_{L_2}$$

or $$\|y\|_{N_{2\sigma}} \leq \sup_{\omega \in (-\infty,\infty)} |\hat{h}(\sigma + j\omega)| \|x\|_{N_{2\sigma}}$$

and $$g_{N_{2\sigma}}(H) \leq \sup_{\omega \in (-\infty,\infty)} |\hat{h}(\sigma + j\omega)|$$

Now the theorem on N_2 boundedness can be stated.

<u>Theorem 1.3</u>

Let the system S.1 be such that:

(i) There exists a constant $k > 0$ with which the element N satisfies assumption 1.1.

(ii) The linear element H satisfies assumption 0.1 and in addition let the Fourier transform $\hat{h}(j\omega)$ of $h(t)$ satisfy the following inequality.

$$\exists\ R > 0 \text{ such that } \sup_{\omega} |h(j\omega)| < R$$

(iii) The functions appearing in the system belong to $L_{2e}[0,\infty)$.

Then $kR < 1$ and $u_1, u_2 \in N_2[0,\infty)$ implies that $e_1, e_2, y_1, y_2 \in N_2[0,\infty)$.

Proof:

Since by lemma 0.1 $N_{2ex}[0,\infty) = L_{2e}[0,\infty)$ all the functions appearing will belong to $N_{2ex}[0,\infty)$. It is then legitimate to use the $N_{2\sigma}$ norm of the truncated functions for $\sigma > 0$.

since $e^{\sigma_0 t} h(t) \in L_1[0,\infty)$

$$\forall\ \sigma \leq \sigma_0\ \exists\ R_\sigma \text{ such that } |\hat{h}(\sigma + j\omega)| \leq R_\sigma \qquad \forall\ \omega \in (-\infty,\infty)$$

using lemma 0.4

$$|\hat{h}(j\omega)| < R \qquad \forall\, \omega \in (-\infty,\infty) \quad \text{implies that}$$

$$\exists\, \sigma_2 > 0 \ni |\hat{h}(\sigma_2 + j\omega)| \leq R \qquad \forall\, \omega \in (-\infty,\infty)$$

then

(1.4) $\quad \forall\, \sigma > 0,\ \sigma < \sigma_2 \qquad g_{N_{2\sigma}}(H) \leq R$

by lemma 1.2.

Since N satisfies assumption 1.1 lemma 1.1 implies that

(1.5) $\quad g_{N_{2\sigma}}(N) < k$

The equations of the system C.1.a to C.1.d imply:

(1.6) $\quad \|e_{1T}\|_{N_{2\sigma}} = \|u_{1T} - y_{2T}\|_{N_{2\sigma}} \leq \|u_{1T}\|_{N_{2\sigma}} + \|y_{2T}\|_{N_{2\sigma}}$

in the same way

(1.7) $\quad \|e_{2T}\|_{N_{2\sigma}} \leq \|u_{2T}\|_{N_{2\sigma}} + \|y_{1T}\|_{N_{2\sigma}}$

the definition of the N_2 gain implies

(1.8) $\quad \|y_{1T}\|_{N_{2\sigma}} \leq g_{N_{2\sigma}}(N)\|e_{1T}\|_{N_{2\sigma}}$

(1.9) $\quad \|y_{2T}\|_{N_{2\sigma}} \leq g_{N_{2\sigma}}(H)\|e_{2T}\|_{N_{2\sigma}}$

replacing $\|y_{1T}\|_{N_{2\sigma}}$ and $\|y_{2T}\|_{N_{2\sigma}}$ by their bound in (1.6) and (1.7) the following inequalities are obtained

$$(1.10)\quad \|e_{1T}\|_{N_{2\sigma}} \leq \|u_{1T}\|_{N_{2\sigma}} + g_{N_{2\sigma}}(H)\|e_{2T}\|_{N_{2\sigma}}$$

$$(1.11)\quad \|e_{2T}\|_{N_{2\sigma}} \leq \|u_{2T}\|_{N_{2\sigma}} + g_{N_{2\sigma}}(N)\|e_{1T}\|_{N_{2\sigma}}$$

replacing $\|e_{2T}\|_{N_{2\sigma}}$ by its bound in (1.10) leads to

$$(1.12)\quad \|e_{1T}\|_{N_{2\sigma}}(1-g_{N_{2\sigma}}(N)\, g_{N_{2\sigma}}(H)) \leq \|u_{1T}\|_{N_{2\sigma}} + g_{N_{2\sigma}}(H)\|u_{2T}\|_{N_{2\sigma}}$$

let u_1 and $u_2 \in N_2[0,\infty)$

then $\exists\ \sigma'$ and $\sigma'' > 0$ such that $u_1 \in N_{2\sigma'}[0,\infty)$ and $u_2 \in N_{2\sigma''}[0,\infty)$

take $\sigma > 0$ $\sigma \leq \min(\sigma_2,\sigma',\sigma'')$

then u_1 and $u_2 \in N_{2\sigma}[0,\infty)$ and from (1.4), (1.5) and (1.12)

$$(1.13)\quad \|e_{1T}\|_{N_{2\sigma}}(1-kR) \leq \|u_1\|_{N_{2\sigma}} + R\|u_2\|_{N_{2\sigma}}$$

since $(1-kR) > 0$ by assumption

$$\|e_{1T}\|_{N_{2\sigma}} \leq (1-kR)^{-1}[\|u_1\|_{N_{2\sigma}} + R\|u_2\|_{N_{2\sigma}}]$$

and $e_1 \in N_{2\sigma}[0,\infty)$ hence $e_1 \in N_2[0,\infty)$

trivially this implies that y_1, e_2 and y_2 also belong to $N_2[0,\infty)$.

It is possible to enlarge the class of functions considered to the space $N_3[0,\infty)$. However it is necessary to restate the assumption on N.

Assumption 1.2

Let N be a relation with domain and range in X.

(i) $\forall\, x \in N_{3ex}[0,\infty) \cap Do(N)$ and $x(t) = x_a(t) + \sum_{i \in I} x_i\, \delta(t-\tau_i)$

the images of x under N are of the type $y(t) = y_a(t) \sum_{i \in I} y_i\, \delta(t-\tau_i)$

where y_i can be zero with the understanding that $y_i = 0$ implies that no δ function occurs at time τ_i.

(ii) there exists a constant $k > 0$ such that

$$\left.\begin{array}{lll} (1.14) & |y_a(t)| < k|x_a(t)| \quad \forall\, t \geq 0 & \forall \text{ images } y \text{ of } x \text{ under } N \\ & y_a = 0 \text{ whenever } x_a = 0 & \\ (1.15) & |y_i| < k|x_i| \quad \forall\, i \in I & \text{and } \forall\, x \in N_{3ex}[0,\infty) \cap Do(N) \end{array}\right.$$

Lemma 1.4

Let N be a relation satisfying assumption 1.2. Then

a) If the domain of N is restricted to $N_{3\sigma}[0,\infty) \cap Do(N)$ with $\sigma > 0$ then its range is in $N_{3\sigma}[0,\infty)$ and given $x \in N_{3\sigma}[0,\infty)$

$$x(t) = x_a(t) + \sum_{i \in I} x_i\, \delta(t-\tau_i)$$

then $\forall$ images y of x under N, $y(t) = y_a(t) + \sum_{i \in I} y_i\, \delta(t-\tau_i)$

(1.16) $\|y_a\|_{N_{2\sigma}} < k\|x_a\|_{N_{2\sigma}}$

$$(1.17)\quad \sum_{i \in I} e^{\sigma t_i} |y_i| < k \sum_{i \in I} e^{\sigma t_i} |x_i|$$

b) If the domain of N is restricted to $N_3[0,\infty) \cap Do(N)$ then its range is in $N_3[0,\infty) \cap Do(N)$.

c) If the domain of N is restricted to $N_{3ex}[0,\infty) \cap Do(N)$ then its range is in $N_{3ex}[0,\infty) \cap Do(N)$.

Proof

a) Let $x \in N_{3\sigma}[0,\infty)$ let y be an image of x under N.

By assumption 1.2

$$|y_a(t)| < k|x_a(t)| \qquad \forall\ t > 0$$

by lemma 2

$$\|y_a\|_{N_{2\sigma}} < k\|x_a\|_{N_{2\sigma}} \quad \text{since } x_a \in N_{2\sigma}[0,\infty)$$

trivially since

$$|y_i| < k|x_i| \qquad \forall\ i \in I \text{ and since } \sum_{i \in I} e^{\sigma t_i} |x_i| < \infty$$

$$\sum_{i \in I} e^{\sigma t_i} |y_i| < k \sum_{i \in I} e^{\sigma t_i} |x_i| < \infty$$

hence $y \in N_{3\sigma}[0,\infty)$

(b) and (c) follow trivially from (a).

Theorem 1.5

Let the system S.1 be such that:

(i) There exists a constant $k > 0$ with which N satisfies assumption 1.2.

(ii) The linear element H satisfies assumption 0.2.

In addition let the Fourier transform $\hat{h}(j\omega)$ of $h(t)$ satisfy the following inequality $\exists\ R > 0$ such that for $\sigma \leq \sigma_0$

$$\sum_{i \in I} |h_i| e^{\sigma t_i} < R; \text{ and } \sup_{\omega} |\hat{h}(j\omega)| < R$$

(iii) The functions appearing in the system belong to $N_{3ex}[0,\infty)$.

Then $k R < 1$ and $u_1, u_2 \in N_3[0,\infty)$ imply that $e_1, e_2, y_1, y_2 \in N_3[0,\infty)$

Proof

The action of the linear element on e_2 is

$$y_2(t) = (h*e_{a,2})(t) + \sum_{i \in I} e_{i,2}(h_a*\delta(t-\tau_i))$$

$$+ \sum_{i \in I} \sum_{j \in J} h_j\, e_{i,2}(\delta(t-\tau_i)*\delta(t-t_j))$$

or

$$y_2(t) = (h*e_{a,2})(t) + \sum_{i \in I} e_{i,2}\, h_a(t-\tau_i)$$

(1.18)

$$+ \sum_{i \in I} \sum_{j \in J} h_j\, e_{i,2}\, \delta(t-\tau_i - t_j)$$

by equating the coefficients of like kind of terms in the equations of the system S.1.

C.1.a will become, using (1.18)

$$\begin{cases} (1.19) \quad e_{a,1}(t) = u_{a,1}(t) - (h*e_{a,2})(t) - \sum_{i \in I} e_{i,2}\, h_a(t-\tau_i) \\ (1.20) \quad \sum_{m \in M} e_{m,1}\, \delta(t-\alpha_m) = \sum_{\ell \in L} u_{\ell,1}\, \delta(t-\beta_\ell) - \sum_{i \in I} \sum_{j \in J} h_j e_{i,2}\, \delta(t-\tau_i-t_j) \end{cases}$$

a) Taking into account the δ function part: (1.20) after truncation will give

$$(1.21) \quad \sum_{m \in M_T} e^{\sigma\alpha_m} |e_{m,1}| \leq \sum_{\ell \in L_T} e^{\sigma\beta_\ell} |u_{\ell,1}| + \sum_{i \in I_T} \sum_{j \in J_T} e^{\sigma(t_j+\tau_i)} |h_j||e_{i,2}|$$

the truncated index sets being obtained in the following way

$$M_T = \{m/m \in M \text{ and } \alpha_m \leq T\}$$

from the hypothesis on H; $\sum_{j \in J} e^{\sigma t_j} |h_j| < R$

then (1.21) becomes

$$(1.22) \quad \sum_{m \in M_T} e^{\sigma\alpha_m} |e_{m,1}| \leq \sum_{\ell \in L_T} e^{\sigma\beta_\ell} |u_{\ell,1}| + \sum_{i \in I_T} R|e_{i,2}|\, e^{\sigma\tau_i}$$

equation C.1.b of the system will become, with respect to the δ functions

$$\sum_{i \in I} e_{i,2}\, \delta(t-\tau_i) = \sum_{\ell \in B} u_{\ell,2}\, \delta(t-\gamma_\ell) + \sum_{j \in A} y_{j,1}\, \delta(t-\delta_j)$$

and become after truncation at T

$$(1.23) \quad \sum_{i \in I_T} e^{\sigma\tau_i} |e_{i,2}| \le \sum_{\ell \in B_T} e^{\sigma\gamma_\ell} |u_{\ell,2}| + \sum_{j \in A_T} |y_{j,1}|\, e^{\sigma\delta_j}$$

using inequality (1.15) with equation (1.23)

$$(1.24) \quad \sum_{i \in I_T} e^{\sigma\tau_i} |e_{i,2}| \le \sum_{\ell \in B_T} e^{\sigma\gamma_\ell} |u_{\ell,2}| + k \sum_{j \in A_T} |e_{j,1}|\, e^{\sigma\alpha_j}$$

by assumption 1.2: $A_T \subset M_T$, then (1.24) becomes

$$(1.25) \quad \sum_{i \in I_T} e^{\sigma\tau_i} |e_{i,2}| \le \sum_{\ell \in B_T} e^{\sigma\gamma_\ell} |u_{\ell,2}| + k \sum_{m \in M_T} |e_{m,1}|\, e^{\sigma\alpha_m}$$

replacing $\sum_{i \in I_T} e^{\sigma\tau_i} |e_{i,2}|$ by its bound in (1.22)

$$(1.26) \quad (1-kR) \sum_{m \in M_T} e^{\sigma\alpha_m} |e_{m,1}| \le \sum_{\ell \in L_T} e^{\sigma\beta_\ell} |u_{\ell,1}| + R \sum_{\ell \in B_T} |u_{\ell,2}|\, e^{\sigma\gamma_\ell}$$

but by the hypothesis on u_1 and u_2 $\sum_{\ell \in L} e^{\sigma\beta_\ell} |u_{\ell,1}| < \infty$ and $\sum_{\ell \in B} e^{\sigma\gamma_\ell} |u_{\ell,2}| < \infty$

since $(1-kR) > 0$

$$\sum_{m \in M_T} e^{\sigma\alpha_m} |e_{m,1}| \leq (1-kR)^{-1} \left[\sum_{\ell \in L} e^{\sigma\beta_\ell} |u_{\ell,1}| + R \sum_{\ell \in B} e^{\sigma\gamma_\ell} |u_{\ell,2}| \right]$$

since the right hand side does not depend on T and is finite

$$(1.27) \quad \sum_{m \in M} e^{\sigma\alpha_m} |e_{m,1}| < \infty,$$

from where it follows that

$$\sum_{i \in I} e^{\sigma\tau_i} |e_{i,2}| < \infty, \quad \sum_{j \in A} e^{\sigma\delta_j} |y_{j,1}| < \infty, \quad \sum_{i \in D} e^{\sigma\theta_i} |y_{i,2}| < \infty$$

b) Looking at the other part of the function equation (1.19) becomes

$$(1.28) \quad \|e_{a,1_T}\|_{N_{2\sigma}} \leq \|u_{a,1_T}\|_{N_{2\sigma}} + \|(h*e_{a,2})_T\|_{N_{2\sigma}} + \left\|\left(\sum_{i \in I} |e_{i,2}| h_a(t-\tau_i)\right)_T\right\|_{N_{2\sigma}}$$

using the same reasoning that in Theorem 1.3

since $\quad |\hat{h}(j\omega)| < R \qquad \forall\ \omega \in (-\infty,\infty)$

$$\exists\ \sigma_2 > 0 \ni |\hat{h}(\sigma_2 + j\omega)| \leq R \qquad \forall\ \omega \in (-\infty,\infty)$$

then $\quad \forall\ \sigma > 0 \quad \sigma \leq \sigma_2 \qquad g_{N_{2\sigma}}(H) \leq R$ by lemma 1.2.

