EXTENSIONS TO INSTABILITY RESULTS GIVEN BY J. C. WILLEMS

by

L. P. Lecoq and A. M. Hopkin

Memorandum No. ERL-M272

3 June 1970

ELECTRONICS RESEARCH LABORATORY

College of Engineering
University of California, Berkeley
94720

EXTENSIONS TO INSTABILITY RESULTS GIVEN BY J. C. WILLEMS*

L. P. Lecoq and A. M. Hopkin+

Introduction. The study of instability differs from the study of stability by the fact that it suffices that one input gives an unbounded or oscillatory output for the system to be instable. Since the Lyapunov approach is well adapted to that type of condition it was used by Brockett and Lee [2] to derive a converse to the circle criterion. The functional analysis approach was used by Willems [1] to reach a similar result. An interesting aspect of both results is the importance of the violation of the encirclement condition. The main limitation of both results is the requirement that the nonlinearity stays in the instable sector. It can be observed experimentally that this is not a necessary condition, and that a nonlinearity which penetrates locally in the instability sector is often instable. In this paper it will be shown that if the slopes of the nonlinearity lie in a certain interval and if the system satisfies a few additional conditions instability will follow. This paper is the generalization of a result of Willems [1]. The method used is the extension to instability criteria of an approach used by Lecoq and Hopkin [3]. The stability considered here requires that error and output functions go to a limit as time increases. In addition their derivatives must go to zero For instance a system which is L_2 stable may be instable according to that requirement.

---

*Received

+Department of Electrical Engineering and Computer Sciences, Electronics Research Laboratory, University of California, Berkeley, California 94720.
Research sponsored by the National Science Foundation, Grant GK-10656x.

Some definitions will be necessary

A relation N with domain and range in X is a subset of the product space X × X.

If (x,y) is a pair belonging to the relation N, y will be said to be an image of x under N.

A mapping M of X into X is a relation with domain and range in X which is such that no two members have the same first coordinate, i.e.

$\forall\, x \in Do(M)$ if $(x,y) \in M$ and $(x,z) \in M$ then $y = z$.

The system S.1 (Fig. 1) which is going to be studied is of a classical type. It has a linear time invariant element H and another element N which behavior is only known through an input-output relationship. This element N can be multivalued, time varying and nonlinear. A great number of practical systems can be modeled in this way.

The equations defining the system S.1 are:

$$\begin{cases} \text{C.1.a} \quad e_1(t) = u_1(t) - y_2(t) \\ \text{C.1.b} \quad e_2(t) = u_2(t) + y_1(t) \\ \text{C.1.c} \quad (e_1, y_1) \in N \\ \text{C.1.d} \quad y_2(t) = (He_2)\ (t) \end{cases}$$

where N is a relation with domain and range in X. And H is a linear mapping with domain and range in X.

Assumption 1. Let H be a linear mapping of X into X such that there exists $h \in X$ and a countable set I such that

$$(0.3) \quad \begin{cases} h(t) = 0 \quad \forall\, t < 0 \\ h(t) = h_a(t) + \sum_{i \in I} h_i \delta(t - t_i) \quad \forall\, t \geq 0 \end{cases}$$

and

$$h_a(t) \in L_1[0,\infty), \quad \sum_{i \in I} |h_i| < \infty$$

defining H in the following way

$$(0.4) \qquad (Hx)\ (t) = (h*x)\ (t) = \int_0^t h(t-\tau)\ x(\tau)\ d\tau \ \forall\ x \in Do(H)$$

in addition let there exists two positive real constants α and β, $0 < \alpha < \beta < \infty$ such that the Nyquist locus of H encircles in the clockwise direction the circle of center $-\frac{\alpha + \beta}{2\alpha\beta}$ and of radius $\frac{\beta - \alpha}{2\alpha\beta}$.

If h(t) has δ-functions then either $t_1 \neq 0$ or in the case when $t_1 = 0$ h_1^{-1} does not lie in to the interval $[\alpha,\beta]$

Assumption 2. Let the nonlinear element N be defined by a linear time varying gain k(t), and let there exists two positive real constants α, β such that

(i) $\alpha < k(t) < \beta$

(ii) k(t) is integrable in the sense of Lebesgue over every finite interval

Assumption 3. Let the nonlinear element N be such that there exists two positive real constants α, β such that

$$\begin{cases} \alpha\, \dot{x}^2(t) < \dot{x}(t)\ \dot{y}(t) < \beta\, \dot{x}^2(t) & \forall\ t \in [0,\infty) \\ \dot{y} = 0 \text{ whenever } \dot{x} = 0 & \end{cases}$$