The equations of the system for the $L_{2e}[0,\infty)$ part of the function become

$$(1.29) \quad \|e_{a,2_T}\|_{N_{2\sigma}} \leq \|u_{a,2_T}\|_{N_{2\sigma}} + \|y_{a,1_T}\|_{N_{2\sigma}}$$

from Lemma 1.4 and the definitions of the gains

$$\left|\begin{array}{ll}(1.30) & \|y_{a,1_T}\|_{N_{2\sigma}} < k\|e_{a,1_T}\|_{N_{2\sigma}} \\ (1.31) & \|(h*e_{a,2})_T\|_{N_{2\sigma}} \leq R\|e_{a,2_T}\|_{N_{2\sigma}} \qquad \sigma > 0 \qquad \sigma \leq \sigma_2\end{array}\right.$$

Equations (1.28) through (1.31) imply

$$(1.32)\quad \|e_{a,1_T}\|_{N_{2\sigma}}(1-kR) \leq \|u_{a,1_T}\|_{N_{2\sigma}} + R\|u_{a,2_T}\|_{N_{2\sigma}} + \left\|\left(\sum_{i \in I} |e_{i,2}| h_a(t-\tau_i)\right)_T\right\|_{N_{2\sigma}}$$

but

$$\left\|\left(\sum_{i \in I} |e_{i,2}| h_a(t-\tau_i)\right)_T\right\|_{N_{2\sigma}}$$

$$\leq \left\{\sum_{i \in I_T} \sum_{j \in I_T} e^{\sigma(\tau_i+\tau_j)} |e_{i,2}\|e_{j,2}|\right\}^{1/2} \|h_{a_T}\|_{N_{2\sigma}}$$

or

$$\left\|\left(\sum_{i \in I} |e_{i,2}| h_a(t-\tau_i)\right)_T\right\|_{N_{2\sigma}}$$

$$(1.33)\qquad \leq \left\{\left(\sum_{i \in I_T} e^{\sigma\tau_i} |e_{i,2}|\right)\left(\sum_{i \in I_T} e^{\sigma\tau_i} |e_{i,2}|\right)\right\}^{1/2} \|h_{a_T}\|_{N_{2\sigma}}$$

$$= \sum_{i \in I_T} e^{\sigma\tau_i} |e_{i,2}| \; \|h_{a_T}\|_{N_{2\sigma}}$$

Replacing the last term by its bound in (1.32)

$$(1.34)\quad (1-kR)\|e_{a,1_T}\|_{N_{2\sigma}} \leq \|u_{a,1_T}\|_{N_{2\sigma}} + R\|u_{a,2_T}\|_{N_{2\sigma}} + \sum_{i \in I} e^{\sigma\tau_i} |e_{i,2}| \|h_{a_T}\|_{N_{2\sigma}}$$

since $h_a \in N_{2\sigma}[0,\infty)$.

hence $u_{a,1}, u_{a,2}, h_a \in N_2[0,\infty)$ and $\sigma', \sigma'', \sigma''' > 0$ such that

$$u_{a,1} \in N_{2\sigma'}[0,\infty),\ u_{a,2} \in N_{2\sigma''}[0,\infty),\ h_a \in N_{2\sigma'''}[0,\infty)$$

let $\sigma > 0$ $\sigma \leq \min(\sigma_2,\sigma',\sigma'',\sigma''')$ then $u_{a,1}, u_{a,2}, h_a \in N_{2\sigma}[0,\infty)$

from the hypothesis $1 - kR > 0$

from part (a) $\displaystyle\sum_{i \in I} e^{\sigma\tau_i} |e_{i,2}| < \infty$

equation (1.34) becomes

$$\|e_{a,1_T}\|_{N_{2\sigma}} \leq (1-kR)^{-1} \left(\|u_{a,1}\|_{N_{2\sigma}} + R\|u_{a,2}\|_{N_{2\sigma}} + \sum_{i \in I} e^{\sigma\tau_i} |e_{i,2}| \|h_a\|_{N_{2\sigma}} \right)$$

The right hand side is finite and does not depend on T.

hence $e_{a,1} \in N_{2\sigma}[0,\infty)$ and $e_{a,1} \in N_2[0,\infty)$

then trivially $y_{a,1}, e_{a,2}, y_{a,2} \in N_2[0,\infty)$

and then $e_1, e_2, y_1, y_2 \in N_3[0,\infty)$

II. Main Stability Criteria

In the previous section criteria for the N_2 and N_3 boundedness of a system of type S.1 were given. In this section these criteria are going to be applied to the behavior of a system with respect to the derivatives of the various functions appearing within the system. Then the use of lemma 0.2 will insure that the functions appearing within the system belong to $L_\infty[0,\infty)$ and have a limit when t goes to infinity. In order to achieve this the behavior of a system of type S.1 with respect to the derivatives of input, output and error functions will be modeled by a system S.2. The system S.2 will be of type S.1. However the elements of S.2 will be obtained from those of S.1 in the following way:

$$(e_1', y_1') \in N' \text{ iff } \exists\ (e_1, y_1) \in N \text{ such that } \dot{e}_1 = e_1' \text{ and } \dot{y}_1 = y_1'$$

the linear element being defined in the same way. An idea on how to do the modeling can be obtained from figure 2.

The equations defining S. 2 are:

(2.1.a) $\quad e_1' = u_1' - y_2'$

(2.1.b) $\quad e_2' = u_2' + y_1'$

(2.1.c) $\quad (e_1', y_1') \in N'$

(2.1.d) $\quad y_2' = H'e_2'$

where $u_1' = \dot{u}_1$ derivative of u_1 in S.1 etc...

Two cases will be considered as in the previous section. First the case when the input functions have no steps and hence their derivatives have no δ functions.

Second the case in which δ functions appear in the derivatives.

A. The input functions have no steps.

An assumption on the element N will be necessary.

Assumption 2.1

Let N be a relation with domain and range in X.

(i) $\forall$ x $\in$ Do(N) which has a derivative the images y of x under N have derivatives.

(ii) There exists a constant k > 0 such that

$$(2.2) \qquad \left.\begin{array}{ll} |\dot{y}(t)| < k|\dot{x}(t)| & \forall\ t \geq 0 \\ \dot{y} = 0 \text{ whenever } \dot{x} = 0 & \end{array}\right\} \quad \begin{array}{l} \forall \text{ images } y \text{ of } x \text{ under } N \\ \forall\ x \in Do(N) \text{ and having a} \\ \quad \text{derivative belonging to } L_{2e}[0,\infty) \end{array}$$

This assumption insures that N' satisfies assumption 1.1.

Theorem 2.1

Let the system S.1 be such that

(i) There exists a constant k > 0 with which N satisfies assumption 2.1.

(ii) The linear element H satisfies assumption 0.1 and the Fourier transform $\hat{h}(j\omega)$ of h(t) satisfies the following inequality

$$\exists\ R > 0 \text{ such that } \sup_{\omega} |\hat{h}(j\omega)| < R$$

(iii) The functions which appear in the system have derivatives which belong to $L_{2e}[0,\infty)$.

If kR < 1 and if the derivatives $\dot{u}_1$, $\dot{u}_2$ of u_1 u_2 belong to $N_2[0,\infty)$.

Then the derivatives $\dot{e}_1$, $\dot{e}_2$, $\dot{y}_1$, $\dot{y}_2$ of e_1, e_2, y_1, y_2 belong to $N_2[0,\infty)$.

Proof

Using the system S.2 to model the behavior of S.1 with respect to the derivatives: since $\exists$ k > 0 with which N satisfies assumption 2.1. N' satisfies assumption 1.1 with k. It, then, satisfies the hypothesis of theorem 1.3.

From L. Schwartz [11] (Vol. 2, Chapt. 6, Thm. 9) it is known that the derivative of a convolution is the convolution of one of the factor with the derivative of the other:

$$(2.5) \qquad \dot{\overline{h*e_2}} = h*\dot{e}_2$$

hence $H' = H$

and H' satisfies the hypothesis of theorem 1.3.

The functions e_1', e_2', y_1', y_2' satisfy the hypothesis of theorem 1.3.

All the hypothesis of theorem 1.3 are then satisfied.

then $\dot{u}_1, \dot{u}_2 \in N_2[0,\infty)$ imply $\dot{e}_1, \dot{e}_2, \dot{y}_1, \dot{y}_2 \in N_2[0,\infty)$.

B. There can be steps in the input functions.

The assumption on N must be restated.

Assumption 2.2

Let N be a relation with domain and range in X.

(i) $\forall$ x $\in$ Do(N) which has a derivative the images y of x under N have derivatives.

(ii) $\forall\ x \in Do(N)$ which has a derivative in $N_{3ex}[0,\infty)$

$$\dot{x}(t) = x_a'(t) + \sum_{i \in I} x_i' \,\delta(t-\tau_i)$$

The derivatives $\dot{y}$ of the images y of x under N are of the type

$$\dot{y}(t) = y_a'(t) + \sum_{i \in I} y_i' \,\delta(t-\tau_i)$$

where y_i' can be zero with the understanding that $y_i' = 0$ implies that no δ function occurs at time τ_i.

(iii) There exists a constant $k > 0$ such that

$$\left.\begin{array}{|l} |y_a'(t)| < k|x_a'(t)| \quad \forall\ t \geq 0 \\ \\ y_a' = 0 \text{ whenever } x_a' = 0 \\ \\ |y_i'| < k|x_i'| \quad \forall\ i \in I \end{array}\right\} \quad \begin{array}{l} \forall \text{ images y of x under N} \\ \\ \forall\ x \in Do(N) \text{ and having a} \\ \\ \text{derivative belonging to } N_{3e}[0,\infty) \end{array}$$

This assumption insures that N' satisfies assumption 1.2.

Theorem 2.2

Let the system S.1 be such that

(i) There exists a constant $k > 0$ with which N satisfies assumption 2.2.

(ii) The linear element H satisfies assumption 0.2.

In addition let the Fourier transform $\hat{h}(j\omega)$ of h(t) satisfy the following inequality $\exists$ R > 0 such that for $\sigma \leq \sigma_0$

$$\sum_{i \in I} |hi| e^{\sigma t_i} < R \text{ and } \sup_{\omega} |\hat{h}(j\omega)| < R$$

(iii) The functions which appear in the system have derivatives which belong to $N_{3ex}[0,\infty)$.

If Rk < 1 and if the derivatives $\dot{u}_1$ and $\dot{u}_2$ of u_1 and u_2 belong to $N_3[0,\infty)$.

Then the derivatives $\dot{e}_1$, $\dot{e}_2$, $\dot{y}_1$, $\dot{y}_2$ of e_1, e_2, y_1, y_2 belong to $N_3[0,\infty)$.

Proof

Using the same procedure as in theorem 2.1 the behavior of the system S.1 with respect to the derivatives is modelled by S.2.

Since $\exists$ k with which N satisfies assumption 2.2, N' satisfies assumption 1.2 with k and then satisfies the hypothesis of theorem 1.5.

Using Schwartz [11] again H' = H and then satisfies the hypothesis of theorem 1.5.

The functions e_1', e_2', y_1', y_2' satisfy the hypothesis of theorem 1.5.

All the hypothesis of theorem 1.5 are then satisfied

then $\dot{u}_1, \dot{u}_2 \in N_3[0,\infty)$ imply $\dot{e}_1, \dot{e}_2, \dot{y}_1, \dot{y}_2 \in N_3[0,\infty)$.

Corollary 2.3

If the hypothesis of either Theorem 2.1 or Theorem 2.2 are satisfied by a system S.1 then the output and error functions belong to $L_\infty[0,\infty)$ and have limits when t goes to infinity.

The proof is a straightforward application of lemma 0.2.

III. Applications

This section deals with the application of the results of section 2. These results can be used either to obtain a L_∞ criterion for systems which could be studied with the L_2 stability, or to obtain a stability criterion for systems which could not be studied with L_2 stability.

A system could be L_2 stable under a sector condition on the nonlinearity and L_∞ stable under another sector condition this time on the slopes. In order to transform the gain conditions in sector conditions for the nonlinearities and in circle or Popov type conditions for the linear element some transformations will be needed.

From the use of these transformations it will be possible to show that hysteresis is a special case of the relation N.

A. The transformation Tc which is going to be used is represented in figure 3. It is a classical transformation which can be expressed analytically by

$$\text{Tc}\begin{cases} N_1 = (N - \Lambda I) & (3.1) \\ H_1 = H(I + \Lambda H)^{-1} & (3.2) \end{cases}$$

Assumption 3.1

Let H be a linear mapping of X into X such that there exists $h \in X$ and $\sigma_0 > 0$

$$(3.3)\quad \begin{cases} h(t) = h' + h_a''(t) + \sum_{i \in I} h_i \, \delta(t-t_i) & \forall\, t \geq 0 \\ h(t) = 0 & \forall\, t < 0 \end{cases}$$

where h' is a real constant

and $\quad e^{\sigma_0 t}\,(h''_a(t) + \sum_{i \in I} h_i\ \delta(t-t_i)) \in L_1[0,\infty)$

defining H in the following way

$\forall\ x \in Do(H)$

(3.4) $\quad (Hx)(t) = (h*x)(t)$

The following lemma is the extension of the circle theorem of Zames [4].

Lemma 3.1

Let S.1 be a system such that

(i) N is a relation with domain and range in X inside the sector (α,β) with $\beta > 0$. i.e.

$$(3.5) \quad \left.\begin{array}{l} \alpha < \dfrac{y(t)}{x(t)} < \beta; \quad \forall\ t \geq 0 \\ y = 0 \text{ whenever } x = 0 \end{array}\right| \begin{array}{l} \forall\ y \text{ images of } x \text{ under } N \\ \forall\ x \in L_{2e}[0,\infty) \text{ or } N_{3ex}[0,\infty) \end{array}$$

(ii) H is a linear element satisfying assumption 3.1. In addition $\exists\ \delta > 0$ with which H satisfies the circle conditions for the sector (α,β). i.e.

a) If $\alpha = 0$

$$(3.6) \quad \mathrm{Re}\ \hat{h}(j\omega) + \frac{1}{\beta} > \delta, \quad \forall\ \omega \in (-\infty,\infty)$$

b) If $\alpha > 0$

$$(3.7) \quad \left|\hat{h}(j\omega) + \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right| > \frac{1}{2}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) + \delta, \quad \forall\ \omega \in (-\infty,\infty)$$

and the nyquist diagram of H does not encircle the point $-\frac{1}{2}(\frac{1}{\alpha} + \frac{1}{\beta})$

c) If $\alpha < 0$ then

$$(3.8) \qquad |\hat{h}(j\omega) + \frac{1}{2}(\frac{1}{\alpha} + \frac{1}{\beta})| < \frac{1}{2}(\frac{1}{\beta} - \frac{1}{\alpha}) + \delta, \ \forall \ \omega \in (-\infty,\infty)$$

Then using the transformation previously defined with

$$(3.9) \qquad \Lambda = \frac{\alpha + \beta}{2}$$

letting

$$(3.10) \qquad k_1 = \frac{\beta - \alpha}{2}$$

N_1 will be a relation inside

$$(3.11) \qquad (-k_1, k_1)$$

and H_1 will be such that

$$(3.12) \qquad |\hat{h}_1(j\omega)| < \frac{1}{k_1}$$

and $\qquad \exists \ \sigma_0 > 0 \ni e^{\sigma_0 t} h_1(t) \in L_1[0,\infty)$

Proof

i) If $h' = 0$ then this is Zames' result.

ii) If $h' \neq 0$ $|\hat{h}_1(j\omega)| < \frac{1}{k}$ is a straight application of Zames results. Using a theorem by Desoer [2] on the general formulation of the Nyquist criterion

$$e^{\sigma_0 t} h_1(t) \in L_1[0,\infty)$$

With the help of lemma 3.1 it is going to be possible to state a modified version of theorems 2.1 and 2.2.

Assumption 3.2

Let N be a relation with domain and range in X

(i) $\forall\, x \in Do(N)$ which has a derivative the images y of x under N have derivatives.

(ii) There exists two real constants α, β, with $\beta > 0$ such that

$$\left\{\begin{array}{l} \alpha\, \dot{x}^2(t) < \dot{y}(t)\, \dot{x}(t) < \beta\, \dot{x}^2(t) \ \forall\, t \geq 0 \\ \\ \text{and } \dot{y} = 0 \text{ whenever } \dot{x} = 0 \end{array}\right\} \quad \begin{array}{l} \forall \text{ images } y \text{ of } x \text{ under } N \\ \forall\, x \in D_0(N) \text{ and having} \\ \text{derivatives belonging} \\ \text{to } L_{2e}[0,\infty). \end{array}$$

This insures that N' is inside (α,β).

Theorem 3.2

Let the system S.1 be such that

(i) There exists two real constants α,β and $\beta > 0$ with which N satisfies assumption 3.2.