$\forall$ images y of x under N, $\forall$ $x \in Do(N)$ and having a derivative belonging to $L_{2e}[0,\infty)$

Assumption 4. Let the nonlinear element N be such that there exists four real constants α, β, D, E with $0 < \alpha < \beta$, $D < E$ such that

$$\begin{cases} \alpha \dot{x}^2(t) < \dot{x}(t)\, \dot{y}(t) < \beta \dot{x}^2(t) & \forall\, t \in \{t \mid D < x(t) < E\} \\ \dot{y} = 0 \text{ whenever } \dot{x} = 0 \end{cases}$$

$\forall$ images y of x under N, $\forall$ $x \in Do(N)$ and having a derivative belonging to $L_{2e}[0,\infty)$

The following lemma is a result of J. C. Willems [1] (theorem 8.1)

Lemma 1. let the system S.1 be such that there exists two positive real constants α and β, $\alpha < \beta$ with which

N satisfies assumption 2

H satisfies assumption 1

Then the system is instable

Corollary 2

Let the system S.1 be such that there exists two positive real constants α and β, $\alpha < \beta$ with which

N satisfies assumption 3

H satisfies assumption 1

Then the system is instable.

In other words if the slopes of the nonlinearity stay inside the interval (α,β) the system is instable.

Proof:

Let the relation N' be defined as follows

$$(x',y') \in N' \text{ iff } \exists\ (x,y) \in N \text{ and } \dot{x} = x',\ \dot{y} = y'.$$

Since N satisfies assumption 3, N' satisfies assumption 2.

Modeling the behavior of the system S.1 with respect to the derivatives by a system S.2 of type S.1, where the nonlinear element is N', the linear element H', the inputs, outputs and errors are the derivatives of the inputs, outputs errors of S.1, it can be noted at once that H' = H since the derivative of a convolution is the convolution of one of the factors with the derivative of the other (L. Schwartz [4] Vol. 2, Chapt. 6, Thm. 9). i.e.

$$\dot{\overline{h*e_2}} = h*\dot{e}_2$$

It can then be seen that the system S.2 satisfies the hypothesis of Lemma 1 hence that S.2 is instable. This implies that S.1 is instable. The concepts used in the proof of this corollary are delineated more carefully in reference [3].

<u>Theorem 3</u>

Let the system S.1 be such that

(i) H satisfies assumption 1

(ii) There exists two real constants D and E with which N satisfies assumption 4

(iii) Denote by S_A the set of points:

$$S_A = \{x | x + \hat{h}(0)y = A\}$$

if there exists an A such that the intersection of N and S_A belong to the interval (D, E) i.e. there is no point in the intersection of N and S_A which is outside the interval (D, E).

Then the system is instable.

Proof:

In order for $e_1(t)$ to have a limit $e_{1\infty}$ as time increases the following condition must be satisfied

$$\text{given} \quad \varepsilon > 0 \ \exists \ T_\varepsilon > 0 \ \ni \ \forall \ t > T_\varepsilon \quad |e_1(t) - e_{1\infty}| < \varepsilon$$

in addition $e_{1\infty}$ must satisfy the following equation

$$u_{1\infty} - \hat{h}(0)u_{2\infty} = e_{1\infty} + \hat{h}(0)y_{1\infty}$$

consider the class of inputs

$$\{(u_1,u_2)\,|\,u_{1\infty} - \hat{h}(0)u_{2\infty} = A\}$$

Assume the system is stable for this class of inputs. The limit $e_{1\infty}$ must belong to (D, E) by hypothesis

$$\text{hence} \quad \exists \ T > 0 \ \ni \ \forall \ t > T \quad e_1(t) \in (D, E)$$

but this implies that $\forall$ $t > T$ the slopes of the nonlinearity stay inside the interval (α,β) and by application of corollary 2 the system is not stable. A contradiction is reached. The system being instable for a class of inputs is instable.

As a consequence of theorem 3 the instability criterion is generalized to cases when the slopes of the nonlinearity do not stay in the instability interval, as well as to cases when the nonlinearity does not stay in an instability sector.

Example 1

Let N be a nonlinearity which stays in an instability sector with slopes in the same interval, around the origin (figure 2) then the system is instable if $\hat{h}(0) > 0$. To see this, it suffices to pick u_1, u_2 such that $u_{1\infty} = u_{2\infty} = 0$ then $e_{1\infty} + \hat{h}(0)y_{1\infty} = 0$ which implies $e_{1\infty} = 0$ a contradiction is reached.

Example 2

Let N satisfy the hypothesis of theorem 3 i.e. there exists an A such that the line

$$y = \frac{A - e_{1\infty}}{\hat{h}(0)}$$ intersect N in the interval (D, E) where the

slopes belong to the instability interval (figure 3).

Remark

Considering the relation N' modeling the behavior of N with respect to the derivatives of the functions in the system, a stability sector can be obtained for N' [3]. The criterion developed here defines an instability sector for N'. It should be noted that these two sectors are not adjacents. As a consequence the fact that often a system is stable for a gain higher than was computed does not contradict the instability criterion. By a careful estimate of both sectors it is possible to obtain the region in which the system becomes instable.

Since saturation plays an important role in the phenomenon of hysteresis the slope is going to be in its highest range around the origin. The hypothesis of theorem 3 will be satisfied and its conclusions will apply. Interestingly enough it is possible to verify this experimentally. It can be observed that with a gain insuring instability a step away from the origin does not trigger oscillations while a step back to the origin or through the origin triggers them.

Conclusion. The requirement that a system needs to stay in the instability sector in order to prove instability can be weakened in two ways. Either there must exist an interval around the origin where the nonlinearity stays in the instability sector, or there must exist a region in which the slopes of the nonlinearity stay in the interval bounded by the slopes of the instability sector and, in addition, $\hat{h}(0)$ must satisfy a certain condition.

The criterion for instability will, then, give an upper bound on the limit gain. Used in conjunction with a stability criterion giving a lower bound on the limit gain, it will define a region in which the limit gain lies. This region can be used to obtain the adjustment range in the design of a system.

References

[1] J. C. Willems, "Stability, Instability, Invertibility and Causality," SIAM J. Control, Vol. 7, No. 4 (November 1969).

[2] R. W. Brockett and H. B. Lee, "Frequency Domain Instability Criteria for Time-Varying and Nonlinear Systems," Proceedings of the IEEE, Vol. 55, No. 5 (May 1967).

[3] L. P. Lecoq and A. M. Hopkin, "A Functional Analysis Approach to L_∞ Stability and its Applications to Systems with Hysteresis," Technical Memorandum No. ERL-M269, Electronics Research Laboratory, University of California, Berkeley, California (January 1970).

[4] L. Schwartz, "Theorie des Distributions," Tome 1 and 2, Hermann and Co., Paris (1950).