(ii) The linear element H satisfies assumption 3.1 and satisfies the circle condition for the sector (α,β).

(iii) The functions which appear in the system have derivatives which belong to $L_{2e}[0,\infty)$.

If the derivatives $\dot{u}_1$ and $\dot{u}_2$ of u_1, u_2 belong to $N_2[0,\infty)$ then the derivatives $\dot{e}_1$, $\dot{e}_2$, $\dot{y}_1$, $\dot{y}_2$ of e_1, e_2, y_1, y_2 belong to $N_2[0,\infty)$.

Proof

Using Lemma 3.1 the transformation Tc with $\Lambda = \frac{\alpha + \beta}{2}$ will transform the system S.1 in another system S.1.1 of type S.1 where N_1 will be such that

$$\left\{ \begin{array}{ll} |\dot{y}_1(t)| < k_1 |\dot{e}_1(t)|, & \forall\, t \geq 0 \\ \dot{y}_1 = 0 \text{ when } \dot{e}_1 = 0 & \end{array} \right\}$$

$\forall$ images y_1 of e_1 under N_1

$\forall e_1 \in Do(N_1)$ and having derivatives

and H_1 will be such that $|\hat{h}_1(j\omega)| < \frac{1}{k_1}$

S.1.1 then satisfies the hypothesis of theorem 2.1.

Assumption 3.3

Let N be a relation with domain and range in X

(i) $\forall\, x \in Do(N)$ which has a derivative the images y of x under N have derivatives

(ii) $\forall\, x \in Do(N)$ which has a derivative in $N_{3ex}[0,\infty)$

$$\dot{x}(t) = x_a'(t) + \sum_{i \in I} x_i' \,\delta(t-\tau_i)$$

the derivatives $\dot{y}$ of the images y of x under N are of the type

$$\dot{y}(t) = y_a'(t) + \sum_{i \in I} y_i' \,\delta(t-\tau_i)$$

where y_i' can be zero with the understanding that $y_i' = 0$ implies that there is no δ function at time τ_i.

(iii) There exists two real constants α, β, and $\beta > 0$ such that

$$\begin{cases} \alpha x_a'^2(t) < y_a'(t)\, x_a'(t) < \beta x_a'^2(t) \;\forall\; t \geq 0 \\ y_a' = 0 \text{ of } x_a' = 0 \\ \alpha x_i'^2 < y_i'\, x_i' < \beta x_i'^2 \;\forall\; i \in I \end{cases}$$

$\forall$ images y of x under N

$\forall \in Do(N)$ and having a derivative belonging to $N_{3ex}[0,\infty)$.

This assumption insures that N' is inside the sector (α,β).

Theorem 3.3

Let the system S.1 be such that

(i) There exists two real constants α, β; $\beta > 0$ with which N satisfies Assumption 3.3.

(ii) The linear element H satisfies assumption 3.1 and $h_a' \in N_{2\sigma_0}[0,\infty)$. In addition it satisfies the circle condition for the sector (α,β),

and $\sum_I |h_i| e^{\sigma t_i} < \frac{1}{\beta}$.

(iii) The functions which appear in the system have derivatives which belong to $N_{3ex}[0,\infty)$.

If the derivatives $\dot{u}_1$, $\dot{u}_2$ of u_1, u_2 belong to $N_3[0,\infty)$ then the derivatives $\dot{e}_1$, $\dot{e}_2$, $\dot{y}_1$, $\dot{y}_2$ of e_1, e_2, y_1, y_2 belong to $N_3[0,\infty)$.

Proof

A transformation and lemma 3.1 are used in the same way as they were in theorem 3.2 and the transformed system satisfies the hypothesis of theorem 2.2.

B. A transformation T_m can be used with some systems in order to have the Popov condition on the linear element. It suffices to take a first order multiplier:

$$Z(j\omega) = \frac{1}{1 + qj\omega} \tag{3.13}$$

The transformation T_m will be expressed analytically by

$$T_m \begin{cases} N_1 = NZ & (3.14) \\ H_1 = Z^{-1} H & (3.15) \end{cases}$$

<u>Assumption 3.4</u>

Let N be 'a relation with domain and range in X

(i) there exists $k > 0$ such that

$$\left.\begin{array}{l} 0 < \dfrac{y(t)}{x(t)} < k \quad \forall\, t \geq 0 \\ y = 0 \text{ whenever } x = 0 \end{array}\right\} \quad \begin{array}{l} \forall\, y \text{ images of } x \text{ under } N \\ \forall\, x \in L_{2e}[0,\infty) \cap Do(N) \end{array} \tag{3.16}$$

(ii) either a) $\int_0^t y(\tau)\dot{x}(\tau)d\tau \geq 0 \quad \forall\, t \geq 0$ $\quad$ $\forall\, y$ images of x under N

$\forall\, x \in L_{2e}[0,\infty) \cap Do(N)$

having a derivative

or b) $\int_0^t y(\tau)\dot{x}(\tau)d\tau \leq 0 \quad \forall\, t \geq 0$ $\quad$ $\forall\, y$ images of x under N

$\forall\, x \in L_{2e}[0,\infty) \cap Do(N)$

having a derivative

The implication of condition 3.4 (ii) will be considered in the examples.

Lemma 3.4

Let S.1 be a system such that: there exists $k > 0$ with which N satisfies assumptions 3.4 (i) and 3.4 (ii, a).

Let H satisfy assumption 3.1.

In addition let H satisfy the Popov condition with $q > 0$

i.e. $\quad \mathrm{Re}[(1 + qj\omega)\, \hat{h}(j\omega) + \frac{1}{k}] \geq \delta > 0 \qquad \forall\, \omega \in (-\infty,\infty)$

Then the transformation T_m with $z(j\omega) = \dfrac{1}{1 + qj\omega}$ will lead to a system N_1, H_1 having the following properties:

a) N_1 is inside the sector $(0, k)$ with respect to the L_2 norm.

i.e. $\quad \langle y_1, y_1 - ke_1 \rangle \leq 0 \qquad \forall\, y_1$ images of e_1 under N_1

$$e_1 \in L_2[0,\infty) \cap Do(N_1)$$

b) $\exists\ \delta' > 0$ with which H_1 satisfies the circle conditions for the sector $(0,k)$.

Proof

a) For N_1 to satisfy $\langle y_1, y_1 - ke_1 \rangle \geq 0$ with $e_1 = x_1 + q\dot{x}_1$

i.e. $\quad x_1 = Ze_1$

it suffices that

$$\langle y_1, y_1 - kx_1 - kq\dot{x}_1 \rangle \leq 0$$

or

$$\langle y_1, y_1 - kx_1 \rangle - kq \langle y_1, \dot{x}_1 \rangle \leq 0$$

by hypothesis

$$\langle y_1, y_1 - kx_1 \rangle \leq 0$$

it suffices then that

$$\langle y_1, \dot{x}_1 \rangle \geq 0$$

This is verified because of assumption 3.4 (ii, a).

b) since $h_1(j\omega) = (1 + qj\omega)\, h(j\omega)$ the hypothesis on H imply trivially that H_1 satisfies the circle condition for the sector (0, k).

Lemma 3.5

Let S.1 be a system such that there exists $k > 0$ with which N satisfies assumption 3.4 (i) and 3.4 (ii, b).

Let H satisfy assumption 3.1.

In addition let H satisfy the Popov condition with $q < 0$.

The transformation T_c with coefficient k and a change of sign i.e.

$$\begin{cases} N_1 = (kI - N) \\ H_1 = -\, H(I + kH)^{-1} \end{cases}$$

will lead to a system having the following properties.

N_1 satisfies assumption 3.4 (i) and 3.4 (ii, a) and

$$\mathrm{Re}[(1 - qj\omega)\, H_1(j\omega) + \frac{1}{k}] \geq \delta > 0$$

Proof

This is the extension of a lemma by Aizerman and Gantmacher [12]

since $\begin{cases} y_1 = kx - y \\ x_1 = x \end{cases}$

$$\int_0^t y_1(\tau)\dot{x}_1(\tau)d\tau = \int_0^t kx(\tau)\dot{x}(\tau)d\tau - \int_0^t y(\tau)\dot{x}(\tau)d\tau$$

hence N satisfying 3.4 (ii, b) implies N_1 satisfies 3.4 (ii, a).

<u>Assumption 3.5</u>

Let N be a relation with domain and range in X

(i) $\forall$ $x \in Do(N)$ which has a derivative the images y of x under N have derivatives.

(ii) there exists a constant $k > 0$ such that

$$\left.\begin{array}{ll} 0 < \dot{y}(t)\dot{x}(t) < k\dot{x}^2(t) & \forall\ t \geq 0 \\ \text{and } \dot{y} = 0 \text{ whenever } \dot{x} = 0 & \end{array}\right\}$$

$\forall$ images y of x under N $\forall$ $x \in Do(N)$ and having derivatives belonging to $L_{2e}[0,\infty)$

(iii) either a) $\int_0^t \dot{y}(\tau)\ddot{x}(\tau)d\tau \geq 0 \quad \forall\ t \geq 0$ $\forall$ images y of x under N $\forall$ $x \in Do(N)$ having a first derivative belonging to $L_{2e}[0,\infty)$ and having a second derivative.

or b) $\int_0^t \dot{y}(\tau)\ddot{x}(\tau)d\tau \leq 0 \quad \forall\ t \geq 0$ $\forall$ images y of x under N $\forall$ $x \in Do(N)$ having a first derivative belonging to $L_{2e}[0,\infty)$ and having a second derivative.

This assumption insures that N' satisfies assumption 3.4.

Theorem 3.6

Let the system S.1 be such that there exists a real constant $k > 0$ with which N satisfies assumption 3.5.

The linear element satisfies assumption 3.1.

There exists a real constant q, $q > 0$ if N satisfies assumption 3.5 (iii, a), $q < 0$ if N satisfies assumption 3.5 (iii, b) with which H satisfies the Popov condition:

$$\mathrm{Re}((1 + qj\omega)\, \hat{h}(j\omega) + \frac{1}{k}) \geq \delta > 0$$

The functions which appear in the system have derivatives which belong to $L_{2e}[0,\infty)$.

In addition $\dot{e}_1(0) = 0$.

If the derivatives $\dot{u}_1$ and $\dot{u}_2$ of u_1, u_2 belong to $N_2[0,\infty)$ then the derivatives $\dot{e}_1$, $\dot{e}_2$, $\dot{y}_1$, $\dot{y}_2$ of e_1, e_2, y_1, y_2 belong to $N_2[0,\infty)$.

Proof

a) $q > 0$ then Lemma 3.4 applies and the transformed system satisfies the hypothesis of theorem 3.2.

b) $q < 0$ then lemma 3.5 applies and (a) applies to the transformed system.

IV. Examples and experiments

In this section the implications of the various assumptions are going to be examined.

A few experiments were made and both theoretical and practical results are given.

a) Let the nonlinearity N be single valued time invariant with the slopes in a finite sector (α,β) $\beta > 0$ (Fig. 4) then assumptions 3.2, 3.3 and 3.5 could apply allowing the use of either a circle condition or the Popov condition on the linear element. In that case assumption 3.4 (ii) on N' would become

$$\int_0^t N'(x')\,\dot{x}'dt = \int_0^{x'(t)} N'(x')dx' \quad \text{which by 3.4 (i) would}$$

always be greater or equal to zero.

b) Let N be multivalued, and let the point x', y' go clockwise around N' (Fig. 4) then assumption 3.2, 3.3 and 3.5 could apply again. Assumption 3.4 (ii, a) on N' being satisfied. This assumption means the area inside the curve y', x' is positive as the point goes around the curve.

c) N can be multivalued and let the point x', y' go counter clockwise around N' (Fig. 4) then assumption 3.2, 3.3 and 3.5 could apply again assumption 3.4 (ii, b) on N' being satisfied. This assumption means the area inside the curve y', x' is negative as the point goes around the curve. Some models of hysteresis and backlash belong to that class.

d) In the case when N is multivalued, time varying and the area inside the curve y', x' may change sign (Fig. 4) only the circle criterion would apply (assumption 3.2 or 3.3).

Experimental Results

A nonlinearity was built using a transformer, a power D.C. amplifier and an integrator (Fig. 5). The need for the power D.C. amplifier arose because the analog computer amplifiers could not drive the transformer.

Several transformers were used in order to observe different hysteresis loops. For all of them the maximum slope varied with frequency. As can be observed in Figure 6 the maximal slope increases as frequency decreases.

The range of variation of this maximal slope will be taken into account.

Another interesting phenomenon occurs sometimes when there is a D.C. offset. No saturation occurs on one side of the hysteresis loop and an infinite slope can be observed. However if the derivative of the output versus the derivative of the input is plotted (Fig. 7) it can be observed that the sector has been shifted away from the origin.

A transformation can be used to deal with this problem (Fig. 8). Let the origin of the sector have the coordinates (α,β). Then let

$$\begin{cases} e' = \dot{e}_1 - \alpha \\ y' = \dot{y}_1 - \beta \end{cases}$$

Then the relation (e', y') obeys the sector condition. α can be transfered into $\dot{u}_1$ and β into $\dot{u}_2$

i.e.
$$\begin{cases} u_1' = \dot{u}_1 - \alpha \\ u_2' = \dot{u}_2 + \beta \end{cases}$$

If α and β are functions of time which obey the assumptions on $\dot{u}_1$ and $\dot{u}_2$ then the stability criterion applies to the transformed system and insures the stability of the original system.

The linear part of the system was simulated on an analog computer. For each case the experimental frequency response was compared to the theoretical one: Figures 9, 11, and 13.

Three linear elements were used. The first had an integrator hence a criterion using $L_2[0,\infty)$ stability could have been applied to it. The second and third had no integrators. As a consequence important D.C. offsets, due to the properties of the nonlinearity appeared. These offsets made it impossible to apply a criterion using $L_2[0,\infty)$ stability. However, the criteria developed in this paper predicted the $L_\infty[0,\infty)$ boundedness of the solutions as well as the absence of sustained oscillations.

Experiment 1

Taking a linear element with an integrator a fourth order denominator and a zero.

$$H(s) = \frac{s + 0.2}{s(s + 0.4)(s + 0.6)(s + 1)}$$

From the frequency response of this system (Fig. 9) it can be seen that $\mathrm{Re}(H(j\omega)) > - 1.535$.

The maximum slope of the nonlinearity is

a) 8.7 if the all range of frequencies is taken into account.

b) 5 if the frequency is close to the frequency of free oscillations of the linear system. It also turns out this is the lowest maximum slope.

The limit gain for the system was found to be 0.213. The theoretical gain would be

a) taking into account the all range of frequencies

$$g = \frac{1}{1.535} \times \frac{1}{8.7} = 0.075$$

b) taking into account the frequencies close to the linear element frequency

$$g = \frac{1}{1.535} \times \frac{1}{5} = 0.13$$

In the worst case the theoretical gain is slightly more than a third of the practical one. In the best case the theoretical gain is more than half of the practical one.

Some of the responses of the system can be seen on Figure 10.

<u>Experiment 2</u>

This linear element has no integrator hence important D.C. offsets occur in the response and L_2 stability cannot be used to study it. The criteria developed in this paper applied and gave a fairly close approximation for the gain.

$$H(s) = \frac{s + 0.2}{(s + 0.25)(s + 0.4)(s + 0.6)(s + 1)}$$

The theoretical and experimental frequency response were plotted (Fig. 11) and it could be seen that

$$\mathrm{Re}(H(j\omega)) > -0.825$$

The nonlinearity was the same as in experiment 1. The limit gain for the system was found to be 0.26. The theoretical gain would be

a) taking into account the all frequency range

$$g = \frac{1}{0.825} \times \frac{1}{8.7} = 0.139$$

b) taking into account the frequencies close to the linear element frequency

$$g = \frac{1}{0.825} \times \frac{1}{5} = 0.242$$

In the worst case the theoretical gain is about half of the practical one. In the best case it is very close to the practical one. The response for the various gains can be observed on Figure 12 a, b.

<u>Experiment 3</u>

A linear element with 2 pairs of conjugate poles was chosen. Once again the lack of integrator introduces D.C. offsets and does not allow the use of the L_2 stability.

$$H(s) = \frac{s + 0.2}{(s + 0.4 + 0.1j)(s + 0.4 - 0.1j)(s + 0.2 + 0.3j)(s + 0.2 - 0.3j)}$$

The theoretical and experimental frequency response were plotted (Fig. 13) and it could be seen that

$$\mathrm{Re}(H(j\omega)) > - 5.443$$

A different transformer was used and different setting of the resistors R_1 and R_2 (Fig. 5) gave two different nonlinearities.

i) Nonlinearity with a maximal slope of 6.22 over the all range of frequencies - and of 3.68 in the frequencies around the oscillatory frequency of the linear system. The response to this system can be seen on Figures 14 a and 15 a, b.

The limit gain was found to be between 0.09 and 0.1. The theoretical gain would be

a) taking into account the all frequency range:

$$g = \frac{1}{5.443} \times \frac{1}{6.22} = 0.0295$$

b) taking into account the frequencies close to the linear element frequency

$$g = \frac{1}{5.443} \times \frac{1}{3.68} = 0.05$$

In the worst case the theoretical gain is slightly less than a third of the practical one. In the best case it is about half of the practical one.

ii) Nonlinearity with a maximal slope of 11.15 over the all range of frequencies, and 5.8 around the linear element frequency.

Its response can be seen on Figure 14 b.

The limit gain was found to be 0.055. The theoretical gain would be

a) taking into account the all frequency range

$$g = \frac{1}{5.443} \times \frac{1}{11.15} = 0.0165$$

Less than a third of the practical gain

b) taking into account the linear element frequencies

$$g = \frac{1}{5.443} \times \frac{1}{5.8} = 0.0317$$

more than a half of the practical gain.

Conclusion of the Experiments

The criteria developed in this paper can be used to predict the L_∞ stability and the lack of sustained oscillations of a system. Even when this system cannot be studied with the classical approach using L_2 stability.

The gains predicted by the criteria are of the order of the real gains since none of them were less than a fourth of the real gains.

If in the computation of the gain, in the case of a gain varying with frequency, the frequency of the system is taken into account a close fit is obtained. Some more investigations of this property should be done.

The following table summarizes the results of the experiments.

Table

	Experiment 1	Experiment 2	Experiment 3 (i)	Experiment 3 (ii)
Lower bound of $ReH(j\omega)$	- 1.535	- 0.825	- 5.443	
Max. slope of N. L. over all freq.	8.7	8.7	6.22	11.15
Max. slope close to syst. freq.	5	5	3.68	5.8
Experimental gain	0.213	0.26	0.09 to 0.1	0.055
Theoretical gain with max. slope over all freq.	0.075	0.139	0.0295	0.0165
Theoretical gain with max. slope around syst. freq.	0.13	0.242	0.05	0.0317

V. Remarks about the Modeling of Hysteresis

Various models have been used for describing the behavior of hysteresis. When a real hysteresis is being observed it can be seen that it has very different properties depending on the type of input and the initial conditions.

For small periodic inputs with initial conditions insuring that there is no D.C. bias the behavior of hysteresis is very close to the one of a linear system with a constant phase shift.

As the amplitude of the input increases the classical hysteresis loop can be observed. Its modeling has been studied in [7] for instance.

As soon as a bias appears the output becomes assymetrical and modelling becomes very difficult.

As far as the D.C. behavior is concerned little work has been done.

The modelling of hysteresis by a double loop going through zero is highly inaccurate - the modeling by a backlash does not show well the saturation which is characteristic of hysteresis, does not take into account the behavior around the origin and the fact that there is not really a dead zone.

Even the assumptions made in this paper: slope boundedness cannot fully describe the phenomenon. Infinite slopes can occur in the region of the origin. By taking the hysteresis in an initial state corresponding to a D.C. bias high enough to be in the saturation region and applying an input close to a step (a step would allow the output to have a step itself), bringing the new input close to zero the output will continue to decrease while the input keeps its value hence there will be a line of infinite slope in the response.

However it seems that the assumption of slope boundedness is the one which allows the closest representation of the phenomenon.

Conclusion

The functional analysis approach allows one to describe the stability of feedback systems for given classes of inputs. The application of this method to the derivatives of the functions appearing within a system eliminates the requirement that the nonlinear element satisfies a sector condition. The new condition being that the slopes of this element be bounded. The assumptions on the linear element are loose enough to allow delays and do not require that this element have an integeator. Two types of criteria are obtained. A circle criterion for general nonlinearities and input functions whose derivatives can have δ-functions. A Popov criterion for a more restricted class of nonlinearities and functions with derivatives without δ-functions.

By adding conditions on the nonlinearity a multiplier criterion could have been obtained, however these conditions would have eliminated hysteresis type nonlinearities hence the criterion was not derived here.

From the results of the experiments it can be seen that the bounds on the gain are fairly close to the experimental gains. The behavior of the experimental hysteresis and its frequency dependence seems to imply a possibility to obtain a closer fit with the criteria. Since that dependence shows a decrease of the maximal slope when the frequency is increased it gives further reasons for the effectiveness of the dithering method [13, 14]. A slow varying signal superimposed on a fast oscillation would then see the maximum slope of the hysteresis decreased, thus allowing a higher gain. The fast oscillations would then be filtered by the linear element. The modeling of hysteresis is not yet satisfactory however the bounded slopes condition seems to lead to results which fit reasonably reality. A way to insure a better approximation would be to replace N' by two elements in parallel: a linear element and a relation N". In this way the displacement of the sector would be accounted for.

References

[1] C. T. Lee and C. A. Desoer, "Stability of Single-Loop Nonlinear Feedback Systems," ERL Report No. ERL-66-13, University of California, Berkeley.

[2] C. A. Desoer, "A General Formulation of the Nyquist Criterion," IEEE Transactions CT-12, No. 2 (June 1965).

[3] I. W. Sandberg, "On the L_2 Boundedness of Solution of Nonlinear Functional Equations," Bell Sys. Tech. J. Vol. 43, pp. 11, July 1964.

[4] G. Zames, "On the Input Output Stability of Time-Varying Nonlinear Feedback Systems, "Part I: IEEE Transactions, AC 11, No. 2 (April 1966); Part II: IEEE Transactions, AC 11, No. 3 (July 1966).

[5] L. O. Chua and D. A. Stromsmoe, "Lumped Circuit Model for Nonlinear Inductors Exhibiting Hysteresis Loops," to be published.

[6] S. Kodama and H. Shirakawa, "Stability of Nonlinear Feedback Systems with Backlash," IEEE Transactions, AC 13, No. 4 (August 1968).

[7] S. Weissenberger, "Application of Results from the Absolute Stability Problem to the Computation of Finite Stability Domains," IEEE Transactions AC 13 No. 1 (February 1968).

[8] S. Weissenberger, "Regions of Ultimate Boundedness for the Problem of Lur'e," Electronics Letters, Vol. 4, No. 18, (September 1968).

[9] J. A. Walker and N . H. McClamroch, "Finite Regions of Attraction for the Problem of Lur'e," Int. J. Control, Vol. 6, No. 4 1967.

[10] V. A. Yakubovich, "The Method of Matrix Inequalities in the Theory of Stability of Nonlinear Controlled Systems III, Absolute Stability of Systems with Hysteresis Nonlinearities," Automation and Remote

Control. Vol. 26, May 1965.

[11] L. Schwartz, "Theorie des Distributions," Tome 1 and 2 Hermann and Co., Paris (1950).

[12] M. A. Aizerman and F. R. Gantmacher," Absolute Stability of Regulator Systems," Holden-Day, Inc., San Francisco (1964).

[13] R. Oldenburger and R. C. Boyer, "Effect of Extra Sinusoidal Inputs to Nonlinear Systems," A.S.M.E. Transactions Journal of Basic Engineering, December 1962.

[14] A. Alexanchovitz and J. Rootenberg, "Dithering as a Factor in Hysteresis Elimination in Rotating Amplifiers," IEEE Transactions AC 13 No. 2 (April 1968).

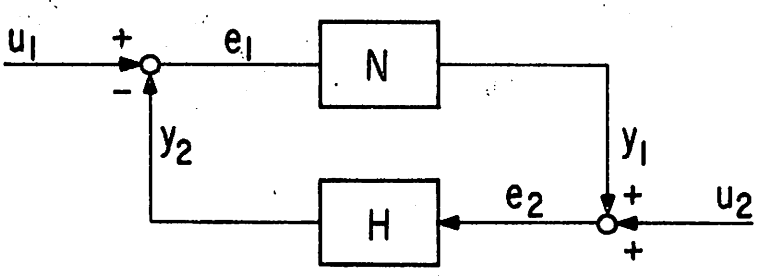


Figure 1   System S.1

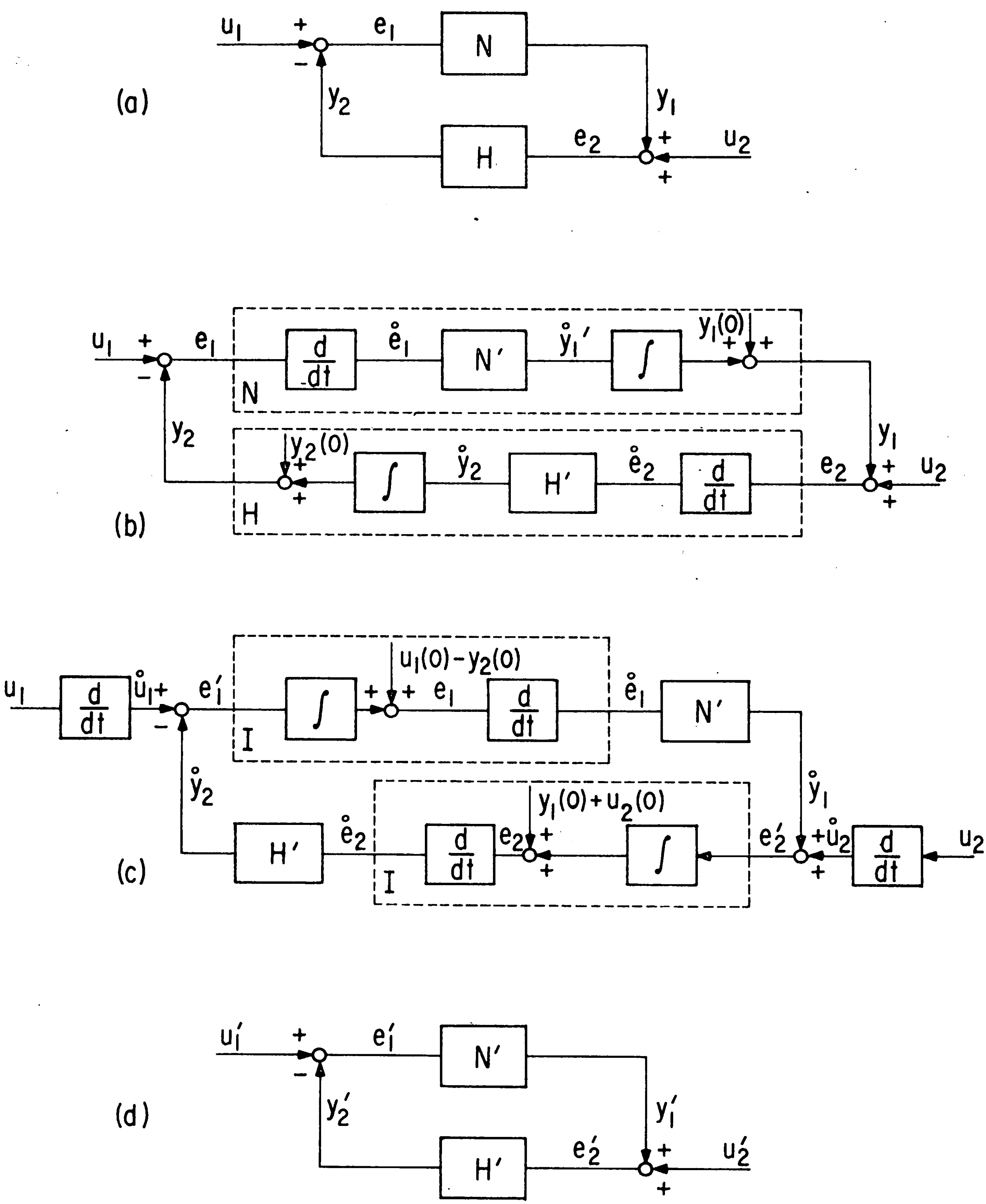


Figure 2 Modeling of the behavior of S.1 with respect to the derivatives by the system S.2