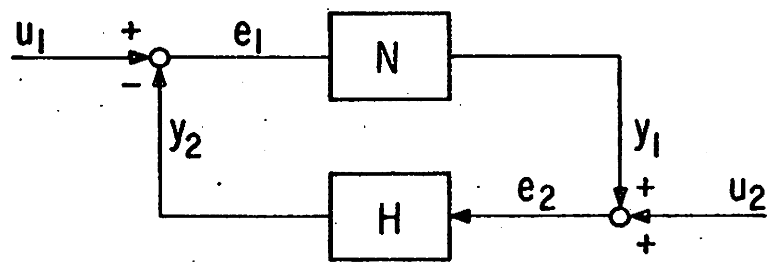

Figure 1 System S.1

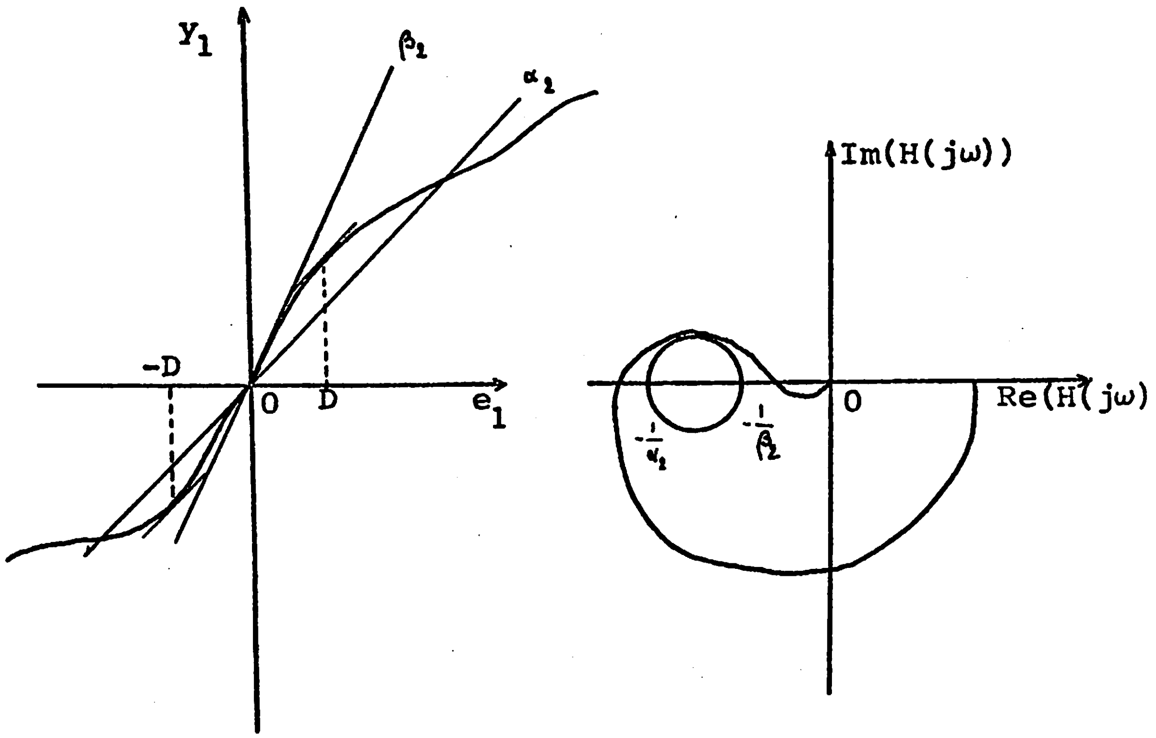

Figure 2

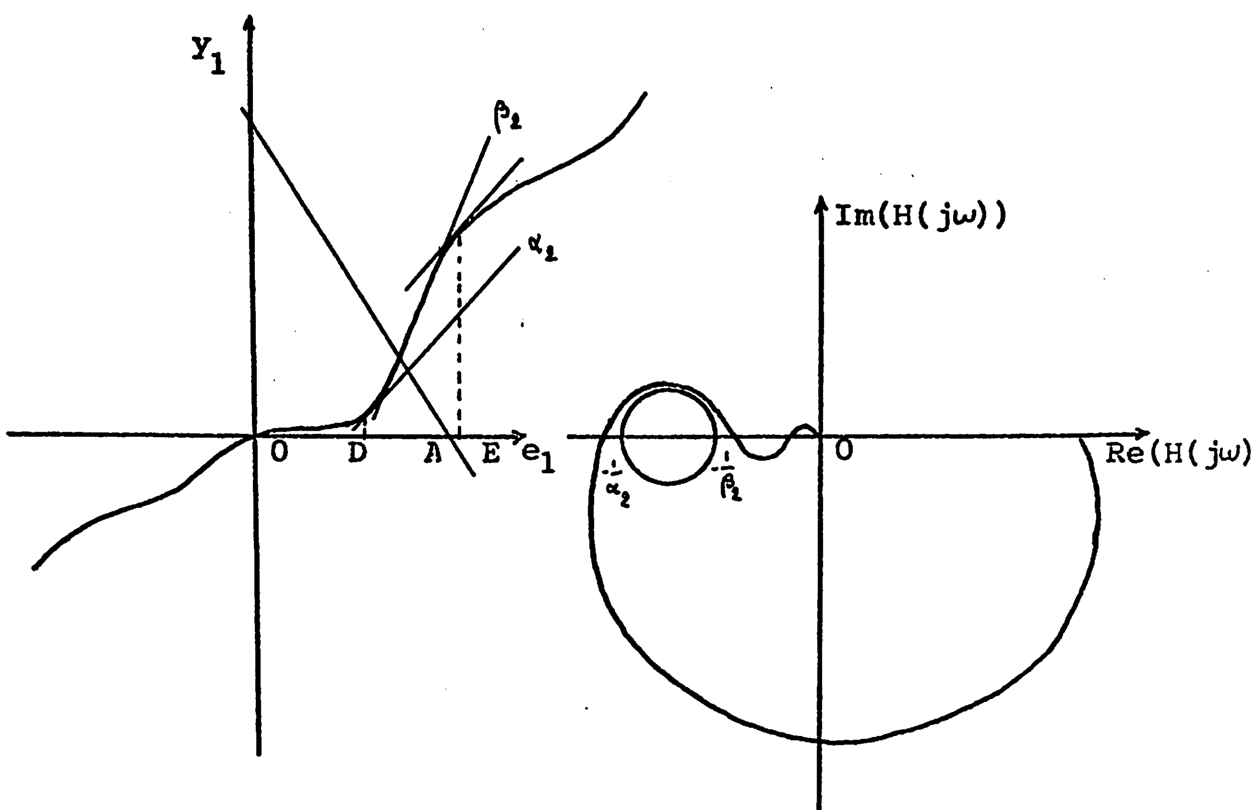

Figure 3