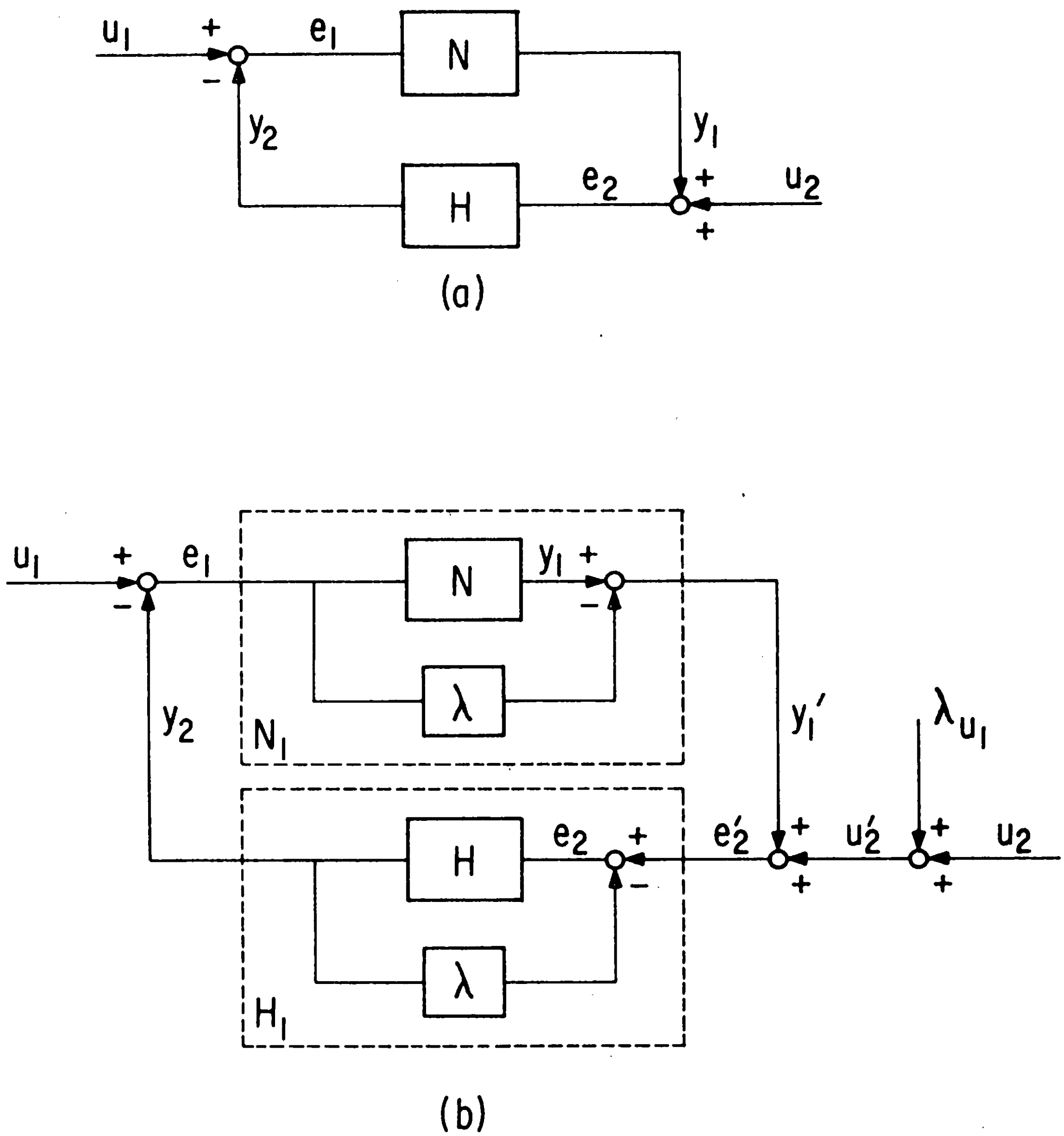


Figure 3 (a) Original system

(b) Transformed system

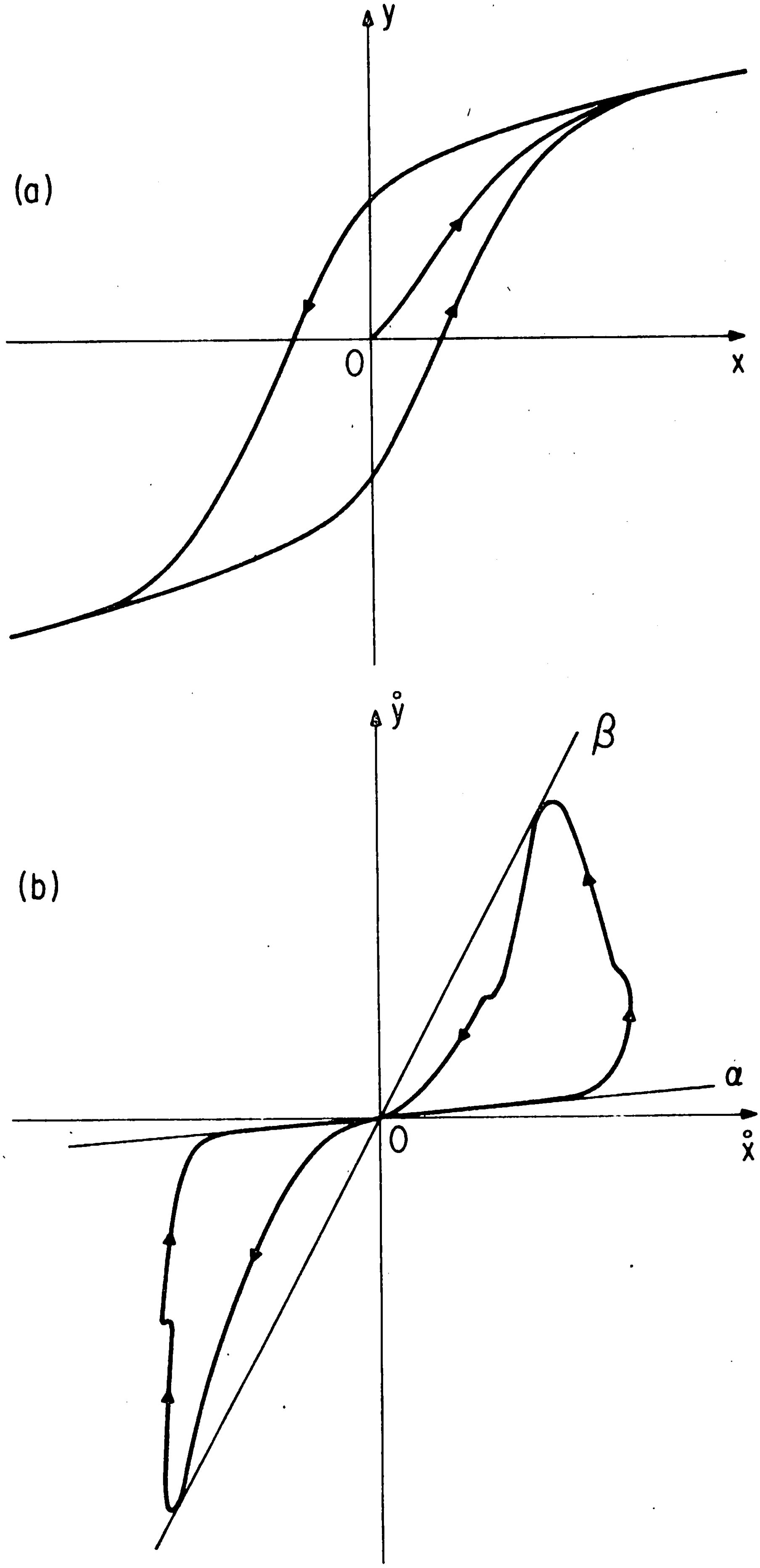


Figure 4 (a) Hysteresis with slopes in the sector (α,β)

(b) Cases to consider when trying to use the Popov-type condition

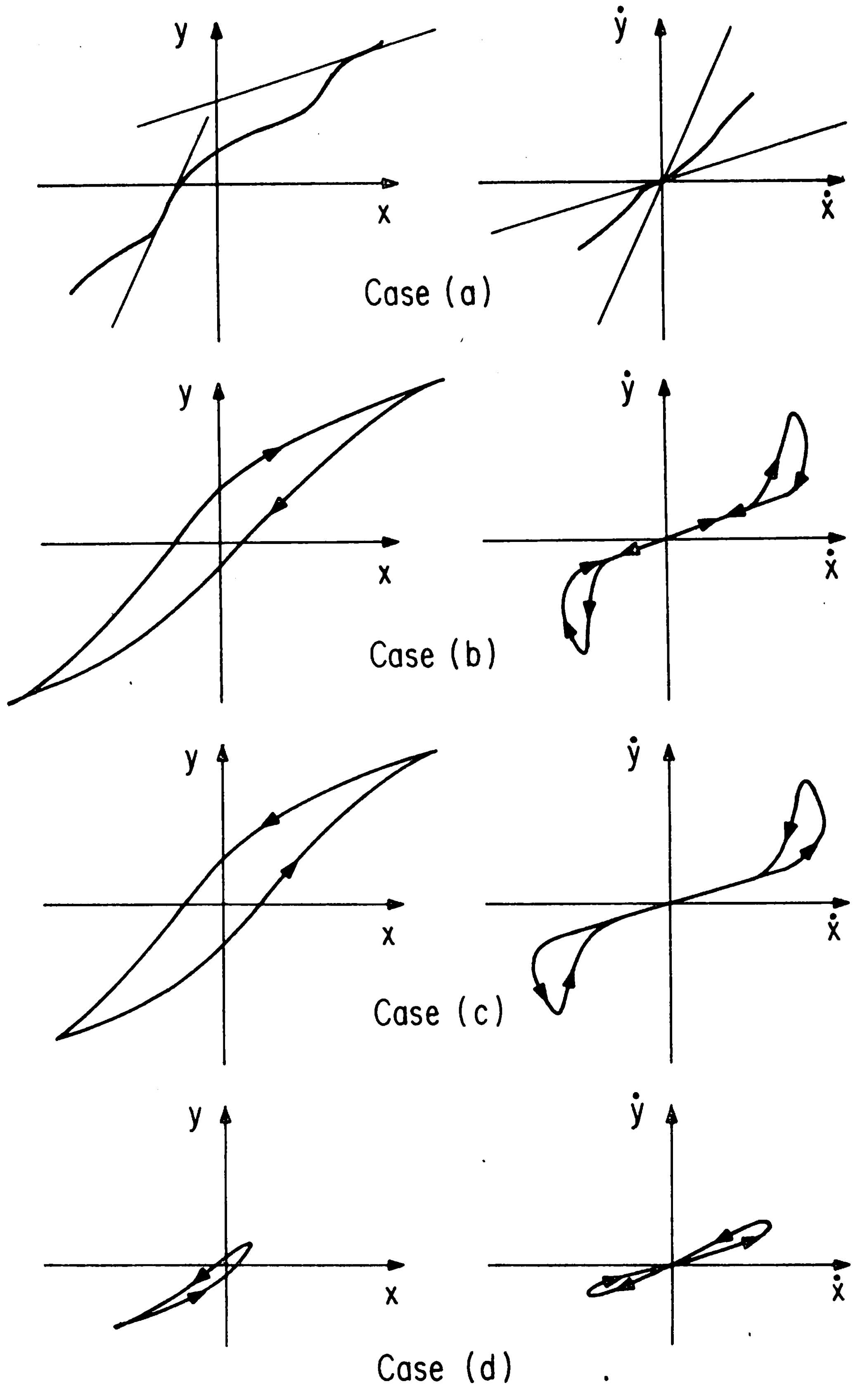


Figure 4-b

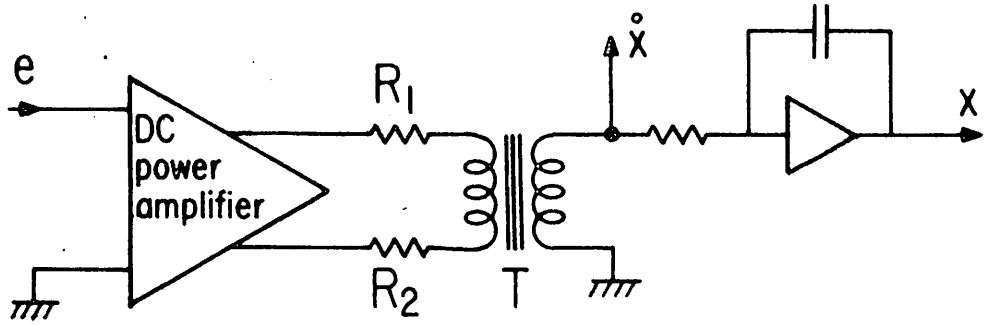


Figure 5 Nonlinearity

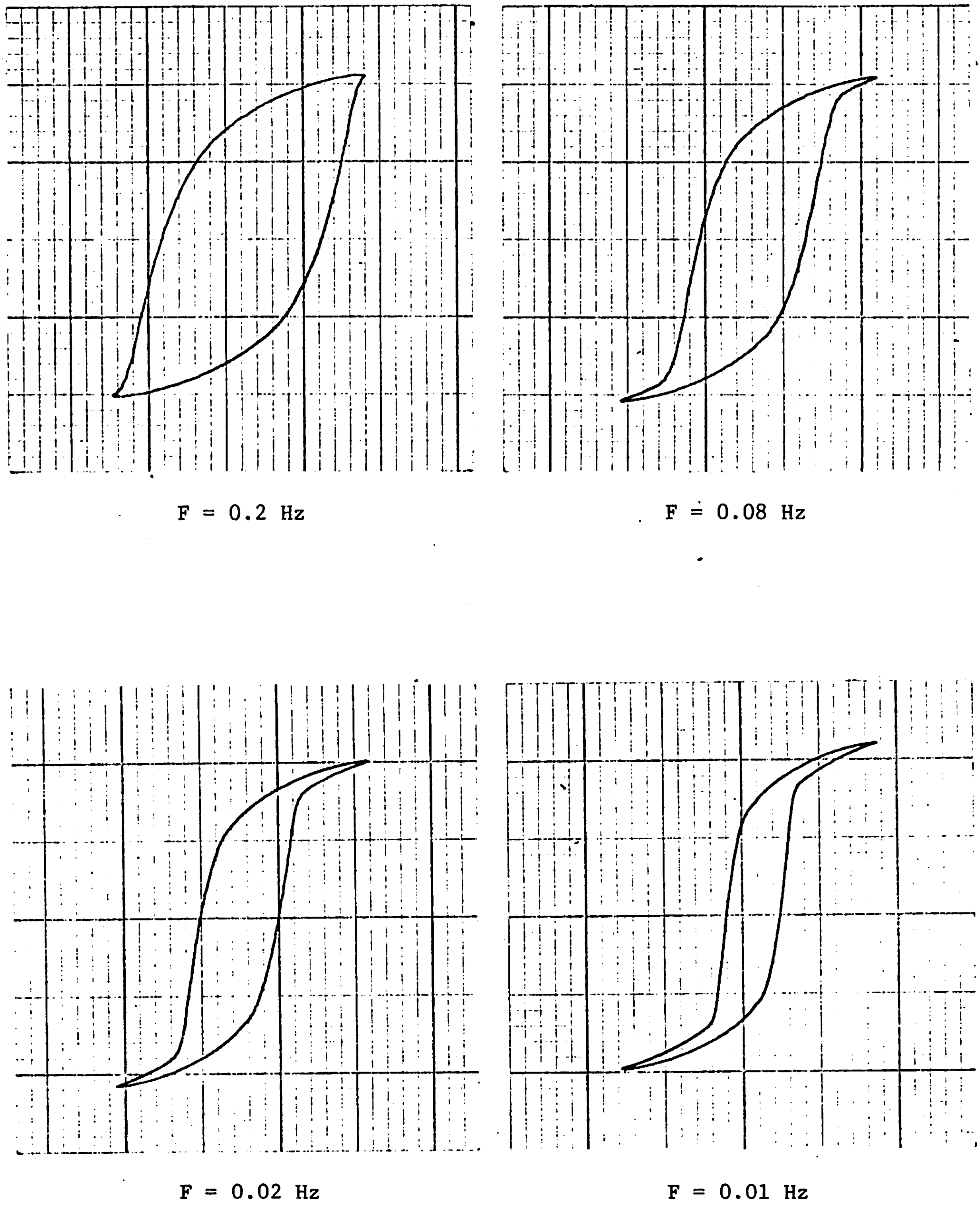


Figure 6 Maximum slope variations

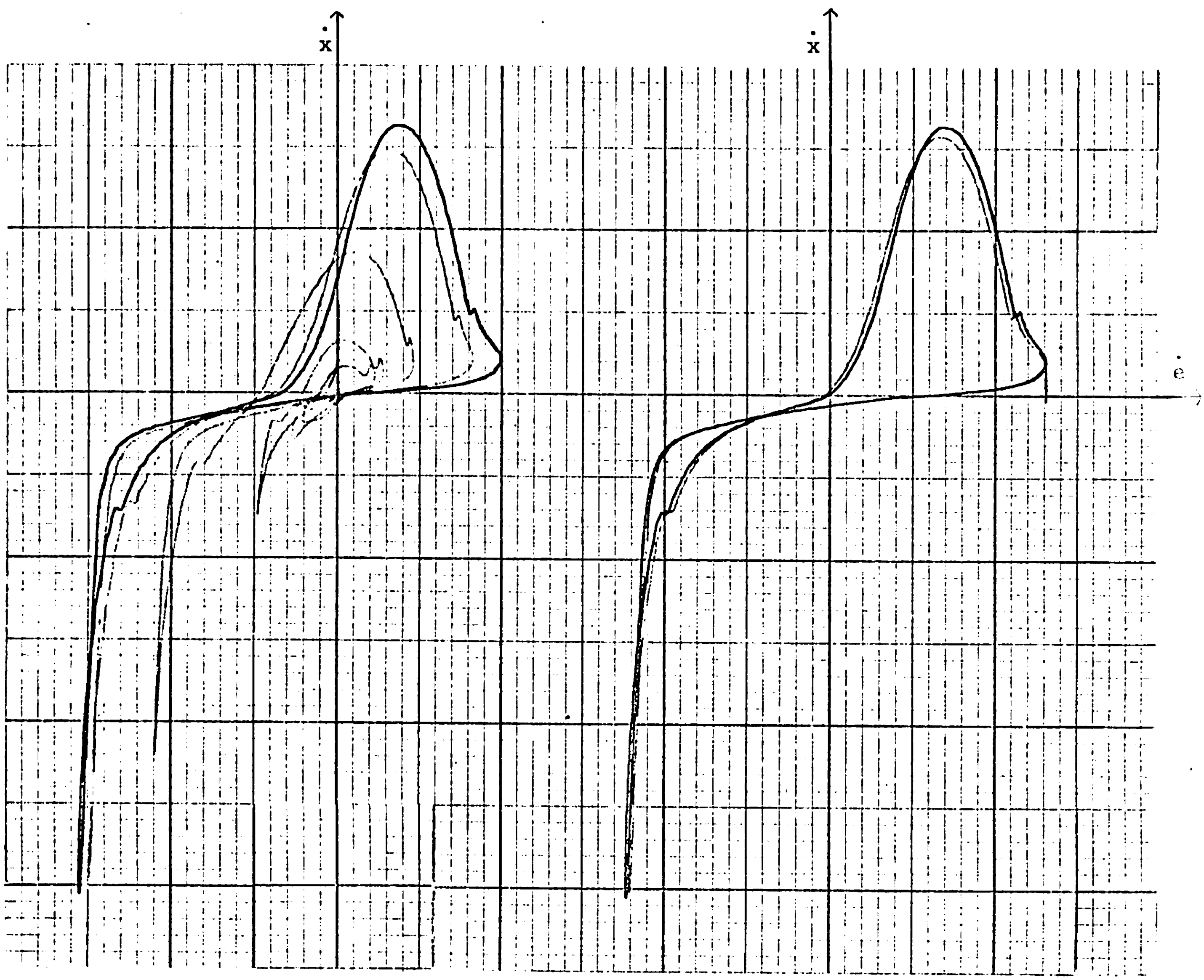


Figure 7

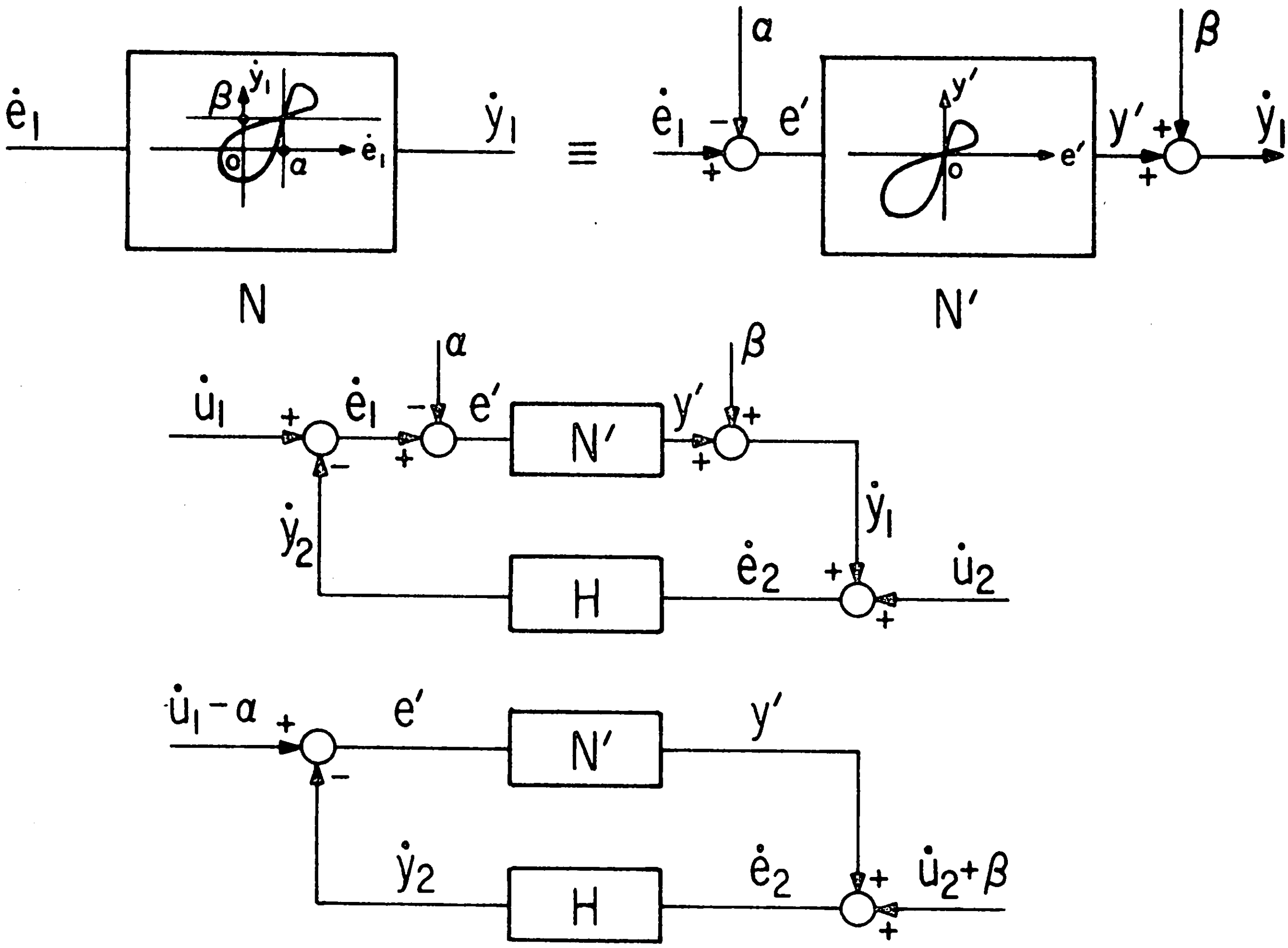


Figure 8 Transformation used to make a system obey the sector condition

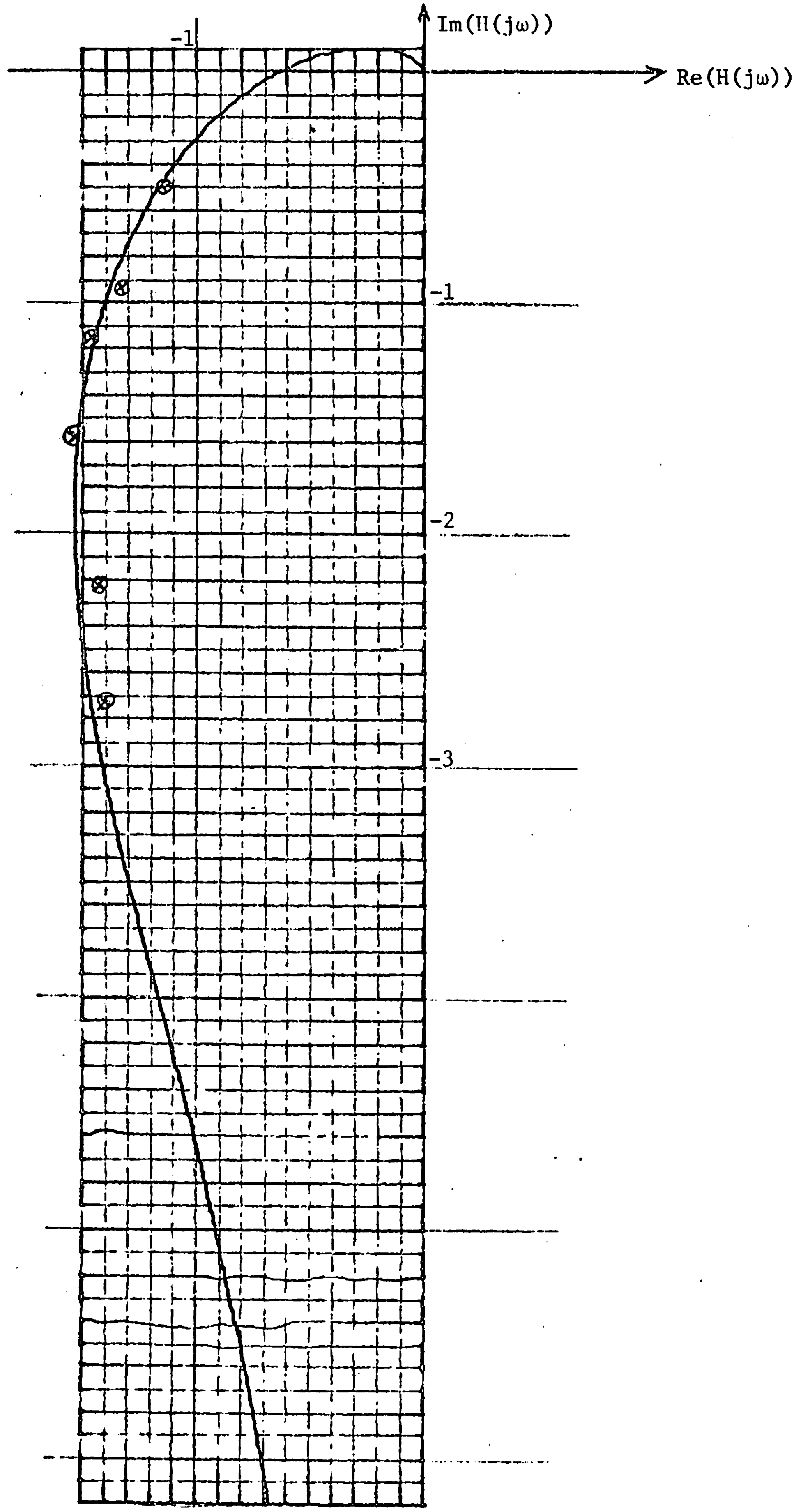
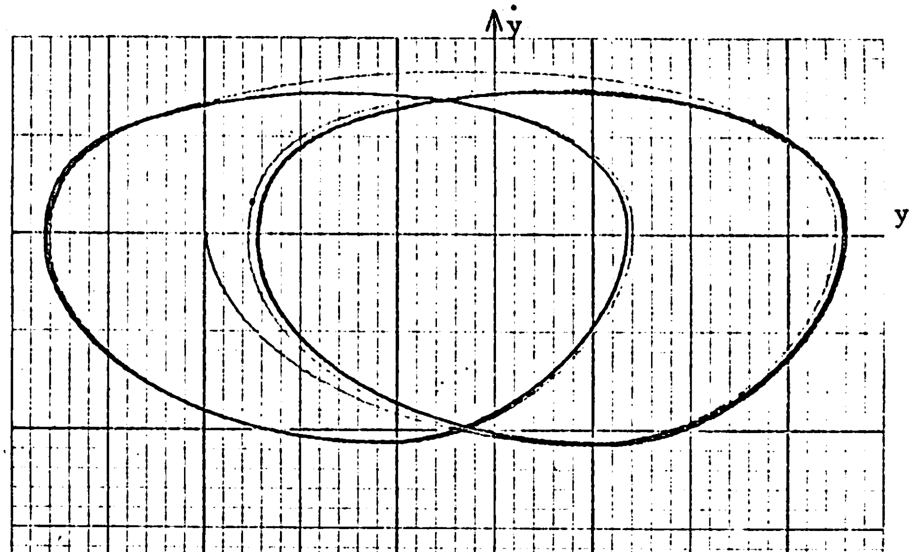


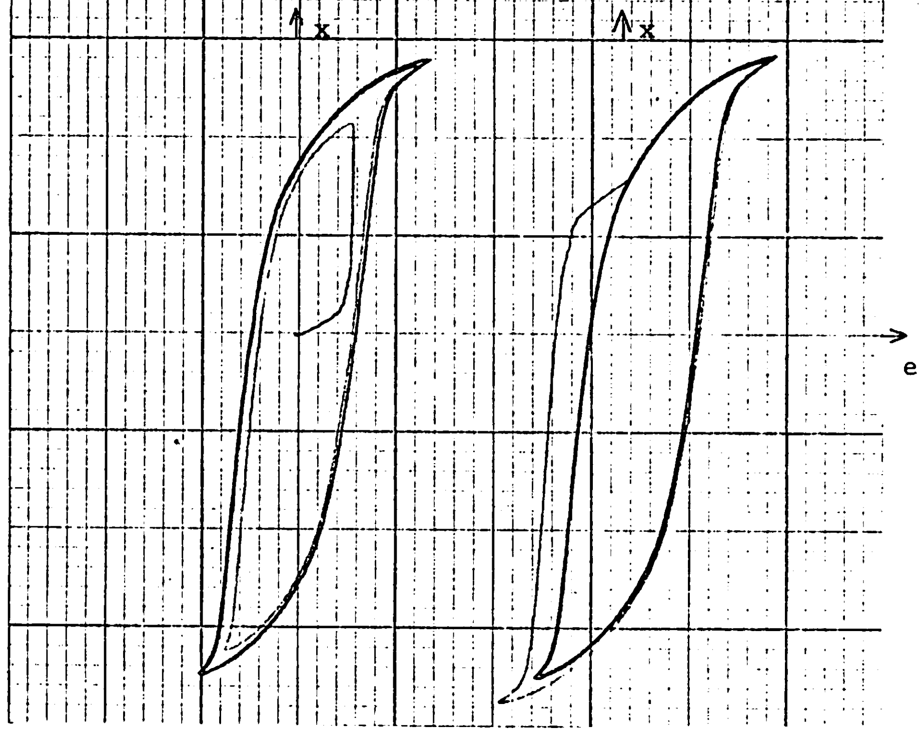
Figure 9

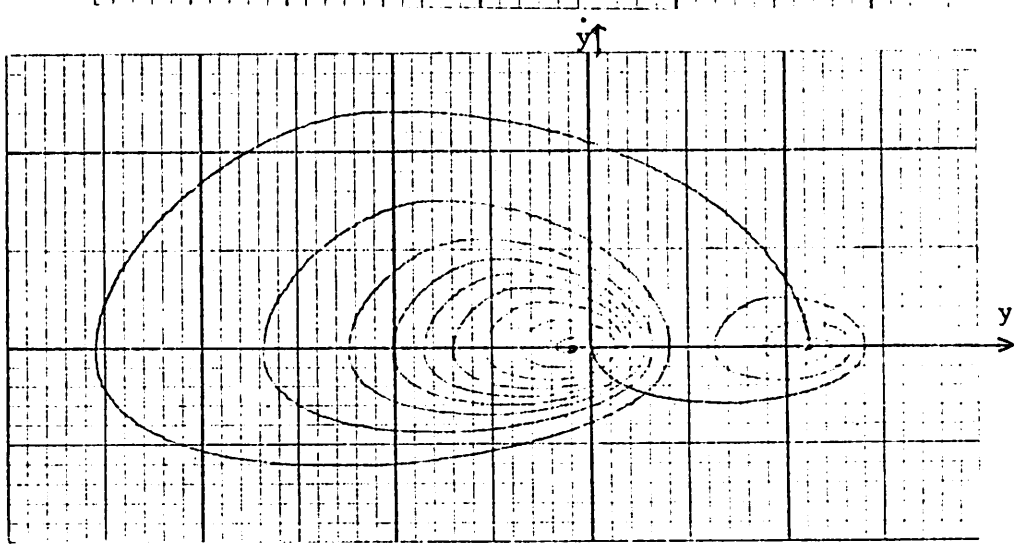
Linear element,
phase plane
gain = 0.213



g = 0.213

Nonlinear element
gain = 0.250





Linear element phase plane, gain = 0.210

Figure 10

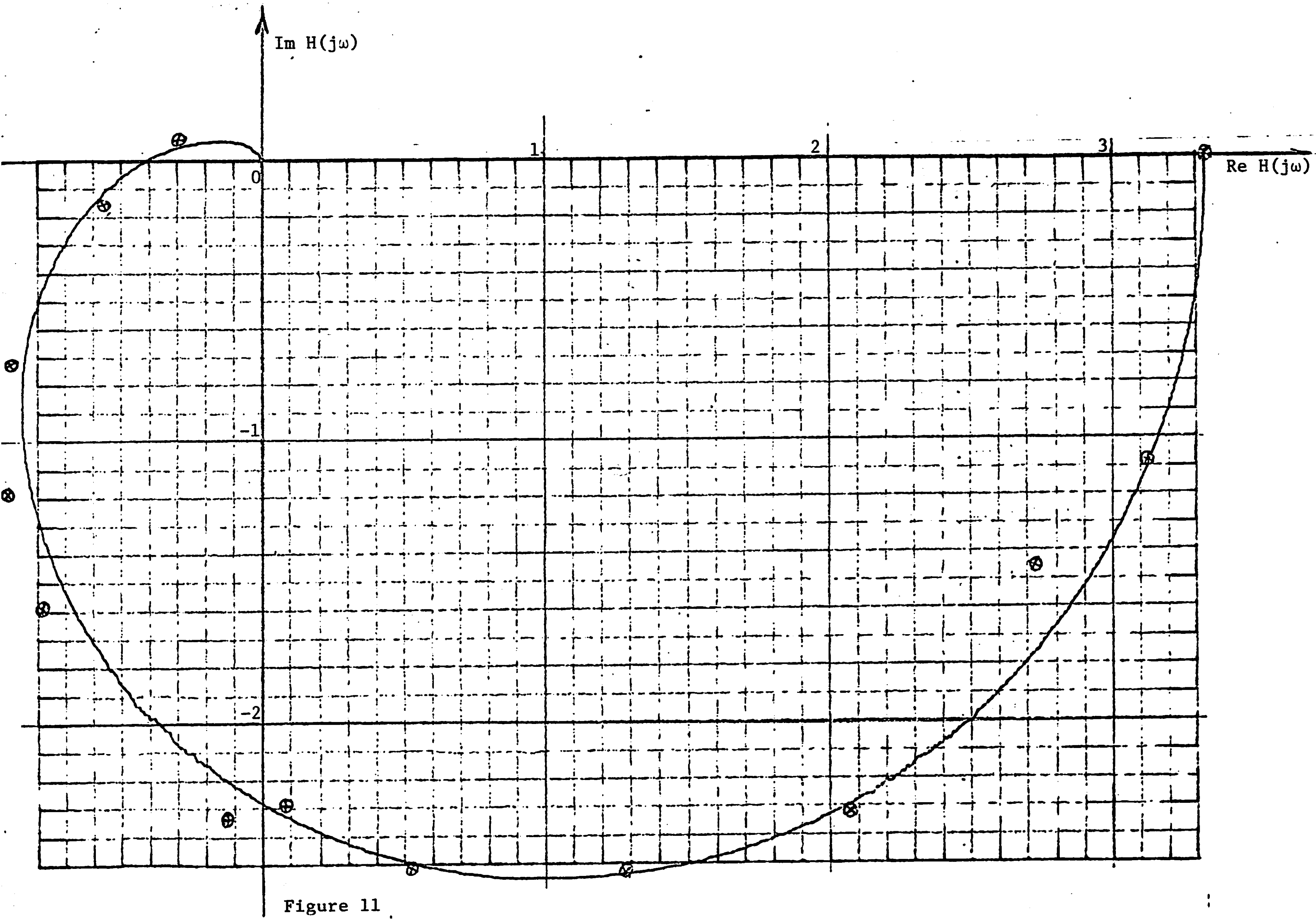
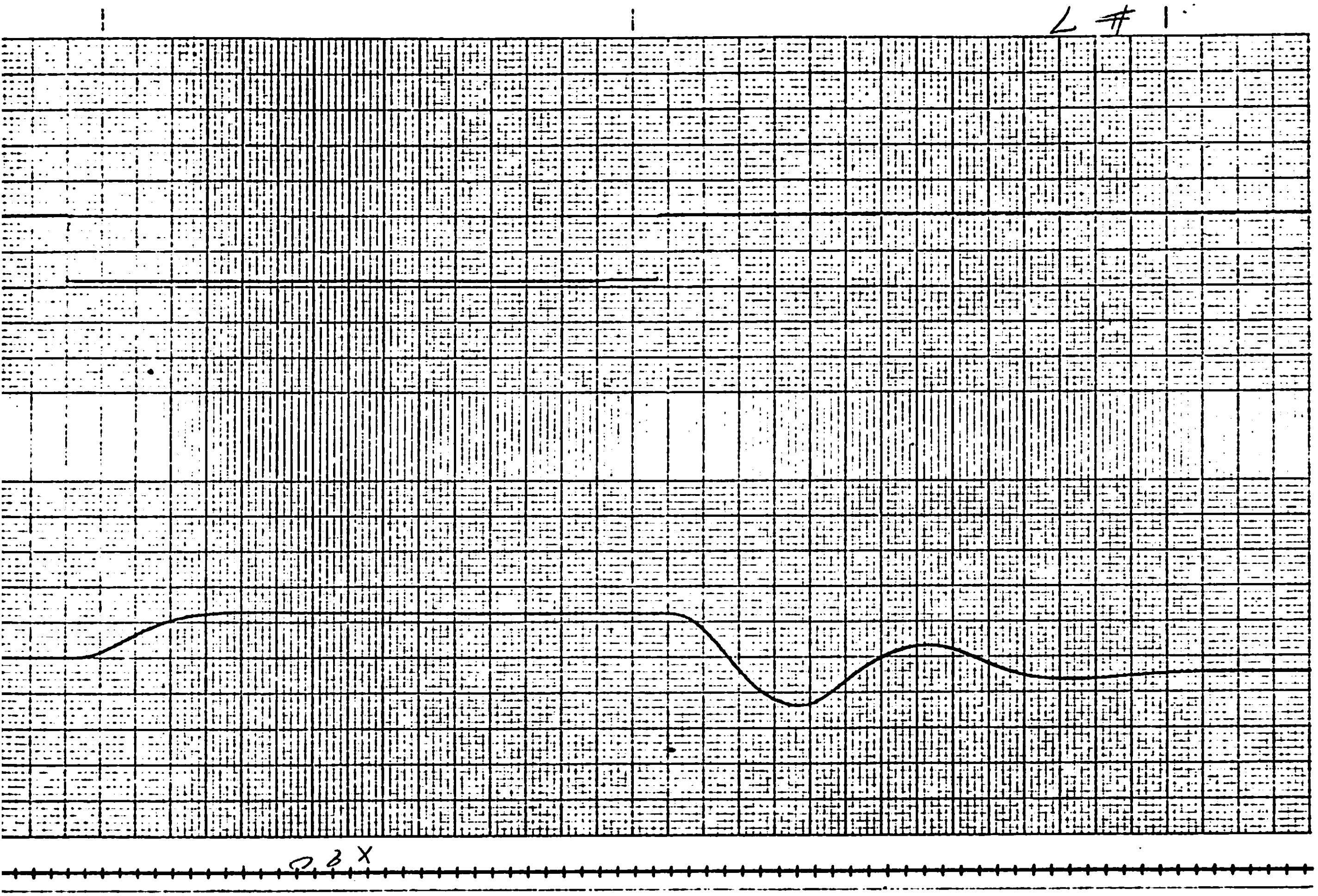


Figure 11



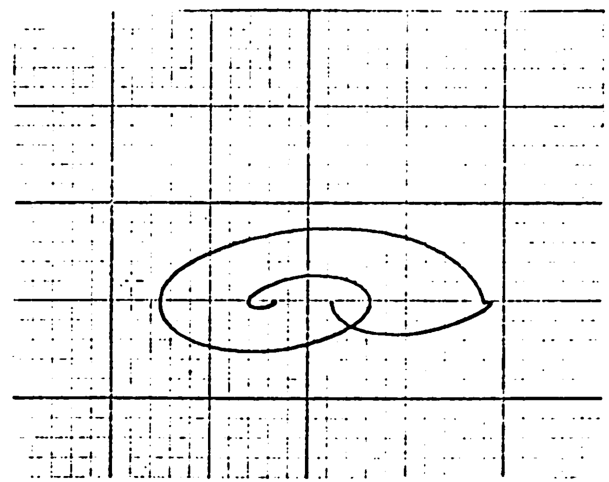
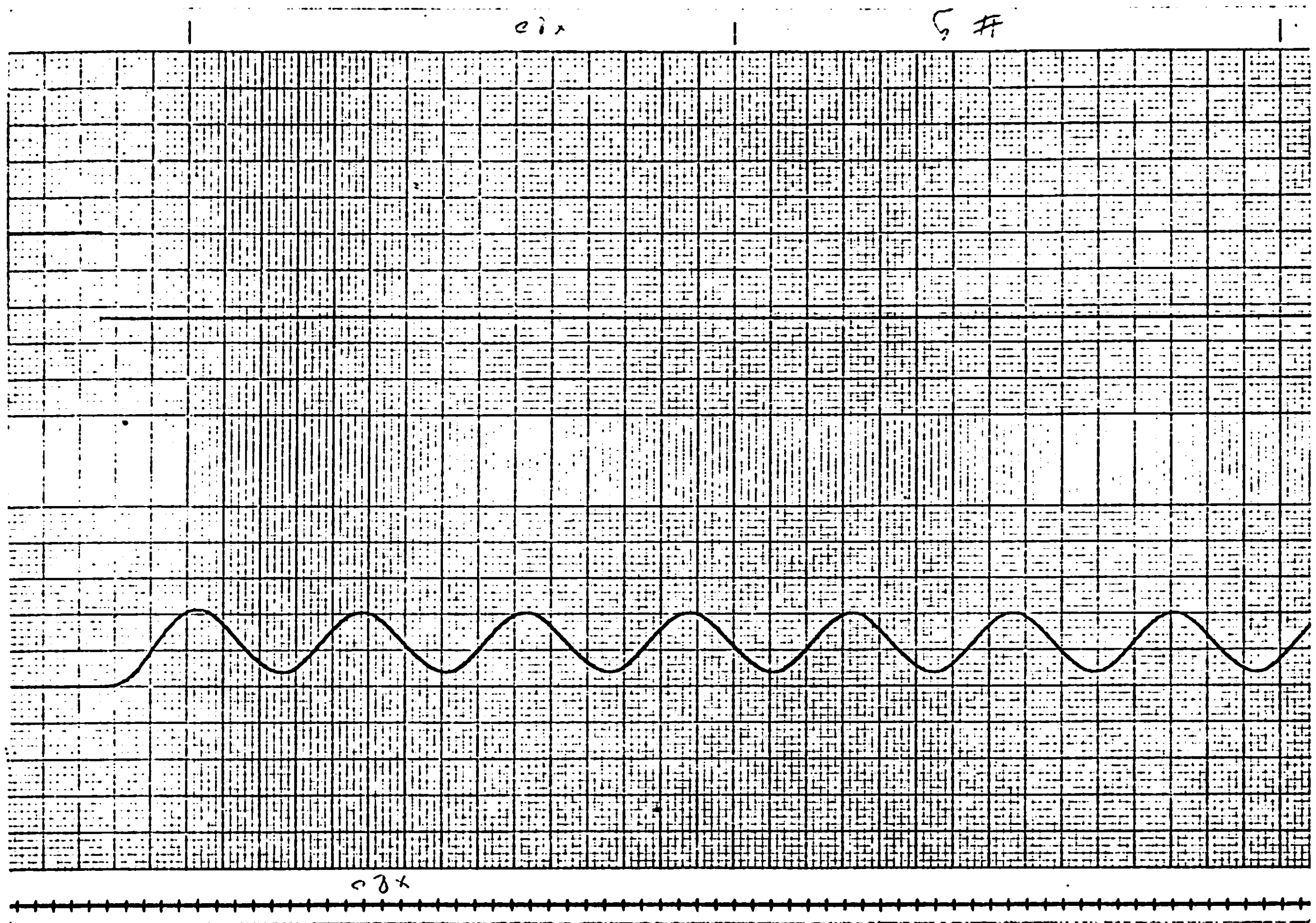


Figure 12.a Response to a step input gain = 0.1

Linear element phase plane



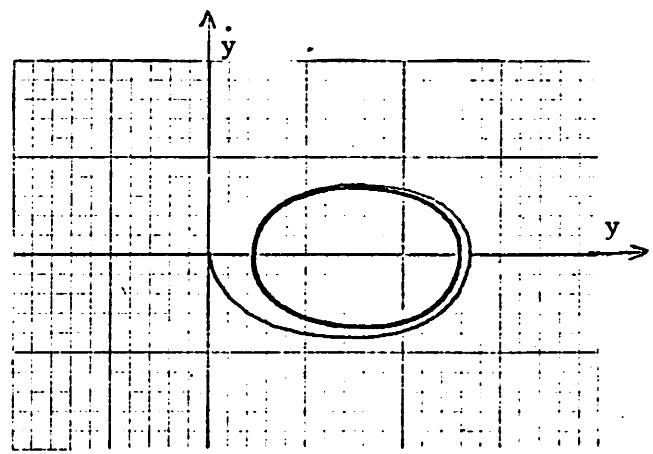


Figure 12.b

gain = 0.4

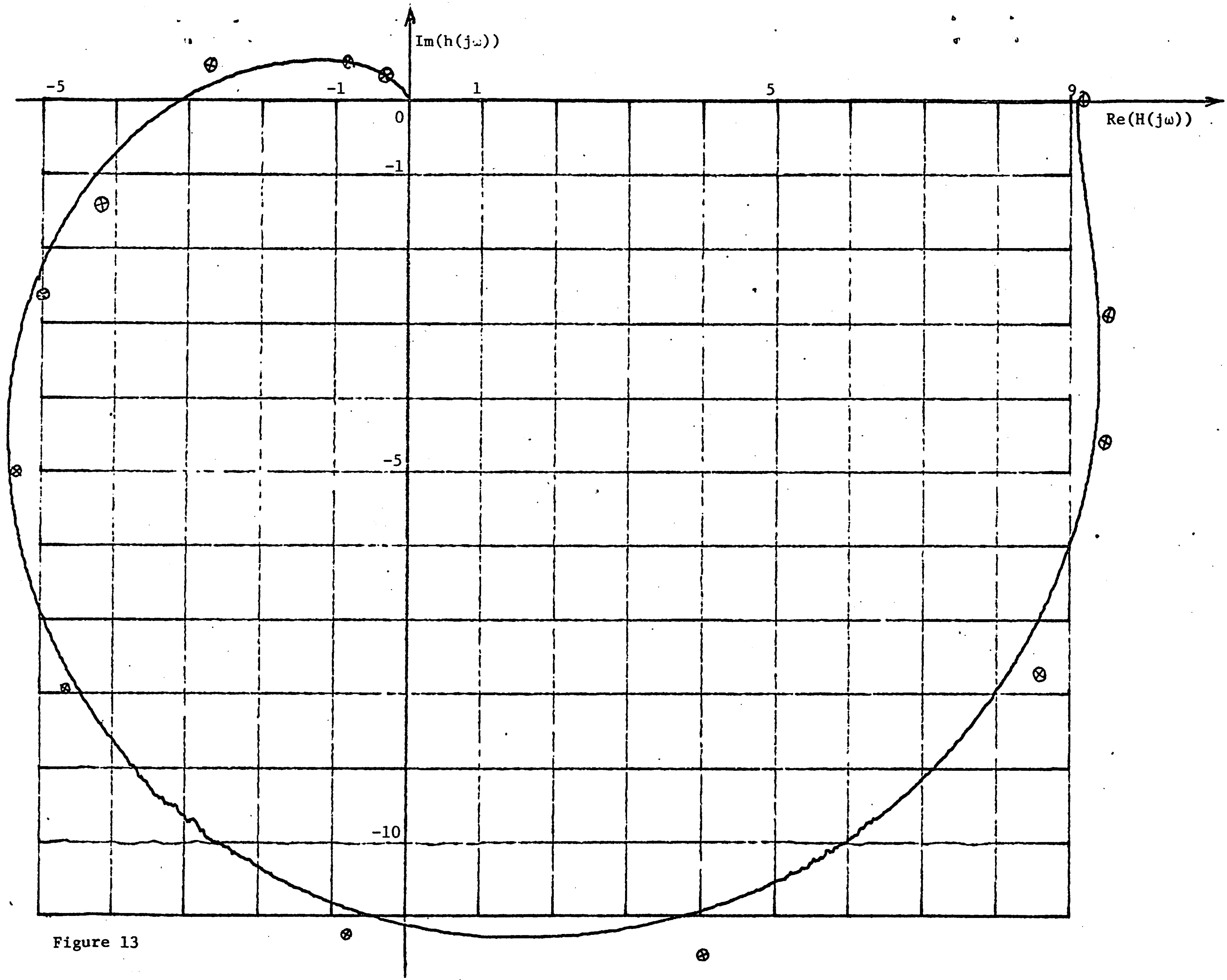
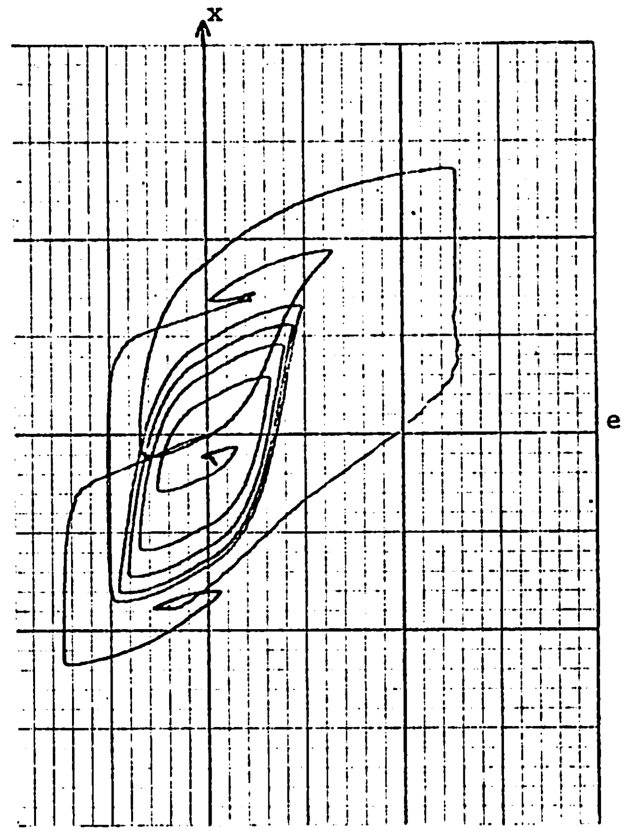
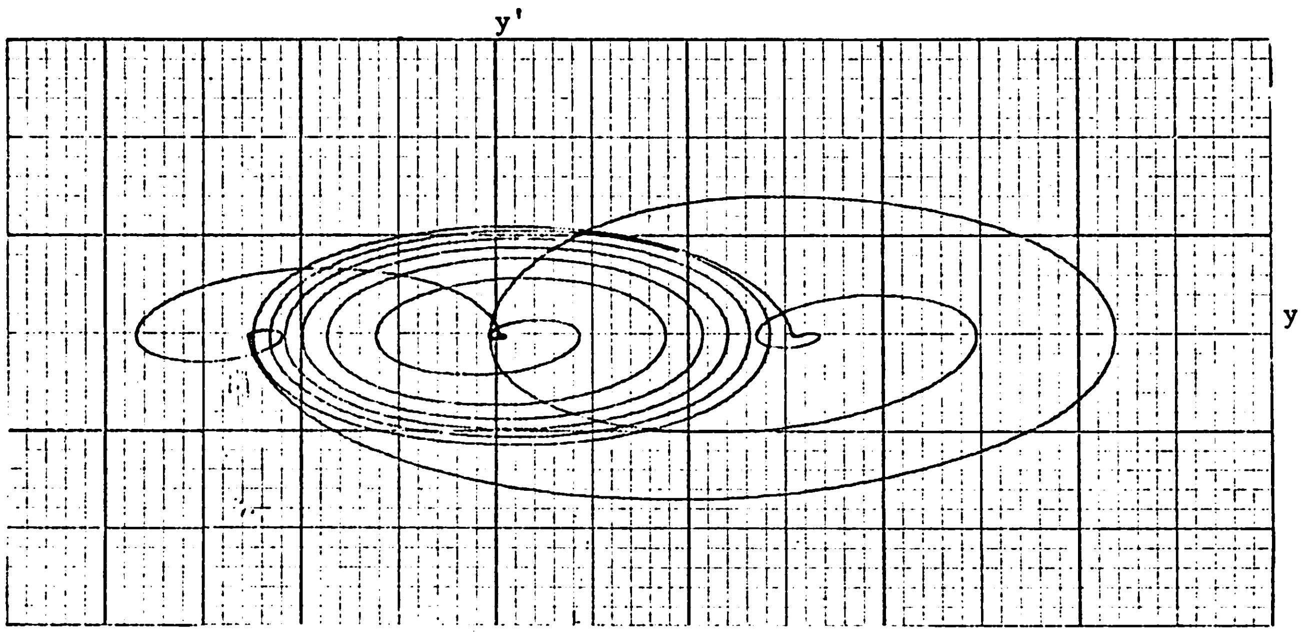


Figure 13

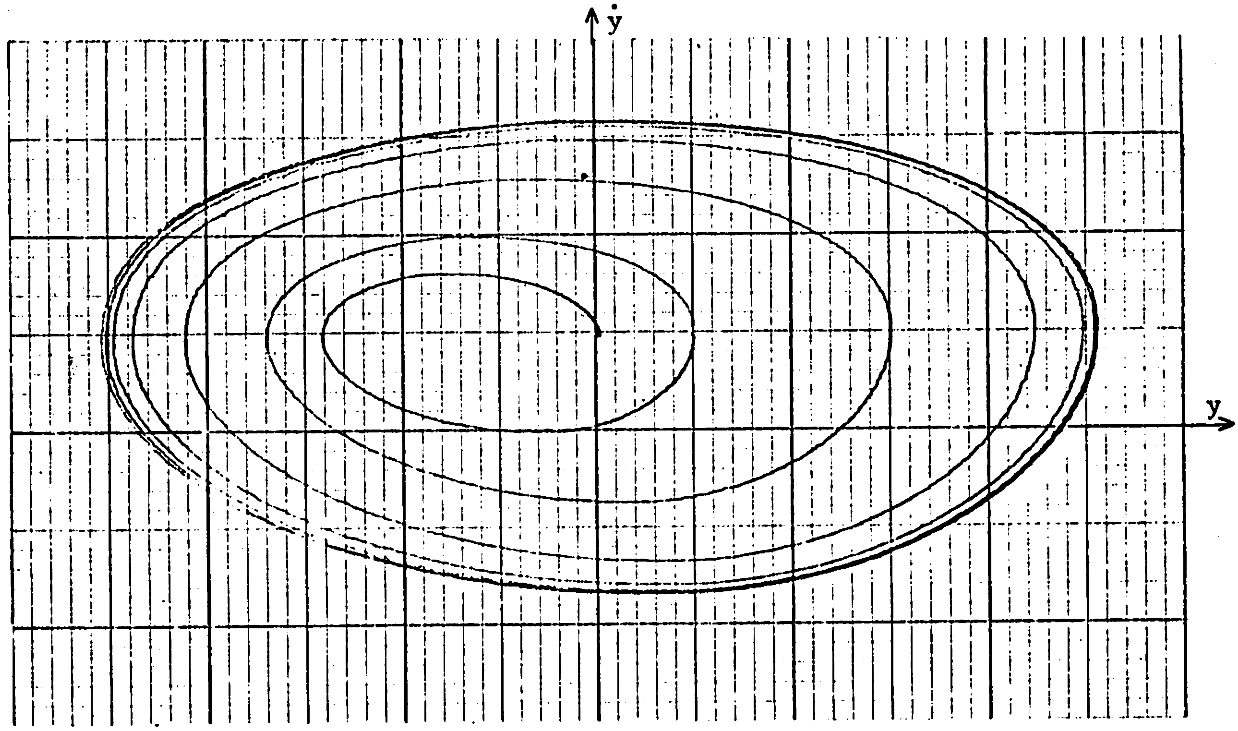


Nonlinear element

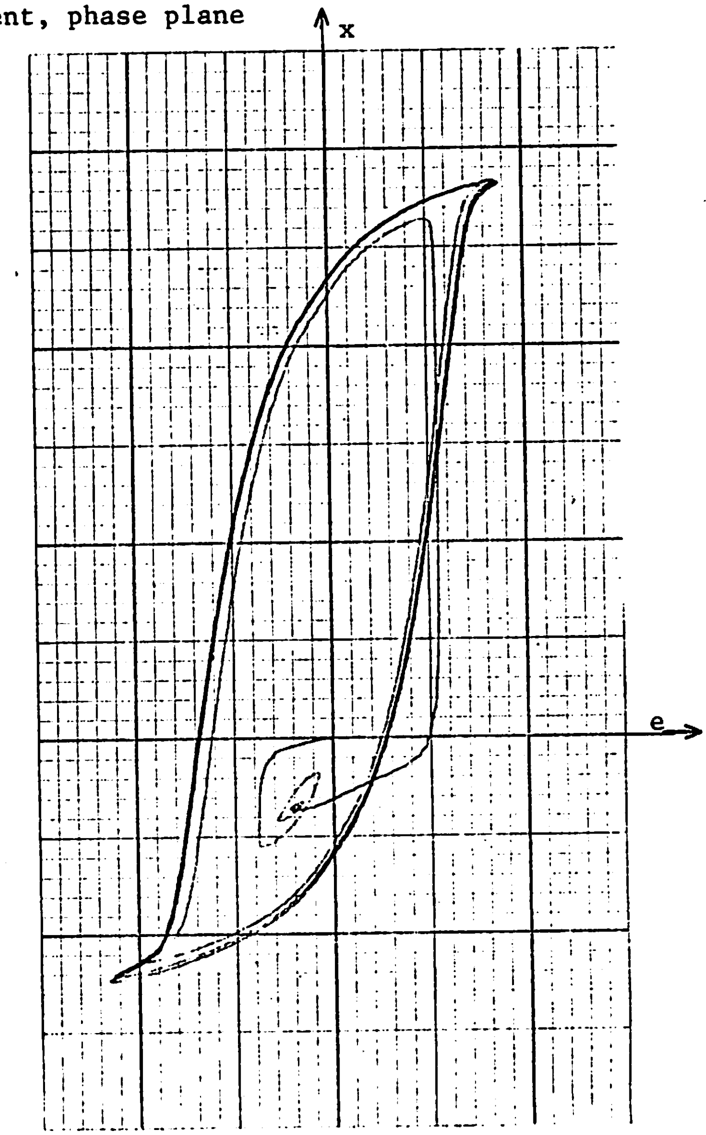


Linear element phase plane

Figure 14.a   Response of the system with a gain of 0.075



Linear element, phase plane



Nonlinear element

Figure 14.b Response of the system with a gain of 0.075

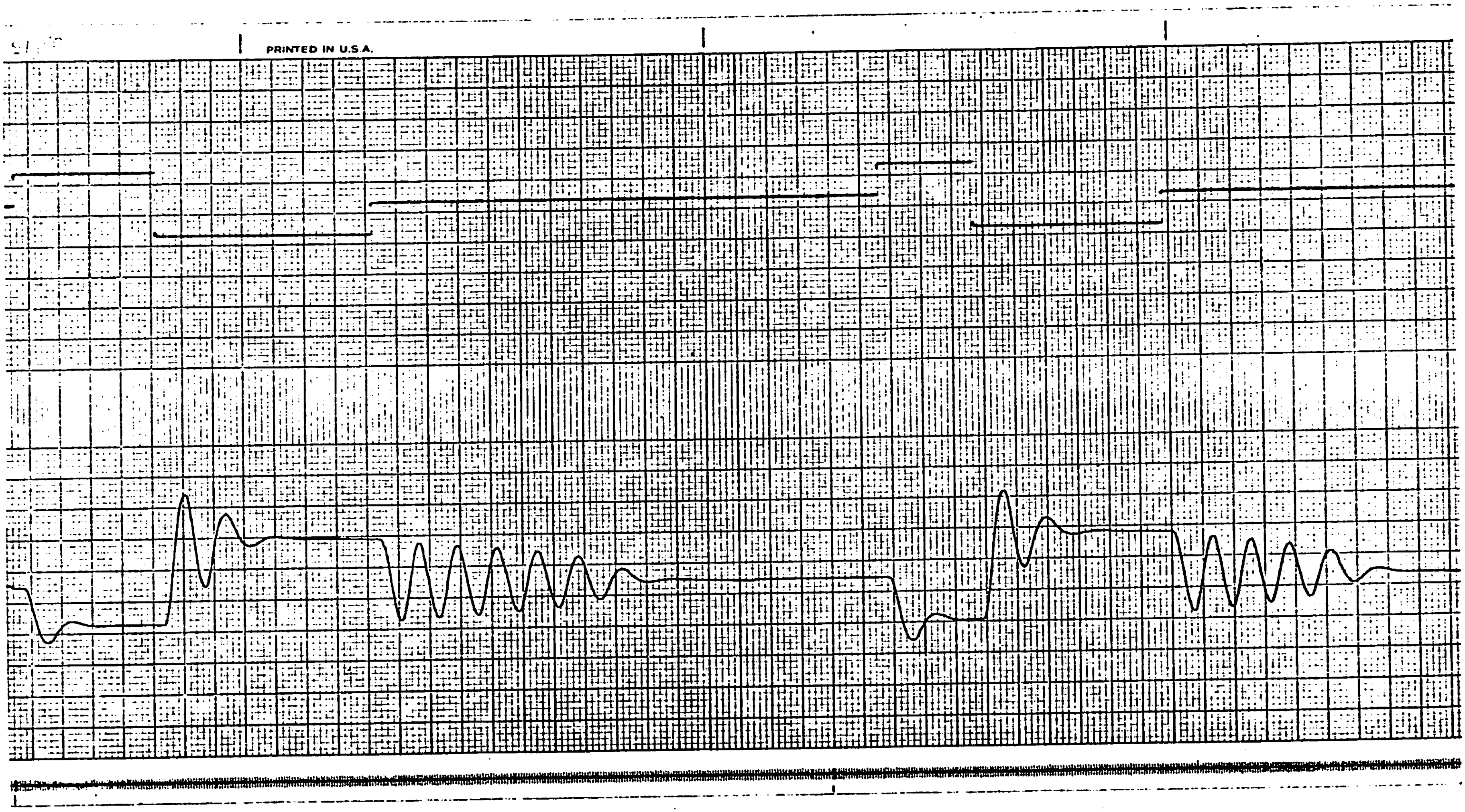


Figure 15.a   Response of the system with gain = 0.075

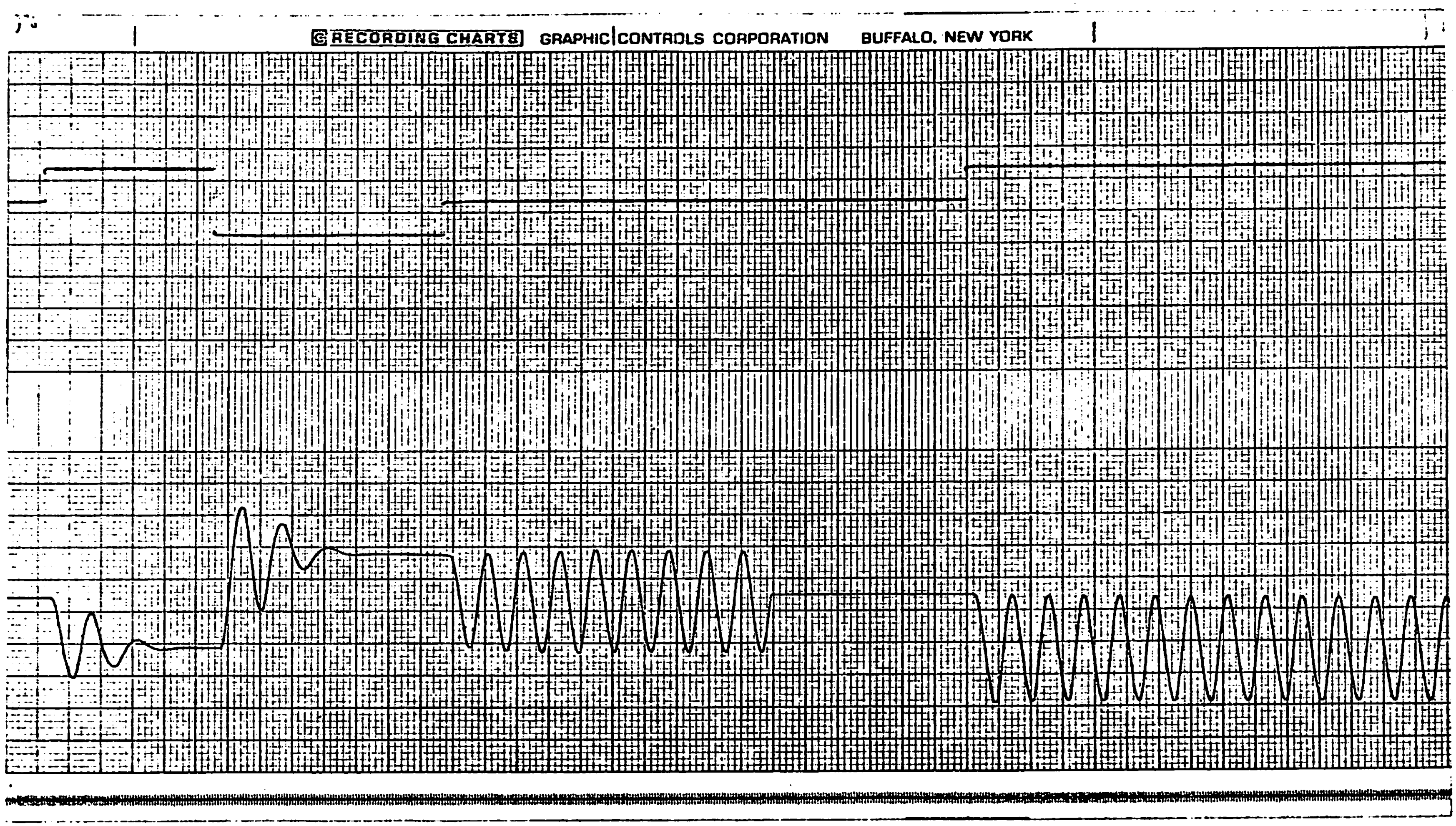


Figure 15.b gain = 0.1